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A.N. Leznov, M.V. Saveliev

FINITE SOLUTIONS OF CLASSICAL YANG-MILLS EQUATIONS

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Лезнов А.Н., Савельев М.В.

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В работе получено в явном виде ограниченное решение для конфигурации двух псевдочастиц (инстантонов) калибровочной $SU(2)$-теории с топологическим зарядом равным двум, которое зависит от тринадцати произвольных параметров.

Abstract

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We present an explicit form of nonsingular solutions for two pseudoparticles (instanton) configuration for $SU(2)$ gauge theory in Euclidean space. Our solutions correspond to the topological charge with value of two, and contains thirteen independent parameters.

1. INTRODUCTION

The solution of the self-duality (anti-self-duality) equations with an arbitrary value of topological charge n obtained in Refs.[1,2], have such obvious disadvantage as the singularities at finite values of the arguments and contain not maximum possible number (=8n-3) of arbitrary independent parameters (a number of degrees of freedom for n pseudoparticles). To define the characteristic quantities of the theory (action, topological charge) it is necessary to put the rules for multiplication of singular function at coinciding arguments as it required by the presence of singularities at finite values of the spatial variables. The assumption is usually made that the singularities may be removed be means of an appropriate gauge transformation. In Ref.[3] a solution without singularities in the finite domain has been obtained for the case of a unity topological charge. The present article is devoted to the solution of a similar problem but with a topological charge equal to two. We hope that the methods developed in this article may be generalized for the case of an arbitrary value of topological charge.

2. EQUATIONS OF MOTION AND GROUPS OF THEIR SYMMETRY

The main assumption made by us here is that the configuration of two pseudoparticles is defined by four-dimensional vector a_i (relative orientation of particles) and thus the structure functions depend on two variables $r = x_i^2$, $s = x_i \cdot a_i$. Therefore the invariance with respect to three-dimensional rotation group takes place that preserves the direction of the vector a_i and consequently it is necessary to find finite solutions for self-duality (anti-self-duality) equations (see below) that would depend on one arbitrary parameter. The remaining twelve (13-1) parameters ("degrees" of freedom) are determined by the parameters of the conformal group (with the exeption of the three parameters of the aforementioned group of three-dimesional rotations).

Under assumptions made, the Yang-Mills field G_i is defined by four structure functions $f_i(r, s)$

$$G_i = f_1 F_{ij} x_j + f_2 F_{ij} a_j + f_3 x_i (x_m F_{mn} a_n) + f_4 a_i (x_m F_{mn} a_n), \qquad (1)$$

where F_{ij} as well as in ref.[2] we consider to be the self-dual antisymmetric tensor ($F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\sigma_\gamma$, $F_{\alpha 4} = \sigma_\alpha$, $1 \leq \alpha \leq 3$), σ_α are generators of SU(2) gauge group*)

The following abriviations will be used: $X_i = F_{ij} x_j$, $A_i = F_{ij} e_j$, $D = x_m F_{mn} e_n$; e_i^2 is put to be equal to unity for simplicity. The gauge constraint for the field $\mathcal{A}_i$ will be chosen further on proceeding from the finiteness condition for the structure functions. Substituting expansion (1) into self-duality (anti-self-duality) equation $\mathcal{F}_{ij} = \pm \tilde{\mathcal{F}}_{ij}$, $\tilde{\mathcal{F}}_{ij} = \frac{1}{2}\epsilon_{ijmn}\mathcal{F}_{mn}$ for the field strength

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j] , \qquad (2)$$

we will come to the system of equations for the structure functions (see Appendix 1). The system of equations we will present in the form convenient for further calculations

$$u_3 = -\frac{\partial}{\partial r} \ln (u_1^2 + u_2^2) + \xi^{-1}\left(2s\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) \operatorname{arctg} \frac{u_1}{u_2} ,$$

$$u_4 = \frac{1}{2}\frac{\partial}{\partial s} \ln(u_1^2 + u_2^2) + \xi^{-1}\left(2r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right) \operatorname{arctg} \frac{u_1}{u_2},$$

$$\xi^2 \left[-\left(2r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)u_3 + \left(2s\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)u_4 - 2u_3 \right] = u_1^2 + u_2^2 - 1, \quad \xi^2 = r - s^2, \qquad (3D)$$

$$u_3 = \frac{\partial}{\partial r} \ln(u_1^2 + u_2^2) + \xi^{-1}\left(2s\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)\operatorname{arctg}\frac{u_1}{u_2},$$

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*) The relation between F_{ij} and σ_α may be chosen in the form $F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\sigma_\gamma$, $F_{\alpha 4} = -\sigma_\alpha$ as well, that is in fact equivalent to space reflection and replacement of the duality condition with antiduality one in all further formulae.

$$u_4 = -\frac{1}{2}\frac{\partial}{\partial s}\ln(u_1^2+u_2^2)+\xi^{-1}\left(2r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s}\right)\operatorname{arctg}\frac{u_1}{u_2},$$

$$-\xi^2\left[-\left(2r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s}\right)u_3+\left(2s\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)u_4-2u_2\right]=u_1^2+u_2^2-1, \qquad (3AD)$$

$$u_1=\xi(sf_1+f_2),\quad u_2=1+\xi^2 f_1,\quad u_3=sf_3+f_4-f_1,$$

$$u_4=rf_3+sf_4+f_2 .$$

(A group interpretation of the introduced combinations u_i will become clear from further coming equations).

The solution of system (3) is greatly simplified by the presence of symmetry groups. First of all it is a three-dimensional O(2,1) subgroup of the conformal group whose algebra consists of dilatations $\hat{D}\,(\equiv -L_2)$, translations $\hat{P}\,(\equiv L_1+L_3)$ and special-conformal transformations $\hat{K}\,(\equiv -L_1+L_3)$ along the direction a_i with the generators (which acts on the column $(f_1,\ldots,f_4)$)

$$\hat{D}=\hat{d}\cdot 1+\begin{Vmatrix}0&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&0\end{Vmatrix},\quad \hat{P}=\hat{p}\cdot 1+\begin{Vmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{Vmatrix},\quad \hat{K}=\hat{k}\cdot 1+\begin{Vmatrix}0&0&0&0\\-r&-2s&0&0\\2&0&0&-2\\0&2&r&2s\end{Vmatrix} \qquad (4)$$

$$\hat{d}=2r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s}+2,\quad \hat{p}=2s\frac{\partial}{\partial r}+\frac{\partial}{\partial s},\quad \hat{k}=2rs\frac{\partial}{\partial r}+(2s^2-r)\frac{\partial}{\partial s}+4s$$

(L_1, L_2, L_3 are generators of the three-dimensional group of pseudo-rotations O(2,1). The second part of symmetry group is a subgroup of a gauge group with infinitesimal transformations of the form

$$\begin{Vmatrix} \delta f_1 \\ \delta f_2 \\ \delta f_3 \\ \delta f_4 \\ 0 \end{Vmatrix} = \begin{Vmatrix} -s\Phi & -\Phi & 0 & 0 & 0 \\ r\Phi & s\Phi & 0 & 0 & \Phi \\ -\Phi & 0 & 0 & 0 & 2\frac{\partial \Phi}{\partial r} \\ 0 & -\Phi & 0 & 0 & \frac{\partial \Phi}{\partial s} \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ 1 \end{Vmatrix}, \qquad (5)$$

where Φ is an arbitrary function of the variables r, s. The expression $\exp \phi = u_1^2 + u_2^2$ is invariant under the gauge transformation; global transformations are two-dimensional rotations of the components u_1 and u_2. The combinations u_3 and u_4 are transformed gradiently under gauge transformations

$$\tilde{u}_3 = \hat{p}\Phi + u_3, \qquad \tilde{u}_4 = (\hat{d} + 1)\,\Phi + u_4. \qquad (6)$$

With respect to the transformations of $O(2,1)$ group, f_1 and $\xi^{-1}(sf_1 + f_2)$ are scalars, and the combinations $\xi^{-1}(sf_3 + f_4 - f_1)$ and $\xi^{-1}(r f_3 + sf_4 + f_2)$ are components of gradient $O(2,1)$ vector.

Excluding u_3 and u_4 from equations (3) we will come to the following nonlinear partial differential equations for the gauge invariant $\exp \phi$;

$$\vec{L}^2 \phi = 2(e^{\phi} - 1), \quad \vec{L}^2 \equiv L_3^2 - L_1^2 - L_2^2. \qquad (7)$$

Hence the expression for $\exp \phi$ and the choice of gauge, defined with formula $\operatorname{tg} \xi \Phi = u_1 \cdot u_2^{-1}$ completely fix the solution of system (3). It is worth mentioning again that we seek for the solution of equation (7) that would depend on one arbitrary parameter and the remaining 12 parameters arise as a consequence of the presence of 12-parametric factor-group of conformal transformations (it takes place, as we shall see, if the solution has not any other additional invariance).

The gauge invariant expressions for action and topological charge are constructed with the gauge invariant in the following way:

$$S = \frac{1}{4\pi^2} \int d^4x \, Sp \mathcal{F}_{ij}^2 = \frac{1}{4\pi^2} \int \frac{d^4x}{\xi^4} \vec{L}^2 (e^{\phi} - \phi), \tag{8}$$

$$q = \frac{1}{4\pi^2} \int d^4x \, Sp \mathcal{F}_{ij} \tilde{\mathcal{F}}_{ij} = \frac{1}{4\pi^2} \int \frac{d^4x}{\xi^4} \{ \vec{L}^2 (e^{\phi} - \phi) - [e^{\phi} - 1 + \xi^2 (2 u_3 + \hat{d} u_3 - \hat{p} u_4)]^2 \},$$

where $\xi^{-4} d^4x$ is an invariant measure with respect to the pseudo-orthogonal three-dimensional group of rotations.

3. SOLUTION OF EQUATION FOR GAUGE INVARIANT

To find a general solution for equation (7) it is convenient to go from the parameters r and s to the angular variables of pseudoorthogonal O(2,1) group via formulae

$$y_1 = \sin\psi \, sh\,\theta = \frac{1 - r}{2\xi}, \quad y_2 = \cos\psi \, sh\,\theta = \frac{s}{\xi}, \quad y_3 = ch\,\theta = \frac{r + 1}{2\xi}, \tag{9}$$

where y_1, y_2, y_3 are the components of the pseudoorthogonal vector

$$y^2 \equiv g_{ij} y_i y_j = -1, \quad g_{ij} = diag(1,1,-1). \tag{10}$$

Transformation properties of the quantities entering the theory under the transformations of O(2,1) group allow one to come to a conclusion that the gauge invariant has a dimensional structure of the type "unity plus the ratio of some component of the 2-nd rank tensor to the square of some component of another 2-nd rank tensor" under these transformations. Therefore it is natural to search for the solution of equation (7) in the form

$$\exp\phi = 1 + \frac{(yCy)}{(yBy)^2}, \tag{11}$$

where C and B are some three-dimensional matrices $(yBy) = y_a B_{a\beta} y_\beta$. The case when matrix C is proportional to B, leads to a known single-instanton solution $e^{\phi} = 1 - ch^{-2}\theta$,

$$f_1 = -(1 + \sin\psi\, th\theta), \quad f_2 = f_3 = f_4 = 0. \tag{11'}$$

The case when matrix C is not proportional to B is considered below; it corresponds to the configuration of two pseudoparticles. Here we will consider only the principle steps in finding the required solution. Intermediate formulae, necessary for the solution of equation (7) are enlisted in Appendix 2.

Substituting formula (11) into equation (7) and concelling the terms containing $(yBy)^{-\mu}$ and $[(yCy)+(yBy)^2]^{-\mu}$, $\mu = 1,2$ respectively, we will find relation between the matrices C and B:

$$C = 4(BgB) - 2\,Sp(Bg)B. \tag{12}$$

and supplementary relation

$$[\,Sp(Bg)\,]^2 = 2Sp(Bg)^2, \quad \text{i.e. } Sp(Cg) = 0. \tag{13}$$

Up to pseudoorthogonal rotation the obtained solution contains one arbitrary parameter since in the expression for $\exp\phi$ only the ratios of the eigen-values of the matrix Bg are essential. The third one is fixed by the condition $\lambda_3 = \lambda_1 + \lambda_2 \pm 2\sqrt{\lambda_1\lambda_2}$.

Taking a diagonal form $Bg = diag(\lambda_1, \lambda_2, \lambda_3)$ for the matrix B we will obtain for $\exp\phi$ the following expression

$$e^{\phi} = \frac{[y_1^2(\lambda_1-\lambda_3)-y_2^2(\lambda_2-\lambda_3)+(\lambda_1-\lambda_2)\,]^2+4y_1^2y_2^2(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}{[\,y_1^2(\lambda_1-\lambda_3)+y_2^2(\lambda_2-\lambda_3)-\lambda_3]^2}, \tag{14}$$

which is a ratio of a coefficient function /4/ to the square of some quadratic form. Finiteness constraint for $\exp\phi$ at finite values of parameters means that the quadratic form in the denominator of (14) should not turn to zero, i.e. $\lambda_1-\lambda_3$, $\lambda_2-\lambda_3$, $-\lambda_3$ should have the same sign.

Let for definiteness $0 > \lambda_1 > \lambda_2$, $\lambda_3 = -(\sqrt{|\lambda_1|}+\sqrt{|\lambda_2|})^2$. The numerator in (14) vanishes at two values of the arguments $y_1 = 0$ $(r = 1)$, $y_1 = \pm\sqrt{\dfrac{\lambda_1-\lambda_2}{\lambda_2-\lambda_3}}$. The presence of zeros in the numerator leads generally speaking to an appearance of logarithmic singularity of ϕ at this point and possible singularities for f_3 and f_4. The condition, that the singularities are absent (or they are completely intercompensated in the functions f_3 and f_4) brings us to an equation for the phase $\operatorname{arc\,tg}\frac{u_1}{u_2}$ whose solution has the form

$$\sum_{\varepsilon_1,\varepsilon_2=\pm 1} \operatorname{arc\,tg}\frac{\sqrt{\lambda_1-\lambda_3}\,s+\varepsilon_1\sqrt{\lambda_1-\lambda_2}}{\sqrt{\lambda_1-\lambda_3}\,\xi+\varepsilon_2\sqrt{\lambda_2-\lambda_3}}\,.$$

(We will remind that the equation for the phase expresses the condition that the phase contribution should compensate all the singularities originated from zero of the logarithm argument in f_3 and f_4).

Thus, with the chosen gauge we obtain the following finite expressions without singularities for the structure functions:

$$f_1=4\,\frac{(\lambda_1-\lambda_2+\lambda_3)-2s^2(\lambda_1-\lambda_3)}{N},\qquad f_2=4s\,\frac{r\,(\lambda_1-\lambda_3)-\lambda_2}{N},$$

$$f_3=-8s\,\frac{\lambda_1-\lambda_3}{N},\qquad f_4=4\,\frac{r\,(\lambda_1-\lambda_3)-\lambda_2}{N},\tag{15}$$

$$N=(r-1)^2(\lambda_1-\lambda_3)+4s^2(\lambda_2-\lambda_3)-4\xi^2\lambda_3\,.$$

(for details of deriving formula (15) refer to Appendix 3). We will note that solutions (15) for system (3) are defined up to "good" (without singularities) gauge transformations.

Substituting solution (11) of equation (7) into expressions (8), (9) for the topological charge and action with account of relations (12), (13) we will obtain the following formula for density

$$2^5\cdot 3\cdot\left[\frac{\prod_{\varepsilon}\left(y_1\sqrt{\frac{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2+\lambda_3)}{2}}+\varepsilon y_2\sqrt{\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2-\lambda_3)}{2}}\right)+\frac{(\lambda_1-\lambda_2)^2-\lambda_3^2}{4}}{\left(y_1^2(\lambda_1-\lambda_3)+y_2^2(\lambda_2-\lambda_3)-\lambda_3\right)^2}\right]^2+$$

$$+2^6 \frac{\lambda_1\lambda_2\lambda_3}{(y_1^2(\lambda_1-\lambda_3)+y_2^2(\lambda_2-\lambda_3)-\lambda_3)^3} \, . \qquad (16)$$

Integration of this expression over the invariant measure $\xi^{-4}d^4x$ brings to the value for charge equal to two. Calculations of a similar integral for $e^{\phi} = 1 - ch^{-2}\theta$ (11') give unity value for the charge. Thus the obtained expessions (15) for the structure functions of the Yang-Mills field describe completely and mathematically correctly a system with two pseudoparticles with a Pontryagin index equal to two (in the *framework of our assumptions*).

In conclusion for the Section we note that in a particular case of $\lambda_1=\lambda_2$ the solution takes a simple form $e^{\phi/2}=3\,sh^2\theta(1+3ch^2\theta)^{-1}$. However in this case additional invariance arises and transformations of the conformal group allow us to obtain only 11-parametric solution.

C O N C L U S I O N

A starting point for the search for self-duality equation solution without singularities was "poor" solutions for an anti-self-dual case, obtained earlier in refs. /1,2/. If we digress from the details, we have in fact made inverse transformation over uncorrect solution of anti-self-duality equations and pointed out (in explicit way) the gauge, which provides the solutions will be finite in the whole domain of the variable changes. We suppose that in the case of the topological charge n to find finite solutions one should also perform inverse transformations and then an appropriate choice for the gauge (with account of the dynamical symmetry) will provide a possibility to remove singularities, as it was done for the solution with a Pontryagin index equal to 1 and 2.

Though the obtained solution depends on the required number(=13) of the independent parameters (a number of degrees of freedom for n pseudoparticles) and satisfy the finiteness conditions, their physical sense is not clear yet. The clarification of the physical sense would make it possible to carry out automatic generalization for the case of finite solutions with arbitrary value of Pontryagin index.

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Appendix 1

Proceeding from parametrization (1) for the field $\mathfrak{a}_i$ and using commutation relations between X_i, A_i, D, F_{mn}

$$[X_i, X_j]_- = r F_{ij} + x_i X_j - x_j X_i, \qquad [A_i, A_j]_- = F_{ij} + a_i A_j - a_j A_i,$$

$$[X_i, A_j]_- = \delta_{ij} D + s F_{ij} + a_i X_j - x_j A_i, \quad [F_{ij}, D]_- = x_i A_j - x_j A_i + a_j X_i - a_i X_j,$$

$$[X_i, D]_- = x_i D + s X_i - r A_i, \quad [A_i, D]_- = a_i D + X_i - s A_i, \tag{p.1}$$

$$[X_i, F_{mn}]_- = -\delta_{im} X_n + \delta_{in} X_m + x_n F_{im} - x_m F_{in}, \quad [A_i, F_{mn}]_- = -\delta_{im} A_n + \delta_{in} A_m + a_n F_{im} - a_m F_{in}.$$

for the field strength $\mathcal{F}_{ij}$ (2) we obtain

$$\mathcal{F}_{ij} = -F_{ij}\,\Lambda + F_{jm} V_{mi} - F_{im} V_{mj} + D W_{ij}, \tag{p.2}$$

where

$$V_{ij} = S_{ij} + R_{ij}, \; W_{ij} = q_{ij} W, \; R_{ij} = q_{ij} R, \; q_{ij} = x_i a_j - x_j a_i,$$

$$\Lambda = 2 f_1 + r f_1^2 + f_2^2 + 2 s f_1 f_2, \quad W = -\frac{\partial f_3}{\partial s} + 2\frac{\partial f_4}{\partial r} + f_2 f_3 - f_1 f_4,$$

$$S_{ij} = x_i x_j f_x + a_i a_j f_a + (x_i a_j + x_j a_i) f_{xa},$$

$$f_x = 2\frac{\partial f_1}{\partial r} - f_1^2 + f_2 f_3 + s f_1 f_3, \quad -f_a = -\frac{\partial f_2}{\partial s} + f_2^2 + f_4 + s f_2 f_4 + r f_1 f_4, \tag{p.2'}$$

$$2 f_{xa} = \frac{\partial f_1}{\partial s} + 2\frac{\partial f_2}{\partial r} + f_2 f_4 + s f_1 f_4 - f_3 - s f_2 f_3 - r f_1 f_3 - 2 f_1 f_2,$$

$$2R = \frac{\partial f_1}{\partial s} - 2\frac{\partial f_2}{\partial r} + f_2 f_4 + s f_1 f_4 + f_3 + s f_2 f_3 + r f_1 f_3.$$

The quantities F_{ij} and $F_{jm} R_{mi} - F_{im} R_{mj}$ are self-dual, $F_{jm} S_{mi} - F_{im} S_{mj}$ is anti-self-dual(up to the trace of S) tensors.

The relation

$$q_{mn} D = \frac{r}{2}(a_n A_m - a_m A_n) - \frac{r}{4} F_{mn} - s(x_n A_m - x_m A_n) + \frac{1}{2}(x_n X_m - x_m X_n) - \frac{1}{2r} D_{mn},$$

also takes place. Here $D_{mn} = (\delta_{mi} r - 2x_m x_i)(\delta_{nj} r - 2x_n x_j)\mathfrak{M}_{ij}$ is a self-dual tensor, which is derived from anti-self-dual one,

$$\mathfrak{M}_{ij} = F_{im}(a_m a_j - \frac{1}{4}\delta_{mj} a^2) - F_{jm}(a_m a_i - \frac{1}{4}\delta_{mi} a^2)$$

by means of the inverse transformation. The indicated properties of the tensors F_{ij} , $F_{jm} V_{mi} - F_{im} V_{mj}$ and $q_{ij} D$ allow one to present the field strength $\mathcal{F}_{ij}$ as a sum of self-dual and anti-self-dual parts,

$$\mathcal{F}_{ij} = \mathcal{F}_{ij}^{(1)} + \mathcal{F}_{ij}^{(2)}, \quad \mathcal{F}_{ij}^{(\mu)} = (-)^{\mu} \tilde{\mathcal{F}}_{ij}^{(\mu)}, \quad \mu = 1,2, \qquad \text{(p.3)}$$

where

$$2\mathcal{F}_{ij}^{(1)} = (F_{jm}\check{S}_{mi} - F_{im}\check{S}_{mj}) - \widetilde{(F_{jm}\check{S}_{mi} - F_{im}\check{S}_{mj})},$$

$$\mathcal{F}_{ij}^{(2)} = -(\Lambda + \frac{r}{4}W + \frac{1}{2}\mathrm{Sp}\,\check{S})F_{ij} + F_{jm}\check{R}_{mi} - F_{im}\check{R}_{mj} - \frac{W}{2r}D_{ij}, \qquad \text{(p.3')}$$

$$\check{S}_{ij} = S_{ij} - \frac{W}{2}[x_i x_j + r a_i a_j - s(x_i a_j + x_j a_i)], \quad \check{R} = R - \frac{s}{2}W.$$

From Eq. (p.3) there directly follow systems of equations for a structure functions in self-dual and anti-self-dual cases:

$$2\frac{\partial f_1}{\partial r} - f_1^2 + f_3(sf_1 + \frac{f_2}{2}) + \frac{1}{2}f_1 f_4 + \frac{1}{2}h = 0,$$

$$-\frac{\partial f_2}{\partial s} + f_2^2 + \frac{r}{2}f_2 f_3 + f_4(1 + sf_2 + \frac{r}{2}f_1) - \frac{r}{2}h = 0, \qquad \text{(p.4D)}$$

$$\frac{\partial f_1}{\partial s} + 2\frac{\partial f_2}{\partial r} - 2f_1 f_2 - f_3(1 + r f_1) + f_2 f_4 - sh = 0,$$

$$h \equiv \frac{\partial f_3}{\partial s} - 2\frac{\partial f_4}{\partial r},$$

$$4f_1 + r f_1^2 + f_2^2 + 2sf_1 f_2 + \hat{d}f_1 + \hat{p}f_2 + \xi^2 (f_2 f_3 - f_1 f_4) - f_4 - sf_3 = 0,$$

$$\frac{\partial f_1}{\partial s} - 2\frac{\partial f_2}{\partial r} + f_3(1 + sf_2 + r f_1) + f_4(sf_1 + f_2) = 0, \qquad \text{(p.4AD)}$$

$$f_2 f_3 - f_1 f_4 - h = 0.$$

Appendix 2

From definition of the generator L_a action on the vectors y_β, $L_a y_\beta = \epsilon^{(a)}_{a\beta\gamma} y_\gamma$ we find

$$\vec{L}^2 (yCy) = 6(yCy) + 2\,Sp(Cg),$$

$$\vec{L}(yCy)\cdot\vec{L}(yBy) = 4(yCy)(yBy) + 2(yCgBy) + 2(yBgCy) \qquad \text{(p.5)}$$

((yCy) contains moments 0 and 2). We will rewrite $\exp\phi$ in the form $\exp\phi = [(yCy) + (yBy)^2]\cdot(yBy)^{-2}$; then with account of (p.5) for $\ln(yBy)$ we will obtain

$$-\vec{L}^2 \ln(yBy)^2 = -4 - 4\,\frac{(yBy)Sp(Bg) - 2(yBgBy)}{(yBy)^2},$$

whereof follows relation (12). Further on, formula

$$\vec{L}^2 \ln[(yCy) + (yBy)^2] =$$

$$= \frac{2[\vec{L}(yBy)]^2 + 2(yBy)[\vec{L}^2(yBy)] + [\vec{L}^2(yCy)]}{(yCy) + (yBy)^2} - \frac{\{2(yBy)[\vec{L}(yBy)] + \vec{L}(yCy)\}^2}{[(yCy) + (yBy)^2]^2}.$$

takes place. We will come to relation (13) if the aforementioned formula be substituted into equation (7). Relation (13) provides the concelling of the terms which contain $[(yCy)+(yBy)^2]^{-\mu}$, $\mu = 1,2$. Thus the function $\exp\phi$ of form (11) satisfies equation (7) under constraint that the matrices B and C are subjected to relations (12) and (13).

Using formula (p.5) we find the following expression for density (8)

$$\vec{L}^2(e^{\phi} - \phi) = 6\left[\frac{(yCy)}{(yBy)^2}\right]^2 + \frac{64\lambda_1\lambda_2\lambda_3}{(yBy)^3}, \qquad \text{(p.6)}$$

which brings us to a charge equal to two.

<u>Appendix 3</u>

To find the phase $\Psi \equiv \operatorname{arc\,tg}\frac{u_1}{u_2}$ we have equations

$$u_3 = -\frac{\partial}{\partial r}\ln e^{\phi} + \xi^{-1}\hat{p}\Psi, \quad u_4 = \frac{1}{2}\frac{\partial}{\partial s}\ln e^{\phi} + \xi^{-1}\hat{d}\Psi. \qquad \text{(p.7)}$$

As was noted $\exp\phi$ turns to zero at the finite point of the plane and to obtain nonsingular expressions for f_3 and f_4 one should completely compensate the singularities, arising from differentiation of $\ln\exp\phi$ with the phase contribution. The properties of coefficient functions /4/ allow to factorize the numerator in the expression $\exp\phi$ (that has zero at the finite point) into a product of four multipliers,

$$\Delta \equiv \{[(r-1)^2 + 4s^2](\lambda_1-\lambda_3) + 4r(\lambda_1-\lambda_2)\}^2 - 16s^2(r+1)^2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) =$$

$$= \prod_{\varepsilon_1,\varepsilon_2=\pm1}[(r+1)\sqrt{\lambda_1-\lambda_3} + 2\varepsilon_2(s\sqrt{\lambda_1-\lambda_2} + \varepsilon_1\xi\sqrt{\lambda_2-\lambda_3})],$$

whereof we obtain an equation used to de-ine the phase

$$\frac{\partial}{\partial s}\Psi_{\varepsilon_1\varepsilon_2} = \frac{\xi\sqrt{\lambda_1-\lambda_3}+\varepsilon_1\varepsilon_2\sqrt{\lambda_2-\lambda_3}}{(r+1)\sqrt{\lambda_1-\lambda_3}+2\varepsilon_2(s\sqrt{\lambda_1-\lambda_2}+\varepsilon_1\xi\sqrt{\lambda_2-\lambda_3})},$$

$$(\hat{p}\Psi_{\varepsilon_1\varepsilon_2}(s,\xi) \equiv \frac{\partial}{\partial s}\Psi_{\varepsilon_1\varepsilon_2}(s,\xi), \text{ as } \hat{p}\xi \equiv 0),$$

From the solution of this equation it follows that

$$\Psi \equiv \sum_{\varepsilon_1,\varepsilon_2}\Psi_{\varepsilon_1\varepsilon_2} = \sum_{\varepsilon_1\varepsilon_2}\operatorname{arctg}\frac{\sqrt{\lambda_1-\lambda_3}\,s+\sqrt{\lambda_1-\lambda_2}\,\varepsilon_1}{\sqrt{\lambda_1-\lambda_3}\,\xi+\sqrt{\lambda_2-\lambda_3}\,\varepsilon_2}. \quad \text{(p.8)}$$

Thus

$$-\frac{\partial}{\partial r}\ln\Delta+\xi^{-1}\hat{p}\Psi=0, \quad \frac{1}{2}\frac{\partial}{\partial s}\ln\Delta+\xi^{-1}\hat{d}\Psi=0, \quad \text{(p.9)}$$

i.e., as is seen from the final result, we have used in fact a non-gauge invariant condition

$$\frac{\partial u_3}{\partial s} = -2\frac{\partial u_4}{\partial r}, \quad \text{(p.10)}$$

that with account of (p.7) leads to the following equation for the phase

$$\vec{L}^2\Psi = 0. \quad \text{(p.11)}$$

Assuming that u_2 and u_1 are real and imaginary parts of some analytical function Z with account of (7) we will obtain

$$\vec{L}^2 \ln Z = |Z|^2 - 1, \quad Z = \exp \frac{1}{2}(\phi + 2i\Psi). \qquad \text{(p.12)}$$

Appendix 4

We will indicate a method to find a solution of equation(7) in a more general form (as compared with expression (14)) which however, contains singularities at finite values of the arguments. For this aim we will use the results obtained in refs./1,2/ where it was shown that a special class of anti-self-dual type solutions satisfies the equation $\chi^{-1}\Box\chi = 0$. Having performed inverse transformations over the solution $\chi(r, s)$, $\chi \to \chi'$, we come to a conclusion that (in the case of two sources) the functions of the form

$$\exp\phi = 1 - \vec{L}^2 \ln \chi',$$

satisfies nonlinear equations (7) while the function χ is a solution of equation $\chi^{-1}\Box\chi = 0$. We have no general proof for this interesting fact, with exeption for a direct check in the indicated particular case.

Цена 9 коп.

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