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RENORMALIZATION GROUP-THEORETIC APPROACH TO ELECTRON LOCALIZATION IN DISORDERED SYSTEMS *

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ABSTRACT

We first study the localization problem for the Anderson tight-binding model with site-diagonal (gaussian) disorder, using a previously established analogy between this problem and the statistical mechanics of a zero-component classical field. The equivalent free-energy functional turns out to have complex coefficients in the bilinear terms but involves a real repulsive quartic interaction. The averaged one-electron propagator corresponds to the two-point correlation function for the equivalent statistical problem and the critical point gives the mobility edge, which we identify with the (real) fixed point energy of the associated renormalization group. Since for convergence reasons the conventional perturbative treatment of Wilson's formula is invalid, we resort to a non-perturbative approach which leads to a physical fixed point corresponding to a repulsive quartic interaction. Our results for the mobility edge in three dimensions and for the critical disorder for an Anderson transition in two dimensions agree well with previous detailed predictions. We also discuss the critical indices describing the approach of the transition at the mobility edge of various physical quantities, within the ε-expansion. We then consider the more general problem where both diagonal and off-diagonal disorder is present in the Anderson hamiltonian. In this case we show that the Hamilton function for the equivalent zero-component classical field model involves an additional biquadratic exchange term. From a simple generalization of Wilson's recursion relation and its non-perturbative solution we obtain explicit expressions for the mobility edges for weak diagonal and off-diagonal disorder in two and three dimensions. Our treatment casts doubts on the validity of recent conclusions about electron localization based on the renormalization group study of the nm-component spin model.

I. INTRODUCTION

The problem of electron localization within the tight-binding model of Anderson with a gaussian distribution of site energies is known to be formally equivalent to the statistical mechanics of a zero-component classical field with a quartic self-interaction[1]. This equivalence is based on the comparison of the configuration-averaged one-electron propagator with the two-point correlation function for the zero-component field as derived by Emery[2]. More precisely, the Anderson problem is found to be equivalent to a statistical-mechanical problem described by a free energy functional with purely imaginary coefficients, r and J, in the bilinear terms and a repulsive coupling constant, $u > 0$, in the quartic self-interaction. In particular, the mobility edge separating the localized and extended electron states is analogous to the critical point of the phase transition, which may be obtained from the fixed point of the associated renormalization group[3),4)] as discussed below.

However, as shown in Ref. 1, the conventional expansion of Wilson's recursion formula[3),4)] for the present problem in powers of u leads to an unphysical fixed point with $u^* < 0$. We note, incidentally, that a similar unphysical fixed point arises also in the study of the so-called nm-model for random systems[2),3)] for the case $n = 0$ and $m = 1$, which is formally similar to the above zero-component classical field model. Since a fixed point with $u^* < 0$ is incompatible with the identification of u with the width of the site energy distribution, it was concluded in Ref. 1 that

the one-electron propagator shows no non-analyticity of the type which is expected near the mobility edge E_c. This was tentatively attributed to the possibility of a decay of correlations in the phase of the wave functions near E_c[5),6)], which could smear out the singularity in the $k = 0$ Fourier component of the electron propagator.

The motivation for the present work thus arose from two circumstances. First, the analysis of Ref. 1 would suggest that the mobility edge cannot be discussed with the averaged one-electron propagator, which implies that it may be necessary to consider the averaged squared propagator. Since the latter is not easily amenable to a detailed analysis, a further study of the one-electron propagator is clearly desirable. Second, in a recent paper by Schuster[7)] it has been tacitly assumed that a real physical fixed point exists which can be identified with the mobility edge for the problem of an electron moving in a continuum in a random potential $\varphi(\vec{r})$ (where $\vec{r}$ denotes spatial coordinates). But, on the other hand, the analysis of Aharony *et al.*[6)] would indicate that such an identification is unwarranted because of the unphysical nature of the fixed point. However, the work of Aharony *et al*[6)] is based on a perturbation expansion of the recursion relation of the renormalization group in powers of the quartic interaction which may be inadequate for the study of the localization problem. In fact, as discussed below, the conventional perturbative treatment of Wilson's recursion relation for the Anderson problem is invalid for reasons of convergence. We have therefore attempted a non-perturbative study of this relation and were able to show that an acceptable fixed point with $u^* > 0$ (for dimensionilities d less than $d_c = 4$) does indeed exist. The mobility edge E_c corresponds to the critical point of

the two-point correlation function of the classical field which we identify with the (real) fixed point energy, $E^* = \mathrm{Im}\, r^*$ [1)]. Since the non-perturbative fixed point for the Anderson problem differs from the fixed point for the usual phase transition problem, the critical indices are expected to be different from those discussed e.g. by Schuster[7)], except for $d > 4$ where they reduce to the mean field values.

In Sec. II we introduce the Anderson model and recall briefly its formal relationship to the statistical mechanics of a zero-component classical field discussed in Ref. 1. In Sec. III we first present our non-perturbative analysis of Wilson's recursion formula for the Anderson problem and discuss the identification of the fixed point energy with the mobility edge. Next we calculate the mobility edge for weak (and intermediate) disorders in three and two dimensions. We also briefly discuss the average density of states, which is found to have a kink at the energy of the mobility edge. Finally we study the critical indices describing the singular behaviour of the localization radius of the wavefunction and of the zero-wavevector one-electron propagator as one approaches the transition at the mobility edge. These results are given in the framework of the ε-expansion[3),4)] for dimensionlities $d \to 4$.

In Sec. IV we turn to the more general problem where both the site energies and the hopping matrix elements in the Anderson hamiltonian involve random components having gaussian probability distributions. We first use Emery's method[2)] to derive the exact expression of the Hamilton function for a zero-component classical field whose two-point correlation function has

the same form as the configuration-averaged one-electron propagator for the above localization problem. Next we discuss a simple generalization of Wilson's recursion formula which incorporates the effect of the higher order (gradient) interaction terms appearing in the new classical field problem. From the non-perturbative solution of the recursion relation at the fixed point we then obtain explicit expressions for the mobility edges as a function of the widths of the distributions of the site energies and of the hopping matrix elements in two and three dimensions. Finally, in Sec. V we summarize our main results and analyze the recent conclusions based on the analogy between the motion of an electron in a continuum in a random potential $\varphi(\vec{r})$ and the nm-model for random systems[6]. We also comment briefly on the relevance of the averaged one-electron propagator for the discussion of electron localization aspects in disordered systems.

II. CORRESPONDENCE BETWEEN PHASE TRANSITION AND ANDERSON LOCALIZATION FOR DIAGONAL DISORDER

In the Anderson model[8] the electronic states of a disordered system are described by a tight-binding Hamiltonian for a perfect lattice

$$H^e = \sum_i^N \mathcal{E}_i |i\rangle + \sum_{i \neq j}^N V_{ij} |j\rangle \langle i| \quad , \qquad (1)$$

where a single orbital $|i\rangle$ of energy $\mathcal{E}_i$ is associated with each site i in a given configuration of the system. The site energies $\mathcal{E}_i$ are assumed

to be independently distributed gaussian variables with mean zero and root mean square deviation Σ . The probability density $p_{\mathcal{E}}(\mathcal{E}_i)$ for the site energies is thus

$$p_{\mathcal{E}}(\mathcal{E}_i) = (2\pi\Sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{\mathcal{E}_i^2}{2\Sigma^2}\right) \quad . \qquad (2)$$

The hopping matrix elements V_{ij} are assumed to be of the form : $V_{ij} = -V\,\delta_{i,j+\delta}$ $(V > 0)$, where δ spans the nearest neighbours. In the absence of disorder, the hamiltonian (1) describes a single band of extended Bloch states of width $\frac{1}{4}ZV$ (Z = lattice coordination number) centered at the energy $E = 0$. When disorder is present the eigenstates of H^e may be divided into localized and delocalized states which are separated by mobility edges at $E = \pm E_c$ $(E_c < 0)$, such that the states with $E < E_c$ or $E > -E_c$ are localized and the states with $E_c < E < -E_c$ are delocalized. In the following we shall be referring exclusively to the lower mobility edge, E_c , for convenience. For the purpose of the analysis of Sec. III we now recall briefly the comparison between the Anderson problem and the statistical mechanics of a zero-component classical field as discussed in detail in Ref. 1.

We define the configuration-averaged propagator in the reciprocal $(\vec{k})$ and direct $(\vec{x})$ space as

$$G^e_{\vec{k}}(E) = \langle\langle \vec{k} \left| \frac{1}{E - H^e} \right| \vec{k} \rangle\rangle \quad , \qquad (3)$$

$$\hat{G}^e_{ij}(E) = \sum_{\vec{k}} G^e_{\vec{k}}(E)\, e^{i\vec{k}(\vec{x}_i - \vec{x}_j)} \quad . \qquad (4)$$

Here the double angular bracket signifies quantum-mechanical expectation and configuration averaging over the distribution (2). For convenience, E is supposed to have a small positive imaginary part. Also, as usual, we replace the summation over $\vec{k}$ by an integration over the first Brillouin zone using the prescription

$$\sum_{\vec{k}} \ldots \to \frac{\Omega_d}{(2\pi)^d} \int d^d \vec{k} \ldots ,$$

where Ω_d is the volume of the d-dimensional lattice under consideration. It is convenient, for the following, to rewrite $\hat{G}^e_{ij}(E)$ explicitly as

$$\hat{G}^e_{ij}(E) = -\frac{1}{V}(2\pi\sigma^2)^{-\frac{N}{2}} \int_{-\infty}^{\infty} \prod_{\ell}^{N} d\,\varepsilon_\ell \, e^{-\sum_{\ell}^{N} \frac{\varepsilon_\ell^2}{2\sigma^2}} \left[(\hat{e} - \hat{v} - \hat{\varepsilon})^{-1}\right]_{ji} , \quad (5)$$

where N is the number of lattice sites and $\hat{e} = -\frac{\hat{E}}{V}$, $\hat{v} = \frac{\hat{V}}{V}$ and $\hat{\varepsilon} = -\frac{\hat{\mathcal{E}}}{V}$ are dimensionless N x N matrices. The matrices $\hat{e}$ and $\hat{\varepsilon}$ are diagonal and $\hat{v}$ is tridiagonal with non-zero matrix elements equal to one. Also $\sigma = \frac{\Sigma}{V}$.

We now compare Eq.(5) with the statistical-mechanical two-point correlation function for an n-component classical field $S_{i\alpha}$ ($\alpha = 1, \ldots, n$) defined on a d-dimensional lattice (with sites $i = 1, \ldots, N$) and having a quartic self-interaction, in the limit n = 0. The n-component field is described by the Hamilton function

$$\mathcal{H} = \sum_{\alpha}^{n} \sum_{i \neq j}^{N} J_{ij} S_{i\alpha} S_{j\alpha} + \sum_{i}^{N} \left[r \sum_{\alpha}^{n} S_{i\alpha}^2 + \frac{u}{4} \left(\sum_{\alpha}^{n} S_{i\alpha}^2 \right)^2 \right] , \; u > 0 , \quad (6)$$

The two-point correlation function for n = 0 is defined by

$$\hat{G}^{n=0}_{ij} = \lim_{n \to 0} \frac{1}{n} \sum_{\alpha}^{n} \langle S_{i\alpha} S_{j\alpha} \rangle . \quad (7)$$

A useful expression for this quantity has been derived by Emery[2] and is given by

$$\hat{G}^{n=0}_{ij} = (\pi u)^{-\frac{N}{2}} \int_{-\infty}^{\infty} \prod_{\ell}^{N} d\,\psi_\ell \, e^{-\frac{1}{u}\sum_{\ell}^{N} \psi_\ell^2} \left[(\hat{J} + \hat{r} + i\hat{\psi})^{-1}\right]_{ji} , \quad (8)$$

where $\hat{r}$ is the unit matrix times r, $\hat{J}$ is a tridiagonal matrix with non-zero elements J (nearest-neighbour coupling constant) and, finally, $\hat{\psi}$ is a diagonal matrix with elements ψ_ℓ . The comparison of Eqs.(5) and (8) leads to the following identifications[1]

$$\hat{r} \leftrightarrow -i\hat{e} \; , \; \hat{J} \leftrightarrow i\hat{v} \; , \; \hat{\psi} \leftrightarrow \hat{\varepsilon} \; , \; u \leftrightarrow 2\sigma^2 \; , \; \hat{G}^{n=0}_{ij} \leftrightarrow -iV\,\hat{G}^{e}_{ij}(E) \; . \quad (9)$$

It follows therefore that, from the point of view of the averaged one-electron propagator, the problem of Anderson localization is equivalent to the statistical mechanics of a zero-component classical field with complex bilinear couplings ($\hat{r} \to -i\hat{e}$, $\hat{J} \to i\hat{v}$).

III. WILSON'S RECURSION RELATION AND THE MOBILITY EDGE

In this section we first determine the fixed point for the classical field model associated with the Anderson localization problem. We will find a fixed point which corresponds to a real value of e and to a real positive value of u. Next we discuss the identification of the mobility edge (E_c) with the fixed point and study in detail its dependence on the width of the disorder. We also study the density of states associated with localized states for energies close to the mobility edge. Finally, we derive the critical indices describing the singular behaviour of the localization radius

and of the zero-wavevector electron propagator as one approaches E_c.

A. Non-perturbative fixed point for the Anderson problem

The usual derivation of Wilson's recursion relation[9] may be readily generalized to the case where the coefficients of the bilinear terms in the Hamilton function, Eq.(6), are complex. Thus, using standard notations, the recursion relation for the present case may be written

$$Q_{\ell+1}(\vec{z}) = - b^d \ln I_\ell \left(b^{-\frac{d}{2}+1} \vec{z}\right) \quad , \tag{10}$$

with the definitions

$$I_\ell(\vec{z}) = \int d\vec{y} \exp\left[- iy^2 - \frac{1}{2} Q_\ell(\vec{z} + \vec{y}) - \frac{1}{2} Q_\ell(\vec{z} - \vec{y})\right] \quad , \tag{11}$$

$$Q_\ell(\vec{z}) = \tilde{r}_\ell z^2 + \tilde{u}_\ell (z^2)^2 \quad , \tag{12}$$

$$\tilde{r}_\ell \equiv - i \left(\frac{r_\ell}{J} + 2d \right) , \tilde{u}_\ell = \frac{u_\ell}{4} , r_\ell \equiv r, u_\ell \equiv u \quad , \tag{13}$$

where the index ℓ refers to the ℓ^{th} recursion step. In the above equations $b > 1$ is an arbitrary scale factor[3),4),9)] and $\vec{y}$ and $\vec{z}$ are real n-component vectors (with $y^2 = \sum_\alpha^n y_\alpha^2$ and $z^2 = \sum_\alpha^n z_\alpha^2$) obtained by scaling the classical field variables $S_{i\alpha}$ with a factor $(\mathrm{Im}\, J)^{1/2}$ in the usual way[9]. It is convenient to rewrite Eq.(11) in the explicit form :

$$I_\ell(\vec{z}) = \exp\left[- \tilde{r}_\ell z^2 - \tilde{u}_\ell (z^2)^2 \right] \int d\vec{y}\, e^{-(\tilde{r}_\ell + i)y^2} e^{-A_\ell} \quad , \tag{14}$$

where

$$A_\ell = \tilde{u}_\ell \left[(y^2)^2 + 4(\vec{z}.\vec{y})^2 + 2 z^2 y^2 \right] \quad . \tag{15}$$

As noted in Ref. 1, the only difference between Eq.(11) and the usual Wilson formula lies in the presence of i in the exponent of the integrand in (11) and in the first of Eqs.(13), which arises as a consequence of J being complex. In Ref. 1 an attempt was made to study the fixed point of Eq.(10) ($\tilde{r}^*_{\ell+1} = \tilde{r}^*_\ell \equiv \tilde{r}^*$, $\tilde{u}^*_{\ell+1} = \tilde{u}^*_\ell \equiv \tilde{u}^*$) by expanding, as usual, the integrand in (14) for small $\tilde{u}_\ell$ through $O(\tilde{u}_\ell^2)$. This procedure led, however, to a fixed point with $u^* < 0$, which is unphysical since the identification $u \leftrightarrow 2\sigma^2$ requires u to be positive. Mathematically, a negative u^* implies that the integrand in Eq.(14) is exponentially growing near the fixed point. This would make the partition function undefined and would preclude an expansion in powers of u . On the other hand, since $\tilde{r}^*$ turned out to be purely imaginary[1)], an expansion of the factor $\exp(-A_\ell)$ in Eq.(14) in powers of u_ℓ cannot be used for reasons of convergence. This convergence difficulty for $y \to \infty$ does not arise, however, if one retains the term in $(y^2)^2$ in the exponent of $\exp(-A_\ell)$ as such, and expands the remaining part of this exponential for small u_ℓ. Thus we shall use the expansion

$$\exp(-A_\ell) = e^{-\tilde{u}_\ell (y^2)^2} \left[1 - \tilde{u}_\ell \left(4(\vec{z}.\vec{y})^2 + 2 z^2 y^2\right) + \frac{\tilde{u}_\ell^2}{2} \left(4(\vec{z}.\vec{y})^2 + 2 z^2 y^2\right)^2 + O(\tilde{u}_\ell^3) \right] \quad , \tag{16}$$

which, incidentally, includes all terms proportional to z^2 and to $(z^2)^2$. The fixed point for the Anderson problem discussed below is a non-perturbative result in the sense of Eq.(16). With the expansion (16), the integrals in Eq.(14) may be performed explicitly in polar coordinates in the n-dimensional space spanned by the vectors $\vec{y}$, using the direction of $\vec{z}$ as the polar axis. In terms of the polar angle θ between $\vec{y}$ and $\vec{z}$ the volume

element $d\Omega_n$ (after integration over the remaining n-2 angular coordinates) in this n-dimensional space is given by

$$d\Omega_n = \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} (\sin\theta)^{n-2}\, y^{n-1}\, d\theta dy \quad , \tag{17}$$

where $\Gamma(x)$ is the gamma function. One easily verifies that Eq.(17) leads to the correct expression for the volume of an n-dimensional hypersphere. Using Eqs.(16) and (17) in Eq.(10) all angular integrals are easily performed and the remaining radial integrals can be carried out exactly in terms of parabolic cylinder functions[10]. Using the standard expressions of the low-order parabolic cylinder functions in terms of error functions[10] and equating successively the coefficients of z^2 and z^4 on both sides of Eq.(10), we obtain the following exact recursion relations for the case n = 0 :

$$\tilde{r}_{\ell+1} = b^2 \left[\tilde{r}_\ell + (\pi\,\tilde{u}_\ell)^{\frac{1}{2}}\, e^{\frac{(\tilde{r}_\ell+i)^2}{4\,\tilde{u}_\ell}}\, \mathrm{erfc}\left(\frac{\tilde{r}_\ell+i}{2\sqrt{\tilde{u}_\ell}}\right) \right] \quad , \tag{18}$$

$$\tilde{u}_{\ell+1} = b^\varepsilon\, \tilde{u}_\ell \left\{ 1 - \left[\frac{5\sqrt{\pi}\,(\tilde{r}_\ell+i)}{2\sqrt{\tilde{u}_\ell}}\, e^{\frac{(\tilde{r}_\ell+i)^2}{4\,\tilde{u}_\ell}}\, \mathrm{erfc}\left(\frac{\tilde{r}_\ell+i}{2\sqrt{\tilde{u}_\ell}}\right) - 5 \right.\right.$$
$$\left.\left. - \frac{\pi}{2}\, e^{\frac{(\tilde{r}_\ell+i)^2}{2\,\tilde{u}_\ell}} \left(\mathrm{erfc}\left(\frac{\tilde{r}_\ell+i}{2\sqrt{\tilde{u}_\ell}}\right)\right)^2 \right] \right\} \quad , \tag{19}$$

where

$$\mathrm{erfc}\, z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt \quad , \tag{20}$$

and $\varepsilon = 4-d$. Having reduced the recursion relations to closed form expressions we now expand them for small $\tilde{u}_\ell$, such that $2\sqrt{\tilde{u}_\ell} \ll |\tilde{r}_\ell + i|$, using the asymptotic expression[10]

$$\mathrm{erfc}\, z = \frac{1}{\sqrt{\pi}\, z}\, e^{-z^2} \left[1 - \frac{1}{2z^2} + \frac{3}{4z^4} + O(z^{-6})\right], \quad |\arg z| < \frac{3\pi}{4} \quad . \tag{21}$$

In this way we obtain from Eqs.(18-19)

$$\tilde{r}_{\ell+1} = b^2 \left[\tilde{r}_\ell + \frac{2\,\tilde{u}_\ell}{\tilde{r}_\ell+i} \left(1 - \frac{2\tilde{u}_\ell}{(\tilde{r}_\ell+i)^2} + O(\tilde{u}_\ell^2) \right) \right] \quad , \tag{22}$$

$$\tilde{u}_{\ell+1} = b^\varepsilon\, \tilde{u}_\ell \left[1 + \frac{12\,\tilde{u}_\ell}{(\tilde{r}_\ell+i)^2} - \frac{68\,\tilde{u}_\ell^2}{(\tilde{r}_\ell+i)^4} + O(\tilde{u}_\ell^3) \right] \quad . \tag{23}$$

As shown by the calculations below, the terms of order $\tilde{u}_\ell^2$ in the square brackets must be retained in order to study the mobility edge to lowest order in the disorder parameter σ^2. This fact implies that, in principle, the Wilson relation (10) is not sufficient for discussing the mobility edge in the limit of weak disorder. Indeed, from the ε-expansion it is known that because of various approximations this relation is not exact beyond order ε^2. However, in view of the simplicity of Eq.(10) and the fact that it gives significantly better values for the critical exponents ν and γ in phase transitions than an improved version of it[11),3)], it seems justified to use this equation in this first study of mobility edges by the renormalization group method.

For continuous dimensionalities d close to 4 the fixed point is given by the linearized form of Eqs.(22-23) and the solution is

$$\tilde{u}^* = \frac{u^*}{4} = \frac{\varepsilon \ell n b}{12} + O(\varepsilon^2) \quad , \quad \tilde{r}^* = \frac{2 i \tilde{u}^*}{1 - b^{-2}} \quad , \qquad (24)$$

which shows that the Anderson localization problem leads to a physical fixed point with $u^* > 0$ for $d < 4$. We note, incidentally, that in the case where the bilinear coupling constants in Eq.(6) would be real, as in the usual phase transition problem, the fixed point obtained from the analogues of Eqs.(18) and (19) would correspond to a negative u^*. This, in turn, would mean that these equations are invalid because the radial integrals in the analogues of (14-16) are defined for $u^* > 0$ only. This demonstrates that different procedures for finding the physical fixed point of Wilson's recursion relation must be used depending on whether the bilinear coefficients in Eq.(6) are real or imaginary : in the former case the usual small u_ℓ, r_ℓ expansion of the integrand on the right-hand side of Eq.(10) yields the correct fixed point[3),4),9)], while in the latter case the above non-perturbative approach (where one does not expand in powers of u_ℓ at an early stage) must be applied. Thus it appears that by making the bilinear terms in the free energy functional complex, one moves out of the domain of stability of the fixed point of the usual linearized recursion relations[3),4)] and goes to another physical fixed point with a new domain of stability.

B. The transition from localized to delocalized electronic states

1. *Mobility edge*

The correspondence relations (9) enable us to identify the mobility edge $e = e_c$ for the Anderson localization problem with the critical point $r = r_c$ of the analogous statistical mechanical problem. More precisely, we know that the zero-wavevector Fourier component $G^{n=o}_{k=o}$ of $\hat{G}^{n=o}_{ij}$ has the form[3)]

$$G^{n=o}_{k=o} \propto \left(\frac{r_c - r}{J} \right)^{-\gamma_o} \quad , \qquad (25)$$

when the critical point is approached from the disordered phase, which in the present case corresponds to $\frac{r}{J} < \frac{r_c}{J}$. Here γ_o is the "susceptibility" index for $n = 0$, which will be discussed below within the ε-expansion. Correspondingly, we must have

$$G^{e}_{k=o}(e) \propto (e - e_c)^{-\gamma_o} \quad , \qquad (26)$$

where $e > e_c$ since the localized states (disordered phase) exist for $E < E_c$. The divergence at $e = e_c$ implies infinite spread of the wavefunction and hence e_c defines the mobility edge.

Now, the Eqs.(22) and (23) at the fixed point ($\tilde{r}^*_p \equiv \tilde{r}^*$, $\tilde{u}^*_p \equiv \tilde{u}^*$) may be viewed as giving a relation between a critical energy, $e^* = ir^*$, and the disorder parameter $\sigma^2 = \frac{u^*}{2}$ obtained by eliminating the arbitrary scale factor[9)] b . This suggests the identification of the fixed point energy e^* with the mobility edge i.e. we put $e_c = e^*$. We note that this definition

of e_c differs somewhat from the conventional *mathematical* definition of a critical point in statistical mechanics. As is well known, the latter definition amounts to identifying the critical point with the point on the critical surface[3),4),9)] (which passes through the fixed point) where the so-called relevant parameter for a given system vanishes. Since this condition provides only *one* relation between the parameters $\tilde{r}_\ell$ and $\tilde{u}_\ell$, for one particular recursion step, it is not sufficient for determining the mobility edge (e_c) in the present case, since the latter should be independent of b . Finally, we note that in the present definition of criticality ($e_c = e^*$) for the Anderson problem the non universal features of the transition enter via a dependence of the fixed point energy on the width of the site energy distribution.

As discussed above we obtain the mobility edge, $e_c = -\frac{E_c}{V} = e^*$, for weak disorder by eliminating the scale factor b between Eqs.(22) and (23) at the fixed point, using the definitions (13) and the identifications (9). The results of this calculation are

$$\tilde{r}^* = i \left[\frac{1}{11} + \frac{7}{36} \tilde{u}^* + O(\tilde{u}^{*2}) \right] \quad , \tag{27}$$

or

$$E_c = -6V - \frac{V}{11} - \frac{7}{72} \frac{\Sigma^2}{V} + O(\Sigma^4) \quad , \tag{28}$$

for d = 3, and

$$\tilde{r}^* = i \left[\frac{1}{5} - \frac{11}{18} \tilde{u}^* + O(\tilde{u}^{*2}) \right] \quad , \tag{29}$$

or

$$E_c = -4V - \frac{V}{5} + \frac{11}{36} \frac{\Sigma^2}{V} + O(\Sigma^4) \quad , \tag{30}$$

for d = 2. Thus it follows that for a gaussian distribution of site energies in three dimensions the mobility edge moves outwards (i.e. towards lower energies) for small values of the width of the disorder. This is in agreement with numerical calculations[12)] based on the Economou-Cohen criterion[13)] for localization as well as with results obtained from a semiclassical percolation argument[14)]. On the other hand, for d = 2 E_c moves inwards with increasing disorder. In fact, if we extrapolate the Eq.(30) in the limit of large disorder (where our actual treatment is, however, invalid) we find a value $\Sigma \equiv \Sigma_c = 2\sqrt{\frac{189}{55}}\,V$ for the critical disorder for localization of all the states of the unperturbed energy band (Anderson transition, defined by $E_c = 0$). We note that, in contrast to the prediction of the original Anderson theory[8)], the above value of Σ_c for d = 2 is in excellent agreement with the results of numerical estimates for a rectangular distribution of site energies[15)], whose width W should be identified with either 2Σ or $2\sqrt{3}\,\Sigma$ (comparison of the r.m.s. deviations for the two distributions). Finally, we note that for $\Sigma \to 0$ the mobility edges (28) and (30) deviate slightly from the positions of the unperturbed band edges at $E = -6V$ and $E = -4V$, respectively. This feature might be related to the fact that the precise value of the fixed point, Eq.(24), depends on the higher order interaction terms which are generated[3)] when going from one point to another in the parameter space of the Hamilton function through successive recursions. This point deserves further study in the future.

The above calculation of E_c is restricted to the limit of weak disorder ($\Sigma \ll V$). For medium disorder, such that $\Sigma \lesssim V$, in three dimensions we may use the fixed point solution (24) since the latter is valid when u^* is comparable to $\varepsilon \lesssim 1$ (for d = 3). Thus by eliminating b

between the two Eqs.(24) and retaining only terms of order ε, we obtain by using (9)

$$E_c = -6V - \frac{\Sigma^2}{V}\left(1 - e^{-\frac{12\,\Sigma^2}{V^2}}\right)^{-1} \quad . \tag{31}$$

We emphasize that this expression is invalid in the limit of very weak disorder. Indeed, in this limit $\tilde{u}$ is intrinsically small while Eq.(24) would require it to be of the order of unity for $d = 3$. Of course, the assumption of small $\tilde{u}$ which underlies the derivation of the recursion relations (19-20) prevents us from discussing the limit where $\Sigma \gg V$.

For dimensionalities $d > 4$ the nontrivial fixed point of Eqs.(22-23) is unstable[3] and the stable solution is the gaussian fixed point $\tilde{u}^* = 0$, $\tilde{r}^* = 0$. This implies that $d = 4$ is the critical dimensionality for the Anderson problem above which the mobility edge coincides with the band edge. This absence of localization for $d > 4$ is also suggested by some recent numerical studies[16].

2. *Density of states*

The average density of states per site is defined by

$$n(E) = -\frac{1}{\pi}\,\mathrm{Im}\,\hat{G}^e_{oo}(E) \quad . \tag{32}$$

In order to calculate this quantity we make use of the identity

$$\frac{1}{n}\,\frac{\partial \mathcal{F}}{\partial r} = \hat{G}^n_{oo} \quad , \tag{33}$$

which is obtained by differentiating the free energy, $\mathcal{F} = -\ln Z_n$ (where $Z_n = \int_{-\infty}^{\infty} \prod_i^N \prod_\alpha^n d S_{i\alpha}\, e^{-\mathcal{H}}$ is the partition function) with respect to r

and using the definition (7) of the correlation function. We note that the limit $n = 0$ of the left-hand side of Eq.(33) is well defined[2]. Furthermore, we use the general scaling relation for the singular part of the free energy[3),9)],

$$\mathcal{F} \sim \xi^{-d} \quad , \tag{34}$$

where the correlation length ξ is defined by[4]

$$\xi \sim \left(\frac{r_c - r}{J}\right)^{-\upsilon_o} \quad , \tag{35}$$

and υ_o is a critical index for the field model with $n = 0$ introduced in Sec. II. Using Eq.(35) and the identifications (9) we get

$$\hat{G}^e_{oo}(E) \sim (e - e_c)^{1-\alpha_o} \quad , \tag{36}$$

where $\alpha_o = 2 - \upsilon_o d$ is the specific heat index appropriate for the $n = 0$ case and the proportionality factor is real. Since $1 - \alpha_o$ is fractional $\hat{G}^e_{oo}(E)$ has a cut along the real axis (corresponding to a continuum of delocalized state poles) extending from $E = E_c < 0$ to $E = -E_c$. Thus from Eqs.(32) and (36) we obtain for the singular part, $n_{sing}(E)$, of $n(E)$

$$n_{sing}(E) \sim |E - E_c|^{1-\alpha_o}\,\theta(E - E_c) \quad , \tag{37}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. A non analytic behaviour of the density of states at $E = E_c$ similar to Eq.(37) has been obtained previously by Ma[17] for the case of a particle moving in a continuum in a random potential $\varphi(\vec{r})$.

3. *Critical indices*

The critical indices describe the non analytic behaviour of various physical quantities for energies close to the mobility edge. We recall that as the critical point is approached the correlation function $\hat{G}_{ij}^{n=o}$ develops an infinite range (which causes the divergence of $G_{k=o}^{n=o}$ in Eq.(25)) due to the fact that $\xi \to \infty$ for $r \to r_c$. In the equivalent localization problem this feature corresponds to the radius of localization of the wavefunction of a localized state becoming infinite at $E = E_c$. Therefore it seems natural to identify ξ with the localization radius, which gives a well-defined meaning to υ_o for the Anderson problem. The interpretation of the indices γ_o and α_o introduced above follows naturally from the physical meaning of the quantities whose behaviour they describe.

For continuous dimensionalities $d \to 4^-$ the critical exponents for the Anderson problem may be determined accurately from the recursion relations (22-23), using their general definition in terms of the standard parameters y_i describing the renormalization of the scaling fields under the renormalization group transformations in the vicinity of the fixed point[3),4),9)]. The parameters y_i of interest for the present case are $y_{\tilde{r}} = (\ell n\, b)^{-1}\, \ell n\, \lambda_{\tilde{r}}$ and $y_{\tilde{u}} = (\ell n\, b)^{-1}\, \ell n\, \lambda_{\tilde{u}}$, where $\lambda_{\tilde{r}}$ and $\lambda_{\tilde{u}}$ are eigenvalues obtained from the recursion relations as follows[3),4),9)]. We linearize the equations (22) and (23) for small deviations $\delta \tilde{r}_p = \tilde{r}_p - \tilde{r}^*$ and $\delta \tilde{u}_p = \tilde{u}_p - \tilde{u}^*$ from the fixed point (24) to obtain

$$\delta \tilde{r}_{\ell+1} = b^2 \left[(1 + 2\tilde{u}^*)\, \delta \tilde{r}_\ell - 2i\, (1 + 4\tilde{u}^* + i\, \tilde{r}^*)\, \delta \tilde{u}_\ell \right] \quad , \qquad (38)$$

$$\delta \tilde{u}_{\ell+1} = b^\varepsilon (1 - 24\, \tilde{u}^*)\, \delta \tilde{u}_\ell \quad , \qquad (39)$$

where the coefficients of $\delta \tilde{r}_\ell$ and $\delta \tilde{u}_\ell$ are given to order $\varepsilon = 4 - d$ up to which the Wilson recursion relation is exact. $\lambda_{\tilde{r}}$ and $\lambda_{\tilde{u}}$ are defined as the eigenvalues of the matrix of the coefficients of the r.h.s. of Eqs. (38-39) and correspond to the equation for $\delta \tilde{r}_{\ell+1}$ and $\delta \tilde{u}_{\ell+1}$, respectively. Using the Eqs.(24), we finally obtain

$$y_{\tilde{r}} = 2\, (1 + \frac{\varepsilon}{12} + O(\varepsilon^2)\,) \quad , \qquad (40)$$

$$y_{\tilde{u}} = -\,\varepsilon + O(\varepsilon^2) \quad , \qquad (41)$$

where the fact that $y_{\tilde{u}} < 0$ indicates that the fixed point (24) is stable[3),4)]. In the language of critical phenomena the parameter $y_{\tilde{r}} > 0$ corresponds to a relevant operator for the critical point, which determines the critical indices through the relations[3),4),9)] $\upsilon_o = \frac{1}{y_{\tilde{r}}}$, $\gamma_o = 2\,\upsilon_o$ ($\eta_o = 0$) and $\alpha_o = 2 - \frac{d}{y_{\tilde{r}}}$. Thus from Eq.(40) we get

$$\upsilon_o = \frac{1}{2} \left[1 - \frac{\varepsilon}{12} + O(\varepsilon^2) \right] \quad , \qquad (42)$$

$$\gamma_o = 1 - \frac{\varepsilon}{12} + O(\varepsilon^2) \quad , \qquad (43)$$

$$\alpha_o = \frac{2}{3}\,\varepsilon + O(\varepsilon^2) \quad . \qquad (44)$$

The values of υ_o and γ_o for $d = 3$ may be compared with the critical exponents for the localization radius, $\upsilon = \frac{3}{5}$, and for the propagator, $\gamma = 1$, obtained from a quite different treatment [18)] for the more general problem (see Sec. V) of an electron moving in a continuum in a random potential $\varphi(\vec{r})$. However, a value of 0.6 for the exponent υ has also been found previously in the framework of the Anderson tight binding model[19)]. The fact that the above

critical exponents for the Anderson problem differ from the corresponding exponents for the usual phase transition problem for $n = 0$[17] is not surprising since the two problems correspond to different stable fixed points, as we have shown above. Thus our analysis invalidates the recent work of Schuster[7] where the existence of a stable fixed point, characterized by the same critical indices as the phase transition problem for $n = 0$, has been implicitly assumed for the problem of an electron moving in a random potential $\varphi(\vec{r})$.

Finally we note that for $d > 4$ the gaussian fixed point, $\tilde{r}^* = 0$, $\tilde{u}^* = 0$, is the only stable fixed point. In this case the mobility edge, $E_c = -2dV$, coincides with the band edge and the critical indices reduce to the mean field values, $\upsilon_0 = \frac{1}{2}$, $\gamma_0 = 1$, $\alpha_0 = 0$, of the ordinary phase transition problem.

IV. LOCALIZATION IN THE PRESENCE OF DIAGONAL AND OFF-DIAGONAL DISORDER

In this Sec. we generalize the treatment of Secs. II and III to the case where the site energies $\mathcal{E}_i$ as well as the hopping matrix elements V_{ij} in the hamiltonian (1) have random components which are independently distributed. Thus we now assume that V_{ij} is of the form

$$V_{ij} = -V\,\delta_{i,j+\delta} + W_{ij} \quad , \tag{45}$$

where V is a constant systematic part and $W_{ij} = W_{ji} = W\,\delta_{i,j+\delta}$ is the random component. More precisely, we assume that the W_{ij} for different pairs of neighbouring sites are independent random variables, such that $2\,W_{ij}$ (the actual hopping matrix element between i and j) has a gaussian distribution with mean zero and r.m.s. deviation Σ_1. Thus the probability distribution

of W_{ij} is

$$P_W(W_{ij}) = (2\pi\Sigma_1^2)^{-\frac{1}{2}} \exp\left(-2\frac{W_{ij}^2}{\Sigma_1^2}\right) \quad . \tag{46}$$

The site energies $\mathcal{E}_i$ are defined by the gaussian distribution (2) as before. The configuration-averaged one-electron propagator in direct space is now given by

$$\hat{G}^e_{ij}(E) = -\frac{1}{V}(2\pi\sigma_0^2)^{-\frac{N}{2}}(2\pi\sigma_1^2)^{-\frac{NZ}{4}} \cdot \int_{-\infty}^{\infty}\prod_{\ell}^{N} d\varepsilon_\ell \prod d W_{[nm]}\, e^{-\sum_{\ell}^{N}\frac{\varepsilon_\ell^2}{2\sigma_0^2}}\, e^{-\sum_{n,m}\frac{W_{nm}^2}{\sigma_1^2}}\, \left[(\hat{e}-\hat{v}+\hat{w}-\hat{\varepsilon})^{-1}\right]_{ji}, \tag{47}$$

where Z denotes the number of nearest neighbours and the notation $\prod d W_{[nm]}$ signifies that one integrates over the variables W_{nm} for distinct pairs of sites only. The quantities $\hat{e} = -\frac{\hat{E}}{V}$, $\hat{v} = \frac{\hat{V}}{V}$, $\hat{w} = \frac{\hat{W}}{V}$, $\hat{\varepsilon} = -\frac{\hat{\mathcal{E}}}{V}$ are N x N matrices defined as in Sec. II and $\sigma_0 = \frac{\Sigma}{V}$ $\sigma_1 = \frac{\Sigma_1}{V}$.

In the same spirit as in Sec. II we shall first discuss the Hamilton function for a zero-component classical field model which leads to a two-point correlation function which is formally similar to Eq.(47). This Hamilton function involves a repulsive biquadratic exchange term in addition to the terms already present in Eq.(6) and our next task will be to generalize the Wilson recursion relation to incorporate the effect of this higher order term. We find that, in the framework of the usual approximations of Wilson's derivation, our more general recursion relation reduces to the same form as

Eq.(10) but the parameters $\tilde{r}_\ell$ and $\tilde{u}_\ell$ are replaced by new parameters depending on the biquadratic exchange constant . It follows therefore that the biquadratic exchange term has the effect of changing the position of the fixed point without changing the critical indices, as expected. Finally we study the detailed form of the mobility edge obtained from the fixed point solution as a function of Σ and Σ_1 for both d = 3 and d = 2.

A. Equivalent classical field model

We consider an n-component field model described by the Hamilton function

$$\mathcal{H}' = \mathcal{H} + \mathcal{H}_1 \quad , \tag{48}$$

where $\mathcal{H}$ is the Hamilton function (6) and $\mathcal{H}_1$ is a biquadratic exchange term of the form

$$\mathcal{H}_1 = \frac{1}{4} \sum_{i \neq j}^{N} K_{ij} \left(\sum_{\alpha}^{n} S_{i\alpha} S_{j\alpha} \right)^2 , \quad K_{ij} = K \delta_{i,j+\delta} \quad , \tag{49}$$

with $K > 0$. The partition function given by

$$Z_n = \int_{-\infty}^{\infty} \prod_{i}^{N} \prod_{\alpha}^{n} S_{i\alpha} \, e^{-\mathcal{H}'} \quad , \tag{50}$$

may be rewritten as usual[2] by introducing integral representations for the fourth order terms. This leads to

$$Z_n = (\pi u)^{-\frac{N}{2}} (2\pi K)^{-\frac{NZ}{4}} \int_{-\infty}^{\infty} \prod_{i}^{N} d\psi_i \prod_{[ij]} d\varphi_{[ij]} \, e^{-\sum_i \frac{\psi_i^2}{u} - \sum_{i,j} \frac{\varphi_{ij}^2}{4K_{ij}}}$$

$$\cdot \prod_{\alpha}^{n} \prod_{i}^{N} d S_{i\alpha} \exp \left\{ -\sum_{\alpha}^{n} \left[\sum_{i,j}^{N} \left(J_{ij} - \frac{i}{2} \varphi_{ij} \right) S_{i\alpha} S_{j\alpha} + \sum_{i}^{N} (r + i\psi_i) S_{i\alpha}^2 \right] \right\} , \tag{51}$$

where we have introduced *one* auxiliary variable $\varphi_{ij} \equiv \varphi_{ji}$ per pair of distinct sites since $K_{ij} = K_{ji}$. The Eq.(51) is of the same general form as the Eqs.(2.2-2.4) of Emery's paper[2]. Thus by following Emery's treatment the two-point correlation function $\hat{G}^n_{ij}$ for n = 0 may now be written

$$\hat{G}^{n=o}_{ij} = (\pi u)^{-\frac{N}{2}} (2\pi K)^{-\frac{NZ}{4}} \int_{-\infty}^{\infty} \prod_{\ell}^{N} d\psi_\ell \prod_{[nm]} d\varphi_{[nm]}$$

$$e^{-\frac{1}{u} \sum_{\ell}^{N} \psi_\ell^2 - \frac{1}{4K} \sum_{nm} \varphi_{nm}^2} \left[\left(\hat{J} - i \frac{\hat{\varphi}}{2} + \hat{r} + i \hat{\psi} \right)^{-1} \right]_{ji} , \tag{52}$$

where the matrices $\hat{J}$, $\hat{r}$, $\hat{\psi}$ are defined as in Sec. II and $\hat{\varphi}$ is a new tridiagonal matrix with non-zero matrix elements φ_{nm} (with $n = m \pm \delta$) . The comparison of Eqs.(52) and (47) now leads to the identifications

$$\hat{r} \leftrightarrow -i\hat{\varepsilon} \, , \; \hat{J} \leftrightarrow i\hat{v} \, , \; \hat{\psi} \leftrightarrow \hat{\varepsilon} \, , \; \frac{\hat{\varphi}}{2} \leftrightarrow \hat{w} \quad ,$$

$$u \leftrightarrow 2\sigma_o^2 \, , \; K \leftrightarrow \sigma_1^2 \, , \; \hat{G}^{n=o}_{ij} \leftrightarrow -i V 2^{\frac{NZ}{2}} \hat{G}^e_{ij}(E) \quad , \tag{53}$$

which demonstrates the equivalence between the electron localization problem in an Anderson model with gaussian diagonal and off-diagonal disorders and the statistical mechanics of a zero-component classical field described by the Hamilton function of Eqs.(48-49). In order to study the mobility edge we must first derive the recursion relation of the renormalization group for this new classical field problem.

B. Generalized Wilson recursion relation

Following the usual derivation of Wilson's recursion relation from an analysis in position-space[9] we first go to the continuum limit of the Hamilton functions (6) and (49), assuming the nearest-neighbour distance δ to be small. The justification for this procedure lies in the fact that close to the critical point $\xi >> \delta$. The manipulations involved in deriving the continuum limit of Eq.(6) by first expanding the quantities $S_{i\alpha} S_{j\alpha} \equiv S_\alpha(\vec{r}) S_\alpha(\vec{r}+\vec{\delta})$ through order δ^2 are standard[4),9)] and their extension to Eq.(49) is straightforward. Therefore we display directly the final result which reads

$$\mathcal{H}' \sim \frac{1}{2} K_o \int d^d r \; | \vec{\nabla} \vec{S}(\vec{r}) |^2 - \frac{1}{2} K \int d^d r \left[\frac{1}{4} |\vec{\nabla} S^2(\vec{r})|^2 + S^2(\vec{r}) \; | \vec{\nabla} \vec{S}(\vec{r})|^2 \right]$$

$$+ \int d^d r \; P_o \left[\vec{S}(\vec{r}) \right] , \; K_o = - 2 J \quad , \qquad (54)$$

where $P_o[\vec{S}(\vec{r})]$ is the local potential given by

$$P_o[\vec{S}(\vec{r})] = (r - d\, K_o)\, S^2(\vec{r}) + \frac{1}{2} \left(\frac{u}{2} + d\, K \right) \left[S^2(\vec{r})\right]^2 \quad . \qquad (55)$$

In these expressions δ has been taken as length unit and the summations over nearest-neighbours in Eqs.(6) and (49) have been performed for a simple cubic lattice in a d-dimensional space as usual[4),9)]. In the above equations $\vec{S}(\vec{r})$ denotes an n-component vector with components $S_\alpha(\vec{r})$ ($S^2(\vec{r}) = \sum_\alpha^n S_\alpha^2(\vec{r})$) and we have used the conventional notation $|\vec{\nabla} \vec{S}(\vec{r})|^2 = \sum_\alpha^n | \vec{\nabla} S_\alpha(\vec{r})|^2$. In the continuum limit the partition function is defined as usual as the functional

integral of $\exp(-\mathcal{H}')$ over the functions $S_\alpha(r)$.

The modifications in Wilson's recursion relation[3),4),9)] will arise from the higher order gradient terms proportional to K in Eq.(54). In the spirit of the Ginzburg-Landau-Wilson expansion of the free-energy functional (where only the simplest possible gradient term is retained[20)] for the description of the most important long-wavelengths fluctuations close to the critical point), we shall first derive an effective hamiltonian, $\mathcal{H}_1'$, which includes the effect of the higher order gradient terms of Eq.(54) through an effective local potential of the same form as (55). We define $\mathcal{H}_1'$ as the effective hamiltonian which results after the *first* of the usual sequence of renormalization group operations which are involved in approaching a fixed point in the parameter space of $\mathcal{H}'$. In order to find the explicit form of $\mathcal{H}_1'$ we shall first rewrite $\mathcal{H}'$ in terms of new field variables corresponding to the separation of $\vec{S}(\vec{r})$ into slowly and rapidly varying components over discrete cells (blocks), defined in the usual way[9)]. Then we will follow the usual steps of eliminating degrees of freedom by integrating out the rapidly varying field components in Z_n and changing the length scale. This will then determine the effective Hamilton function $\mathcal{H}_1'$.

Thus we start by defining

$$\vec{S}(\vec{r}) = \vec{S}_o(\vec{r}) + \vec{S}'(\vec{r}) \quad , \qquad (56)$$

where $\vec{S}_o(\vec{r})$ is slowly varying and $\vec{S}'(\vec{r})$ fluctuates rapidly over the width, a, of discrete cells (blocks) m centered at $\vec{r}_m$, into which the system is assumed to be subdivided. The rapidly varying part $\vec{S}'(\vec{r})$ is integrated out in the partition function after expressing it as a linear combination of functions $\psi(\vec{r} - \vec{r}_m)$ centered around $\vec{r}_m$,

$$\vec{S}'(\vec{r}) = \sum_m \vec{S}'_m \; \psi(\vec{r} - \vec{r}_m) \quad , \qquad (57)$$

and making the following standard assumptions[3),9)]. First one assumes that the functions $\psi(\vec{r} - \vec{r}_m)$ for different cells form an orthonormal set and that $\psi(\vec{r} - \vec{r}_m)$ is so well localized that it vanishes outside the cell m. Similarly one assumes that $\int d^d r\, \vec{\nabla}\psi(\vec{r} - \vec{r}_n)\, \vec{\nabla}\psi(\vec{r} - \vec{r}_m) = \rho_o\, \delta_{n,m}$, where ρ_o is a constant such that[9)] $b^{-2} \ll \rho_o \leq 1$. This assumption together with the orthonormality condition for the functions $\psi(\vec{r} - \vec{r}_m)$ implies that

$$\vec{\nabla}\psi(\vec{r} - \vec{r}_m).\vec{\nabla}\psi(\vec{r} - \vec{r}_n) = \rho_o\, \psi(\vec{r} - \vec{r}_m).\psi(\vec{r} - \vec{r}_n) \quad . \tag{58}$$

In order to estimate the local terms in (54), as well as the higher order gradient terms after substituting (56-58), one further assumes that $\psi(\vec{r} - \vec{r}_m)$ has the value $a^{-d/2}$ in one half of the cell m and $-a^{-d/2}$ in the other half. This is consistent with the assumption that $\vec{S}'(r)$ varies rapidly over the width of a cell while $\vec{S}_o(\vec{r})$ varies over much larger distances only (i.e. $\vec{S}_o(\vec{r}) \approx \vec{S}_o(\vec{r}_m)$ inside a cell m). A final approximation, to which we shall return later, consists in neglecting all terms in $\mathcal{H}'$ involving gradients of $\vec{S}_o(\vec{r})$ except the lowest order Ginzburg-Landau interaction. With the above assumptions the spatial integrations in Eq.(54) may be replaced explicitly in terms of summations over the cell variables. This leads to the expression

$$\mathcal{H}' = \frac{K_o}{2} \int d^d r\, \left| \vec{\nabla}\vec{S}_o(\vec{r}) \right|^2 + \frac{1}{2} K_o \rho_o \sum_m S'^2_m + \frac{a^d}{2} \sum_m \left(P_o\left[\vec{S}_o(\vec{r}_m) + a^{-\frac{d}{2}} \vec{S}'_m\right] + P_o\left[\vec{S}_o(\vec{r}_m) - a^{-\frac{d}{2}} \vec{S}'_m\right] \right)$$

$$- \frac{K \rho_o}{2} \sum_m \left[\left(\vec{S}_o(\vec{r}_m).\vec{S}'_m\right)^2 + S_o^2(\vec{r}_m)\, S'^2_m + 2\, a^{-d} \left(S'^2_m\right)^2 \right] \quad . \tag{59}$$

Now, by integrating out the cell variables from the partition function in a standard way[9)] and changing the length scale we obtain for the effective Hamilton function $\mathcal{H}'_1$ after the first step of renormalization (with $\vec{S}_1(\vec{r}) = b^{\frac{d}{2}-1} \vec{S}_o(\vec{r})$)

$$\mathcal{H}'_1 = \frac{K_o}{2} \int d^d r\, \left| \vec{\nabla}\vec{S}_1(\vec{r}) \right|^2 + \int d^d r\, P_1\left[\vec{S}_1(\vec{r})\right] \quad , \tag{60}$$

where

$$P_1\left[\vec{S}_1(\vec{r})\right] = -\, a^{-d}\, b^d\, \ln I_o\left[\left(\frac{\mathrm{Im}\, K_o \rho_o a^d}{2} \right)^{\frac{1}{2}} b^{-\frac{d}{2}+1} \vec{S}_1(\vec{r}) \right] + \text{constant} \quad , \tag{61}$$

and

$$I_o(\vec{z}) = \int d\vec{y}\, \exp\left(- iy^2 - \frac{2K}{K_o^2} \left[(\vec{z}.\vec{y})^2 + z^2 y^2 + 2(y^2)^2 \right] - \frac{1}{2} Q_o(\vec{z} + \vec{y}) - \frac{1}{2} Q_o(\vec{z} - \vec{y}) \right) \quad , \tag{62}$$

$$Q_o(\vec{z}) = \tilde{r}_o z^2 + \tilde{u}_o (z^2)^2 \quad , \tag{63}$$

$$\tilde{r}_o = -\, i \left(\frac{r}{J} + 2d \right) \quad , \qquad \tilde{u}_o = \frac{1}{4} (u + 2dK) \quad . \tag{64}$$

Here $\vec{y}$ and $\vec{z}$ are real n-component vectors defined as in Sec. III. In general the Eqs.(63-64) and the coefficient of the terms proportional to K in Eq.(62) involve factors of $\rho_o a^d$ which we have put equal to unity [thus assuming both lengths $1/\sqrt{\rho_o}$ and a (constant block size) to be comparable to δ] on the usual ground that the end results of the treatment should be independent

of these ad hoc parameters. Of course, in the present case another possibility would be to fix a relation between the parameters a and $1/\sqrt{\rho_o}$ at the end of the treatment by requiring the fixed-point energy (mobility edge) to coincide with the band edge in the ordered limit. The factors of i in (62) and (64) appear in the same way as in Sec. III due to the fact that J in Eqs.(54-55) is purely imaginary (see (53)). The Eq.(60) which is of the same form as the usual Ginzburg-Landau-Wilson functional defines our new effective hamiltonian in which the effect of the higher order gradient terms of (54) is taken into account through a renormalized local interaction $P_1[\vec{S}_1(\vec{r})]$.

We now use the Ginzburg-Landau Hamilton function (60) (in which K_o is purely imaginary) as our starting hamiltonian for a renormalization group study involving the determination of the fixed point from Wilson's recursion formula. The recursion formula of the renormalization group for an initial Hamiltonian of the form of (60) is given by[3),4),9)]

$$Q'_{\ell+1}(\vec{z}) = - b^d \ln I'_\ell (b^{-\frac{d}{2}+1} \vec{z}) \quad , \tag{65}$$

with

$$I'_\ell (\vec{x}) = \int d\vec{y} \exp \left[- iy^2 - \frac{1}{2} Q'_\ell (\vec{z} + \vec{y}) - \frac{1}{2} Q'_\ell(\vec{z} - \vec{y}) \right] , \tag{66}$$

where the starting expression of $Q'_\ell(\vec{z})$ in the recursion procedure is

$$Q'_1(\vec{z}) = a^d P_1 \left[\left(\frac{2}{\mathrm{Im}\, K_o \rho_o a^d} \right)^{\frac{1}{2}} \vec{z} \right] + \text{constant} \quad , \tag{67}$$

$$\vec{z} = \left(\frac{1}{2} \mathrm{Im}\, K_o \rho_o \right)^{\frac{1}{2}} \vec{S}_1 \quad . \tag{68}$$

Since the bilinear terms in $\mathcal{H}'_1$ are purely imaginary the Eq.(65) must be solved non-perturbatively to find the fixed point, from which the mobility edge can then be obtained as in Sec. III.

Before discussing the explicit solution of Eqs.(65-67) we wish to analyze further the validity of the procedure which has led from the general Hamilton function (54) for our problem to the effective Ginzburg-Landau expression (60). Perhaps the most drastic assumption made in obtaining $\mathcal{H}'_1$ has been the neglect (in (59)) of the higher order interaction terms resulting from the replacement of $\vec{S}(\vec{r})$ by $\vec{S}_o(\vec{r})$ in the second term of (54). This assumption lies, however, in the spirit of Wilson's original method where one tries to incorporate, via successive renormalizations, the effect of fluctuations of larger and larger wavelengths without departing from the simple Ginzburg-Landau form for the renormalized hamiltonians. In this way one is ignoring all the higher order gradient and polynomial terms which are generated when carrying out the renormalization group procedure in wavevector space[3)]. As emphasized, for example, by Ma[3)] such a procedure requires nevertheless some further justification. In fact, Ma has analyzed in detail the effect of higher gradient terms of precisely the same form as the second term in (54) in the framework of the perturbative expressions for the renormalization-group equations derived in wavevector space, for the usual phase transition problem. In particular, by studying the form of the recursion relations close to the gaussian fixed point, Ma has verified that the parameter K is more irrelevant than u . Since close to the gaussian fixed point the recursion relations for K in an enlarged parameter space are the same for the localization problem and for the usual phase transition problem, Ma's conclusion applies in the present case as well. Ma has further shown (see also ref. 11) that the second term in Eq.(54) does not affect the fixed

point and the critical exponents to order ε for $d < 4$. This fact provides an indirect justification for the use of the effective Hamilton function $\mathcal{H}_1'$ instead of (54) since the formal similarity of the Wilson recursion relations (10) and (65) implies that the critical exponents for the localization problem with off-diagonal disorder are identical (to order ε) to those, studied in Sec. III, for the diagonal disorder-case. However, in general, the higher order gradient terms are expected to affect the fixed point equations (and hence the mobility edge) beyond quadratic order[3] in the amplitude of the quartic term in (54) and this has been taken into account approximately through renormalizations of the local potential in Eq.(60) (see Eqs.(69-71) below).

C. Fixed point and mobility edge

We first study the detailed form of the renormalized local potential term in the effective Hamilton function $\mathcal{H}_1'$, which is required for the analysis of the recursion relation (65). Due to the presence of i in Eqs. (62-63) we must determine $I_o(\vec{z})$ non-perturbatively as in Sec. III. Proceeding in the same way as in Sec. III we isolate the factor $\exp\left[-(\tilde{r}_o + i)\, y^2 - (\tilde{u}_o + \frac{K}{J^2})(y^2)^2\right]$ in the integrand of the r.h.s. of Eq.(62) and expand the remaining $\vec{y}$-dependent part through quadratic order in the parameters $\tilde{u}_o$ and K , and finally perform the $\vec{y}$-integrals exactly in polar coordinates. Then, after expanding further the logarithm on the r.h.s. of Eq.(61) for small $\tilde{u}_o$ and small $\vec{k}$ and identifying the result with a polynomial expression for $Q_1'(\vec{z})$ in (67), of the form

$$Q_1'(\vec{z}) = r_1' z^2 + u_1' (z^2)^2 \quad , \tag{69}$$

we obtain the following expressions for the renormalized parameters of the effective Hamilton function $\mathcal{H}_1'$:

$$r_1' = b^2 \left[\tilde{r}_o + \frac{u_o''}{2\sqrt{2u_o'}}\, f(x) \right] \quad , \tag{70}$$

$$u_1' = b^\varepsilon \left\{ \tilde{u}_o - \frac{u_o''}{4u_o'} \left[(5\,\tilde{u}_o + \frac{7}{8}\frac{K}{J^2})\,(x\, f(x) - 1) - \frac{u_o''}{4} f^2(x) \right] \right\} , \tag{71}$$

where

$$f(x) = \sqrt{\frac{\pi}{2}}\; e^{\frac{x^2}{2}}\; \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad , \tag{72}$$

$$x = \frac{\tilde{r}_o + i}{\sqrt{2u_o'}} \quad , \tag{73}$$

$$u_o' = \tilde{u}_o + \frac{K}{J^2} \quad , \quad u_o'' = 4\,\tilde{u}_o + \frac{K}{2J^2} \quad . \tag{74}$$

The Eqs.(70-71) may now be expanded further since $\tilde{u}_o$ and K are assumed to be small. Using the asymptotic expression (21) we get

$$r_1' = b^2 \left[\tilde{r}_o + \frac{u_o''}{2(\tilde{r}_o + i)} \left(1 - 2\,\frac{u_o'}{(\tilde{r}_o + i)^2} \right) + \ldots \right] \quad , \tag{75}$$

$$u_1' = b^\varepsilon \left[\tilde{u}_o + \frac{u_o''}{(\tilde{r}_o + i)^2} \left(3\,\tilde{u}_o + \frac{K}{2J^2} \right) \ldots \right] \quad , \tag{76}$$

and we note that, like $\tilde{r}_o$, r'_1 is purely imaginary. With the definitions (75) and (76) of the parameters of the potential function $Q'_1(\vec{z})$ we may now study in detail the recursion relations for r'_ℓ and u'_ℓ, which coincide with the recursion relations for $\tilde{r}_\ell$ and $\tilde{u}_\ell$ given explicitly in Sec. III. In particular, the fixed point solutions ($r'_{\ell+1} = r'_\ell \equiv r'^*$, $u'_{\ell+1} = u'_\ell = u'^*$) for small ε have the same form as Eq.(24), namely

$$u'^* = \frac{\varepsilon \ln b}{12} \quad , \quad r'^* = \frac{2 i u'^*}{1 - b^{-2}} \quad . \tag{77}$$

As discussed in Sec. III, the fixed point (77) enables us to determine the mobility edge for medium disorder (Σ/V and $\Sigma_1/V \lesssim 1$) in three dimensions by eliminating the arbitrary scale factor b . Using the identifications (53) we obtain from (64), (74-77) and retaining systematically terms of order ε only,

$$E_c = -6V - \frac{1}{V}\left(\Sigma^2 + 3\Sigma_1^2\right)\left[1 - e^{-\frac{12}{V^2}\left(\Sigma^2 + 3\Sigma_1^2\right)}\right]^{-1} + \frac{\Sigma_1^2}{4V} \quad , \; d=3 \quad . \tag{78}$$

On the other hand, in order to study the weak disorder case for d = 3 and d = 2 we must go back to the analogs of Eqs.(22-23) for the present case and find the exact solutions for the fixed point r'^* to lowest non-vanishing order in u'^* . These solutions are given by the analogs of Eqs.(27) and (29) that is

$$r'^* = i\left[\frac{1}{11} + \frac{7}{36}u'^* + O(u'^{*2})\right] \quad , \; d = 3 \quad , \tag{79}$$

$$r'^* = i\left[\frac{1}{5} - \frac{11}{18}u'^* + O(u'^{*2})\right] \quad , \; d = 2 \quad . \tag{80}$$

By combining (79) and (80), respectively, with the definitions (74-76), (64) and using the identifications (53) we obtain for the mobility edges

$$E_c = -(6V + \frac{V}{11}) - \frac{1}{8V}\left(\frac{7}{9}\Sigma^2 + \frac{1}{2}\Sigma_1^2\right) \quad , \; d = 3 \quad , \tag{81}$$

$$E_c = -(4V + \frac{V}{5}) + \frac{1}{4V}\left(\frac{11}{9}\Sigma^2 + \frac{177}{54}\Sigma_1^2\right) \quad , \; d = 2 \quad . \tag{82}$$

The Eq.(81) for d = 3 shows that off-diagonal disorder makes the mobility edge move outwards from the original band edge. This agrees with physical expectation since the main effect of off-diagonal disorder should be to broaden the original unperturbed band by an amount depending on the width of the distribution of the random component of the hopping matrix element. However, this intuitive argument is not supported by the Eq.(82) for the two-dimensional case. The Eqs.(78) and (81-82) provide the first explicit results for mobility edges for the Anderson model with gaussian diagonal and off-diagonal disorders for two and three-dimensional systems. For completeness we mention that the mobility edge in the Anderson model with off-diagonal randomness has recently been studied numerically, using the Economou-Cohen approach[13] combined with the coherent potential approximation for a semi-circular distribution of W_{ij}[21].

V. CONCLUDING REMARKS

From the point of view of the properties of the configuration-averaged one-electron propagator, the Anderson localization problem in the presence of gaussian disorder, in the diagonal terms or in both the diagonal and off-diagonal terms of the hamiltonian, is equivalent to the statistical mechanics of zero-component classical field models, whose Hamilton functions involve repulsive quartic interactions but imaginary couplings in the bilinear terms. For the case of site-diagonal disorder the quartic interaction is the ordinary local self-interaction term while for combined diagonal and off-diagonal disorders an additional biquadratic exchange term appears in the equivalent classical field model. We have shown that Wilson's recursion relations for these problems lead to stable fixed points (u^*, r^*) with $u^* > 0$ (for $d < 4$) as required physically. An essential point of our treatment is that the conventional expansion of the Wilson formula for small u at an early stage has been avoided. In this sense our treatment is a non-perturbative one.

We have derived explicit expressions for the mobility edge E_c from the fixed-point equations for three- and two-dimensional systems in the cases of diagonal and both diagonal and off-diagonal disorders. Our result for E_c for $d = 3$ in the case of diagonal disorder is consistent with numerical calculations based on a different approach. The extrapolation of our E_c for $d = 2$ in the case of large disorder yields a critical disorder for the Anderson transition in excellent agreement with previous numerical estimates. We have also discussed the behaviour of the density of states close to the mobility edge. Finally from the study of the renormalization group equations close to the fixed point, we have obtained the critical indices for the localization radius and other quantities of interest in the Anderson problem. These critical indices are given within the ϵ-expansion since $d = 4$ appears as a critical dimensionality for the Anderson problem.

In conclusion we wish to comment briefly on the relation between recent renormalization group studies of the nm-component classical field model for systems with random quenched impurities[3] and the field model of Sec. II. It is well known [22),3),6)] that the nm-model on a lattice in the limit $n = 0$ and $m = 1$ leads to a fixed point $u^* < 0$ (in our notation) which is unphysical since u is identified with the second cumulant of the random impurity potential. Since formally the two-point correlation function derived from the nm-model in the above limit[2] is similar to the configuration-averaged electron propagator, it has been concluded[6] that there is no real energy at which the range of the propagator becomes infinite. Thus, according to the authors of Ref. 6, the mobility edge cannot be discussed in the framework of the averaged one-electron propagator. The treatment of Sec. III casts doubts on this conclusion. An essential point for our treatment has been the observation that the comparison of the average propagator for the Anderson tight-binding model with the equivalent correlation function leads to a complex free energy functional (see Sec. II). As shown in Sec. III, a proper non-perturbative study of the recursion relations, which is sine qua non for this case, leads to a physical fixed point and hence to the existence of a mobility edge. If, however, we were to define new fields, $S'_{i\alpha} = \sqrt{i}\ S_{i\alpha}$, the complex free energy functional would appear to describe an attractive self-interaction with real bilinear terms[23] and, in fact, the connection between the random electron problem and the classical field problem has usually been stated in this form[3),6),17)]. However, in the study of the fixed point for this case one must explicitly note the fact that the fields are complex when performing partial

integrations. Inadequate appreciation of this point has led to the erroneous conclusions mentioned above. Finally, we recall that our discussion has been in the framework of the Anderson problem with site-diagonal disorder. The problem of an electron moving in a continuum with random potential $\varphi(\vec{r})$ is qualitatively different in that it has both diagonal and off-diagonal disorders defined with respect to the complete set of eigenfunctions of the position operator. Therefore, this problem cannot be reached as a limiting case of the Anderson tight-binding model as the lattice constant goes to zero. However, the qualitative conclusions concerning localization are not expected to be different in this case. This is supported by the analysis of Sec. IV for the case where off-diagonal disorder is also included in the Anderson hamiltonian.

After the present work was completed a paper appeared[24] which also discusses the motion of an electron in a continuum in a random potential $\varphi(\vec{r})$ using a renormalization group-theoretic approach. Since the arbitrary constant a introduced in Ref. 24 is purely imaginary, the Hamilton function in the partition function for the equivalent statistical-mechanics problem [Eq.(5.11) of Ref. 24] involves purely imaginary bilinear terms (for Im E = 0) and a repulsive quartic interaction, just like in the classical field model discussed in Sec. II. Thus for convergence reasons similar to those discussed above the use of the perturbative form of Wilson's recursion relation[24] is invalid for this case and the conclusions drawn in Ref. 24 about the behaviour of the averaged Green's function are incorrect.

At the end we comment briefly on the use of the averaged one-electron propagator $\tilde{G}^{e}_{ij}(E)$ in the limit $R_{ij} \to \infty$, for obtaining the mobility edge. The underlying argument is that the large-distance behaviour of the true propagator (or, equivalently, the small-wave vector form of its Fourier transform) is expected to be unaffected by the configuration-averaging as far as the non-

analytic behaviour is concerned. This is because for large separations the intervening potential profile samples all possible complexions of the random system. What is implied here is that the probability distribution function for the true propagator is sharply peaked at the mean value, $\tilde{G}^{e}_{ij}(E)$, in the limit $R_{ij} \to \infty$. This is supported by the recent findings of Sadovskii[25].

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