# Analytic, High $\beta$, Flux Conserving Equilibria for Cylindrical Tokamaks

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ORNL/TM-6040
Dist. Category UC-20 g

Contract No. W-7405-eng-26

FUSION ENERGY DIVISION

# ANALYTIC, HIGH β, FLUX CONSERVING EQUILIBRIA FOR CYLINDRICAL TOKAMAKS

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Date Published - September 1978



Prepared by the
OAK RIDGE NATIONAL LABORATORY
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UNION CARBIDE CORPORATION
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DEPARTMENT OF ENERGY

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ABSTRACT

Using Grad's theory of generalized differential equations, the temporal evolution from low to high $\beta$ due to "adiabatic" and nonadiabatic (i.e., neutral beam injection) heating of a cylindrical tokamak plasma with circular cross section and peaked current profiles is calculated analytically. The influence of shaping the initial safety factor profile and the beam deposition profile and the effect of minor radius compression on the equilibrium is analyzed.

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## I. INTRODUCTION

Recently, Clarke[1] observed that if one starts with an ohmically heated low $\beta$ tokamak plasma into which intense neutral beams are injected, the plasma $\beta$ can be increased in a flux conserving manner. That is, if the energy deposition time of the beams is shorter than the resistive skin time, then the magnetic flux and the safety factor profile (which is a function of poloidal flux) will be conserved during the energy deposition.

The determination of such a sequence of flux conserving tokamak equilibria (and, more generally, the adiabatic problem) yields a non-standard mathematical problem whose well-posedness has been investigated by Grad[2,3] in his theory of generalized differential equations. Numerical toroidal flux conserving equilibria have been found by Dory and Peng.[4] Clarke and Sigmar,[5] using a virial theory approach, have investigated global properties of flux conserving equilibria assuming the safety factor profile $q(\psi)$ = const for a circular shell in fully toroidal geometry. Mizoguchi and Kammash[6] have recently generalized the work of Ref. 5 to an elliptical shell. Dory and Peng[4] also solved the most elementary problem of a flux conserving cylinder with a uniform $q(\psi)$ profile analytically and a peaked current profile case (similar to ours) purely numerically. Related calculations have been performed by Bateman.[7] These authors prescribed a fixed $q(r)$ profile and generated a series of somewhat arbitrary pressure or current profiles such that flux was conserved.

In this paper, an analytic solution is presented for the physically more meaningful case in which neighboring pressure profiles are connected via an adiabatic equation of state, driven by some heat source profile (due to auxiliary heating). (Such a problem was recently solved purely numerically by Nelson.[8])

We find it worthwhile to close the gap in analysis between the most degenerate analytic example given as an introduction to the numerical work of Ref. 4 and the most involved numerical cases of Ref. 8. In the process, we investigate the nature of the nonlinearity (with rising pressure) of the equilibrium problem and the possibility of creating dia- or paramagnetic equilibria under the constraints of flux conservation and auxiliary "adiabatic" heating. The flux conserving equilibria for circular flux surfaces in cylindrical geometry with nonconstant q and $p'(\psi)$ profiles are found by performing perturbation expansions in $\varepsilon << 1$, where $\varepsilon$ is the ratio of poloidal to toroidal magnetic fields (tokamak ordering). The sequence of equilibria connects an initially low $\beta \sim O(\varepsilon^2)$ equilibrium to high $\beta \sim O(\varepsilon)$ equilibria conserving total flux and the $q(\psi)$ profile. It should be noted that while no upper limit on $\beta$ has been found in the numerical codes of Dory and Peng,[4] there does appear to be a critical $\beta$ of 5-10% in the linear stability codes of Bateman and Peng[9] and others. This motivates our perturbation technique in the smallness of $\varepsilon$, suitable even for so-called "high" $\beta$ tokamaks.

In Sec. II, we briefly review Grad's theory[2] of generalized differential equations. The flux conserving heating problem is formulated in Sec. III together with the basic dimensionless variables. In Sec. IV, flux conserving equilibrium solutions are constructed for arbitrary safety factor and heat source profiles. A class of $q(\psi)$ profiles is then found which avoids the peeling instability at high beta. In Sec. V, the special case of neutral beam heating (described by keeping a heat source term in the entropy equation) and the case of adiabatic compression heating are explored. Finally, in Sec. VI we summarize our results.

## II. BRIEF REVIEW OF GRAD'S THEORY OF GENERALIZED DIFFERENTIAL EQUATIONS[2,3]

For a cylindrical plasma with axial ("toroidal") symmetry, we can introduce the poloidal magnetic flux function $\psi$ such that, in cylindrical polar coordinates,

$$B_\theta = \frac{\partial \psi}{\partial r} . \tag{1}$$

The ideal magnetohydrodynamic equilibrium equations

$$\begin{aligned} \nabla p &= \underset{\sim}{J} \times \underset{\sim}{B} \\ \nabla \times \underset{\sim}{B} &= 4\pi \underset{\sim}{J} \\ \nabla \cdot \underset{\sim}{B} &= 0 \end{aligned} \tag{2}$$

can then be written as

$$\nabla^2 \psi = -4\pi \frac{dp}{d\psi} - B_z(\psi) \frac{dB_z}{d\psi} . \tag{3}$$

Because of the axial symmetry, the hyperbolic part of the mixed elliptic-hyperbolic equations (2) can be integrated out, giving rise to the two arbitrary functions $p = p(\psi)$ and $B_z = B_z(\psi)$ on the right-hand side of Eq. (3). When these two profiles are specified, Eq. (3) becomes a standard elliptic partial differential equation for $\psi$. The problem becomes well-posed by the application of, for example, a Dirichlet boundary condition on $\psi$.

However, if one is interested in the adiabatic problem — in particular, in determining a sequence of flux conserving equilibria -- one wishes to specify not only that the safety factor profile $q(\psi) = d\phi/d\psi$ (where $\phi$ is the toroidal flux) remain invariant, but also $\psi$ on the boundary (i.e., a Dirichlet boundary condition) as well as $\psi$ at an interior point. Grad has shown that such a nonstandard problem can indeed be well-posed. His idea is to specify two auxiliary profiles rather than $p(\psi)$ and $B_z(\psi)$ so that Eq. (3) becomes

$$\nabla^2\psi = f\left(V, \psi, \frac{d\psi}{dV}, \frac{d^2\psi}{dV^2}\right) \tag{4}$$

such that the right-hand side of Eq. (4) involves the second derivative $d^2\psi/dV^2$. V is the volume enclosed, for example, by simply nested $\psi$ = const surfaces. It is the presence of this $d^2\psi/dV^2$ term on the right-hand side of Eq. (4) which changes the nature of the well-posed problem from that for elliptic or qualitatively elliptic equations.

To find the well-posedness conditions for such a "generalized differential" equation, Grad performs a flux surface average over Eq. (4) to obtain

$$\overline{\nabla^2\psi} = \frac{d}{dV}\left[K(V)\frac{d\psi}{dV}\right] = f\left(V, \psi, \frac{d\psi}{dV}, \frac{d^2\psi}{dV^2}\right) , \tag{5}$$

where $K(V) = \langle|\nabla V|^2\rangle$ is a geometric factor determined by the family of $\psi$ = const surfaces. For simply nested topologies, $K(0) = 0$. In particular, for a circular cylinder,

$$\left\langle\nabla^2\right\rangle = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) = \frac{d}{dV}\left(4\pi V\frac{d}{dV}\right) ,$$

revealing that $K(V) = 4\pi V$ for this case. The original two-dimensional operator $\nabla^2\psi$ does not allow the specification of $\psi$ at an interior point of the domain (for example, at $V = 0$), while the flux-averaged elliptic operator $\nabla^2\psi$ becomes a one-dimensional operator with $V = 0$ a singular point. But because of the presence of the $d^2\psi/dV^2$ term in the right-hand side of Eq. (5), the averaged generalized differential equation now becomes a second order ordinary differential equation with $V = 0$ a regular point. Thus, Eq. (5) can be solved with the boundary conditions $\psi$ at the boundary and $\psi$ at $V = 0$ given. This, together with an invariant $q(\psi)$ profile, determines a sequence of flux conserving equilibria. We note that only if the initial equilibrium state possesses circular cylindrical geometry, from Eq. (3) one cannot break this symmetry by increasing the pressure $p(\psi)$. Thus, for the circular cylinder it suffices to solve the one-dimensional Eq. (5) for $\psi(V)$. (This is not so for an elliptic cylinder or a torus of any cross section.)

## III. EQUATION OF STATE AND PROFILE CONSERVATION OF $q(\psi)$

Grad et al.[3] have considered the adiabatic equation of state

$$\frac{d}{dt}(pn^{-\gamma}) = 0$$

where n is the number density and the convective derivative d/dt is that derivative following the motion of a flux surface as one proceeds parametrically in time through neighboring states. That is,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla, \quad \frac{d\psi}{dt} = 0 \tag{6}$$

where $\underset{\sim}{u}$ is the velocity of a flux surface.

In view of the importance of auxiliary heating of tokamaks (e.g., by neutral beam injection), we generalize the adiabatic equation of state to allow for external heating sources. We have, on time scales faster than all effective collisional time scales, in the frame moving with velocity $\underset{\sim}{u}$, the energy balance equation[10]

$$\frac{\partial}{\partial t}(pV'^{\gamma}) = (\gamma - 1)V'^{\gamma}Q(\psi) \tag{7}$$

where $\frac{\partial}{\partial t}$ is now understood to be at constant $\psi$, $V' = dV/d\psi$, and Q is the beam heating source. Defining the "adiabate"

$$\alpha \equiv p\left(\frac{d\psi}{dV}\right)^{-\gamma} \tag{8}$$

Eq. (7) can be written as

$$\left.\frac{\partial}{\partial t}\right|_{\psi} \ln \alpha = (\gamma - 1)\frac{Q(\psi)}{p(\psi,t)} \tag{9a}$$

or, if (8) is used

$$\frac{\partial \alpha}{\partial t}\Big|_\psi = (\gamma - 1)QV^{-\gamma} \tag{9b}$$

For convenience, we will choose $\gamma = 2$ rather than 5/3, cf. Grad.[2,3b]

The safety factor profile $q(\psi)$ has been shown[1, 11] to be invariant under a flux conserving pressure rise: One obtains, solely from Faraday's law and axisymmetry,

$$4\pi^2 \frac{\partial q(\psi, t)}{\partial t} = \frac{\partial}{\partial \psi}(V' \langle \underset{\sim}{E} \cdot \underset{\sim}{B} \rangle) = 0 \tag{10}$$

where the right side vanishes on the faster than resistive time scale considered here. Starting from a low pressure equilibrium at $\beta_p = 1$ and a given $q(\psi)$, one obtains from the definition of $q = d\phi/d\psi$ (for circular flux surfaces with enclosed two-dimensional volume $V = \pi r^2 L$, where L is the length of the cylinder)

$$B_z(\psi) = 2\pi L q(\psi)\frac{d\psi}{dV} . \tag{11}$$

Equations (9) and (11) serve to express $p(\psi)$ and $B_z(\psi)$, for a given heating profile $Q(\psi)$, through the two free functions $\alpha(\psi)$ and $q(\psi)$ of the flux conserving problem.

The profiles $\alpha$ and $q(\psi)$ convert the Grad-Shafranov equation, Eq. (3), into a generalized differential equation. The flux-averaged generalized differential equation, Eq. (5), becomes

$$\frac{d}{dV}\left[K(V)\frac{d\psi}{dV}\right] = -4\pi\frac{d\alpha}{d\psi}\left(\frac{d\psi}{dV}\right)^2 - 8\pi\frac{d^2\psi}{dV^2} - 4\pi^2L^2q\frac{dq}{d\psi}\left(\frac{d\psi}{dV}\right)^2 \qquad (12)$$

$$- 4\pi^2L^2q^2\frac{d^2\psi}{dV^2} .$$

For circular flux surfaces, $K(V) = 4\pi V$. Now Eq. (12) can be written in a more convenient form by treating $\alpha$ and $q$ as functions of $V$, i.e.,

$$(V + 2\alpha + \pi L^2q^2)\frac{d^2\psi}{dV^2} + (1 + \frac{d\alpha}{dV} + \pi L^2q\frac{dq}{dV})\frac{d\psi}{dV} = 0 \qquad (13)$$

Thus for $\gamma = 2$, the flux conserving "adiabatic" equilibrium equation has become a linear one, an important fact concerning the otherwise well known bifurcation problem.[12] Note that $V = 0$ is a regular point of this ordinary differential equation. It is convenient to introduce dimensionless quantities (denoted by tildes)

$$\psi = \psi_o\tilde{\psi},\ p = p_o\tilde{p},\ q = q_o\tilde{q},\ V = V_o\tilde{V},\ \alpha = \alpha_o\tilde{\alpha}, \qquad (14)$$

where $\psi_o$ is the poloidal flux in the cylinder of total volume $V_o$. $p_o$ and $q_o$ are respectively the pressure and the safety factor at the magnetic axis $V = 0$. Thus $\alpha_o = \alpha(V = 0)$, $p_o = \alpha_o\psi_o^2V_o^{-2}$, and

$$\tilde{p} = \tilde{\alpha}(\frac{d\tilde{\psi}}{d\tilde{V}})^2 . \qquad (15)$$

where $\alpha$ is now a $O(1)$ quantity, a fact to be used in finite $\beta$ expansion ordering, below. For tokamak plasmas operating below the Kruskal-Shafranov (and ballooning) stability limit, it is also convenient to introduce the small parameter $\varepsilon^2$,

$$\varepsilon^2 \equiv \frac{V_o}{\pi L^2 q_o^2} \simeq \frac{B_\theta^2}{B_z^2} << 1 \ , \tag{16}$$

with

$$\beta_p \equiv \frac{V_o p_o}{\psi_o^2} \simeq \frac{p_o}{B_\theta^2} \tag{17}$$

and

$$\beta \equiv \varepsilon^2 \beta_p . \tag{18}$$

We will consider a sequence of equilibria with $1 \leq \beta_p \leq \varepsilon^{-1}$, where $\varepsilon$ and $\beta$ will be used as the perturbation parameters. Note that $q_o$ and $\psi_o$ are conserved for all neighboring flux conserving equilibria. Equation (13) becomes (we drop the tilde notation)

$$(q^2 + 2\beta\alpha + \varepsilon^2 V)\frac{d^2\psi}{dV^2} + (q\frac{dq}{dV} + \beta\frac{d\alpha}{dV} + \varepsilon^2)\frac{d\psi}{dV} = 0 \ , \tag{19}$$

which allows for the specification of flux conserving boundary conditions

$$\psi = 0 \text{ at } V = 0, \quad \psi = 1 \text{ at } V = 1. \tag{20}$$

When we are discussing the case of adiabatic compression heating, the final plasma volume is $V^* < 1$ so that the flux conserving boundary condition is $\psi = 1$ at $V = V^*$.

Equation (19) has the first integral (with c being a constant of integration)

$$\frac{d\psi}{dV} = \frac{c}{(q^2 + 2\beta\alpha + \varepsilon^2 V)^{1/2}} \exp\left(-\frac{\varepsilon^2}{2}\int^V \frac{dV}{q^2 + 2\beta\alpha + \varepsilon^2 V}\right)$$

$$= \frac{c}{q}\left(1 - \beta\frac{\alpha}{q^2}\right) + O(\beta^2, \varepsilon^2) \ . \tag{21}$$

Thus, $\psi(V)$ can be found if q and $\alpha$ are specified as functions of V. However, it is simpler to solve $V(\psi)$, for a given $q(\psi)$, on specifying $q(\psi)$ and $\alpha(\psi)$:

$$\frac{\partial V(\psi,\ t)}{\partial\psi} = \frac{q(\psi)}{c}\left[1 + \beta\frac{\alpha(\psi,\ t)}{q^2(\psi)}\right] + O(\beta^2, \varepsilon^2). \tag{22}$$

Combined with Eq (9b) for the adiabate we obtain:

$$\frac{\partial\alpha(\psi,\ t)}{\partial t} = Qq^2\left[1 + 2\beta\frac{\alpha(\psi,\ t)}{q^2}\right]/c^2 + O(\beta^2, \varepsilon^2) \tag{23}$$

Together with the flux conserving constraint (10) we will solve this coupled system of first order equations for $V(\psi, t)$ and $\alpha(\psi, t)$ by expanding in the smallness of $\beta$. Once these functions are known the pressure follows from Eq. (9a), the axial magnetic field from Eq. (11), the poloidal magnetic field from (1), and the total axial current and the current density from Ampere's law.

## IV. FLUX CONSERVING EQUILIBRIA

### (A) General Solution

Eq. (23) has the solution

$$\alpha(\psi, t) = \alpha_i(\psi) \exp(th) + \frac{Qq^2}{c^2 h} [\exp(th) - 1] \tag{26a}$$

where $h \equiv 2\beta Q/c^2$, showing that the external power source Q changes the adiabate exponentially. This closed form solution is not analytically useful in Eq. (22), however.

Thus, introducing the $\beta$-expansion

$\alpha = \alpha^{(0)} + \alpha^{(1)}, \quad c^2 = c_o + c_1$

and using the boundary condition

$V(\psi = 1, t) = V^*$

where $V^* = 1$ for constant volume "beam heating" and $V^* < 1$ for adiabatic compression heating, and the initial condition

$$\alpha(\psi, t = 0) = \alpha_i(\psi) \tag{26b}$$

the solution of Eqs. (22), (23) is

$$\alpha^{(0)}(\psi, t) = \alpha_i + (Qq^2/C_o^2)t \tag{27a}$$

$$\alpha^{(1)}(\psi, t) = (2\beta Q/C_o^2)\{ (\alpha_i - \Gamma_1 q^2)t + \tfrac{1}{2}q^2 [(Q/C_0^2) - 2\Gamma_2)]t^2\} \tag{27b}$$

where

$$\Gamma_1 = \int_0^1 d\psi(\alpha_i/q)/ \int_0^1 d\psi q \tag{28a}$$

$$C_o^2\Gamma_2 = \int_0^1 d\psi qQ/ \int_0^1 d\psi q \tag{28b}$$

and

$$V^* C_0 = \int_0^1 d\psi q \tag{29a}$$

$$(C_1/C_o) = \beta(\Gamma_1 + \Gamma_2 t) \tag{29b}$$

Using this solution for $\alpha(\psi, t)$ in Eq. (22) there results to first order in $\beta$

$$\frac{V(\psi, t)}{V^*} = \frac{\int_0^\psi d\psi q + \beta \int_0^\psi d\psi(\alpha^{(0)}/q)}{\int_0^1 d\psi q + \beta \int_0^1 d\psi(\alpha^{(0)}/q)} \tag{30}$$

where $a^{(0)}(\psi, t)$ was given in (27a). From this, one concludes that our perturbation expansion will break down when

$$\beta Q t q / C_o^2 \simeq 1 \tag{31}$$

depending on the heat source $Q(\psi)$. From physical reasons, $\beta$ is bounded from above by the ballooning mode threshold and t by the resistive time scale.

From $\Delta\psi = 4\pi j_z$ one obtains the current density

$$j_z = \frac{d}{dV} V \frac{d\psi}{dV} \tag{32}$$

and using (21) and (27a), one finds

$$j_z = 1/V' + c \frac{V}{V'} \frac{d}{d\psi} [q^{-1} - \beta\alpha^{(0)}/q^3]$$

Thus, a <u>localized</u> heat source $Q \sim \delta(\psi - \psi_1)$, driving $\alpha^{(0)}$ as given in (27a) would lead to a $\delta'(\psi - \psi_1)$ singularity for $j_z$. Because of the resistive diffusion equation $\frac{\partial\psi}{\partial t} = \eta j_z$ an immediate local violation of flux conservation would result, even for vanishingly small resistivity $\eta$.

(B) Stability Constraint on $q(\psi)$

While $Q(\psi)$ and the initial adiabatic $\alpha_i(\psi)$ can be chosen rather arbitrarily the $q(\psi)$ profile should be such that the axial current density Eq. (32) vanishes at the wall where $\psi = 1$, at least to zeroth order in $\beta$. Using (30), (32) becomes

$$q^2(1) = q'(1) \int_0^1 q(\psi)\, d\psi \tag{33}$$

which is satisfied by the class of functions (acceptable from the stability standpoint)

$$q(\psi) = 1 + k\psi^n, \qquad n \geq 3 \tag{34}$$

where $k \geq 0$ describes the magnetohydrodynamically stable tokamak. The result (34) is significant for plasma stability. Since it has been shown that $j_z(\psi = 1) \neq 0$ leads to the "peeling" instability[13] we conclude that appropriate q-profile tailoring of the low beta initial state is desirable to avoid this instability during the flux conserving pressure rise.

## V. HIGH BETA EQUILIBRIUM SOLUTIONS

To further explore the fundamental results Eqs. (29) (27a, b) for $\alpha(\psi, t)$ and Eq. (30) for $V(\psi, t)$ we shall treat the nonadiabatic and adiabatic case separately. However first some general observations about the total plasma current I and $B_z$ are possible for both cases.

From (30) and Ampere's law, one has

$$8\pi^2 I = K \frac{d\psi}{dV} \tag{35a}$$

where the inverse inductance coefficient is determined as

$$K = \langle|\nabla V|^2\rangle = 4\pi V \tag{35b}$$

Thus, using (30)

$$2\pi I = V(\partial V/\partial\psi)^{-1} = q^{-1}(\psi)\int_0^{\psi} d\psi q[1 + \beta F(\psi)] \tag{35c}$$

where F is a well defined function of O(1) to be determined. One concludes that under the constraint of $q(\psi)$ - conservation the pressure driven current rise is only of $O(\beta)$ in cylindrical tokamak geometry, in contrast to the toroidal tokamak where, as was shown in Ref. 5, this current rise is of O(1). This is explained noting that the conservation of azimuthal symmetry in the straight cylinder does not permit large changes of K. In the toroidal plasma, the definition (35b) of K is generalized to $K = \langle|\nabla V|^2/R^2\rangle$ with R being the major toroidal radius. There, $|\nabla V|^2 = V'^2|\nabla\psi|^2$ has been shown[5] to assume a strong poloidal angle dependence thus producing O(1) changes in K and therefore the current.

The normalized axial field $B_z = q/V'$ becomes to first order

$$V^*B_z(\psi, t) = [1 - \beta\alpha^{(0)}/q^2 + \frac{C_1}{C_0}] \int_0^1 d\psi q \quad (36)$$

showing diamagnetic/paramagnetic and compression effects to occur depending on the choice of profiles and V*. This will be discussed in special limits.

(A) Nonadiabatic Heating

By this we understand auxiliary heating proportional to $Q(\psi)$ at constant volume $V^* = 1$ with vanishing initial pressure [and thus $\alpha_i(\psi) = 0$].

From Eqs. (8) and (9b) the pressure evolves according to

$$p/Qt = \alpha/(t\frac{\partial\alpha}{\partial t})$$

where $Q(\psi)$ is the auxiliary power input. With the results (26, 27) for $\alpha$ one obtains for

$$p(\psi, t) \equiv p^{(0)} + p^{(1)} \quad (37a)$$

$$p^{(0)} = Qt \quad (37b)$$

which determines the final value of $\beta$, and

$$p^{(1)}/p^{(0)} = -t\beta Q(\psi)/\left(\int_0^1 d\psi q\right)^2 \quad (37c)$$

Let the profile shape parameter

$$S \equiv \int_0^1 d\psi qQ/\int_0^1 d\psi q \quad (37d)$$

One can discriminate between two fundamental types of heating profiles, namely

core heating: $Q(1) = 0, \quad Q'(\psi) < 0$ (38a)

edge heating: $Q(1) > 0, \quad Q'(\psi) > 0$ (38b)

the latter being due to insufficient beam penetration, for example. From Eqs. (37) and (38) one infers that to leading order the pressure profile follows the heating profile as expected. The $O(\beta)$ pressure contribution is flattening the pressure profile for the core heating case and for the edge heating case.

For $B_z$, one finds from Eq. (36) and (27a)

$$B_z = \left\{ 1 - \beta t[Q(\psi) - S] \left( \int_0^1 d\psi q \right)^{-2} \right\} \int_0^1 d\psi q \tag{39}$$

where, for core heating, one has $Q - S > 0$ in the central region, thus indicating a diamagnetic equilibrium for this case, while for edge heating, where $Q - S < 0$, a paramagnetic equilibrium results. A specific example with $q = 1 + 2\psi^3$, $\alpha_i = 1 - \psi^2$ and $Q = 1 - Q_1\psi$ or $Q = Q_1\psi$ is given in the appendix.

(B) Adiabatic Heating

For this case, $Q(\psi) = 0$ in Eq. (9), and $q(\psi)$ and the initial adiabate $\alpha_i(\psi)$ are given (invariant) profiles. Heating occurs by volume compression:

$V(\psi = 1) = V^* < 1$

In order not to violate the perturbation solution (22) we restrict the compression to $1 - V^* < 1$.

The solution of Eq. (25) is, naturally for this case,

$$\alpha(\psi, t) = \alpha^{(0)} = \alpha_i(\psi)$$

thus immediately determining the solution (30) for $V(\psi)$. From Eq. (8) one then finds for the pressure

$$V^{*2}p = \frac{\alpha_i(\psi)}{q^2(\psi)} \left(\int_0^1 d\psi q\right)^2 \left[1 + 2\beta\left(\Gamma_1 - \frac{\alpha_i}{q^2}\right)\right] \tag{40}$$

where $\Gamma_1$ was defined in Eq. (28a).

The zeroth order pressure profile follows $\alpha_i/q^2$ and, as expected, increases with the square of the adiabatic compression. Typically, the first order pressure correction factor proportional to $(\Gamma_1 - \alpha_i/q^2)$ is an increasing function of $\psi$, thus spreading out the pressure profile.

From Eq. (36) one obtains immediately for the axial magnetic field

$$V^*B_z = \left(\int_0^1 d\psi q\right) \left[1 + \beta\left(-\frac{\alpha_i}{q^2} + \Gamma_1\right)\right] \tag{41}$$

showing the expected behavior of $B_z$ with respect to compression.[14] With increasing compression the factor $\beta(\Gamma_i - \alpha_i/q^2)$ adds more magnetic field to the outer region in $\psi$ and depresses $B_z$ in the center. This leads to a jump of $B_z$ at the plasma edge which can be relaxed effectively by a small expansion, as mentioned before.[7] Eq. (41) displays the effect of the initial q- and $\alpha_i$ profiles on this jump.

Finally, from Eq. (35) the plasma current is given by

$$2\pi I(\psi) = q^{-1} \int_0^{\psi} d\psi' q(\psi') \cdot \left\{1 + \beta \frac{\int_0^{\psi} d\psi' \alpha_i/q}{\int_0^{\psi} d\psi' q} - \beta\, \alpha_i/q^2\right\} \tag{42}$$

For constant or decreasing $\alpha_i(\psi)$ and the class of profiles (34) for $q(\psi)$ the factor in brackets increases with radius since

$$q^2 \int_0^{\psi} d\psi/q - \int_0^{\psi} d\psi q > 0, \ 0 \leq \psi \leq 1 \tag{43}$$

A specific example with $q = 1 + 2\psi^3$, $\alpha_i = 1 - \psi^2$ is presented in the appendix. Overall, there is a pressure (i.e., compression) driven current increase of $O(\beta)$, similar to that discussed for the beam heating case. The value of $\beta$ in Eq. (41) is determined by the amount of compression, see Eq. (40).

In Figs. 1-4 we explicitly show the $p(\psi)$ and $B_z(\psi)$ profiles before and after heating using the specific example given in the appendix.

VI. SUMMARY

Analytic investigation of the flux conserving equilibrium evolution with auxiliary or compression heating provides a number of insights useful for numerical treatments in more realistic geometries. In view of these computer results for special cases[8,9] the results of this paper are predominantly expressed in terms of the general profile functions $q(\psi)$, the initial adiabate $\alpha_i(\psi)$, the external heat source $Q(\psi)$ and $V^*$, the variable volume of the plasma column. The appendix contains one special (though rather representative) case for $q(\psi)$ and $\alpha_i(\psi)$, however.

Our main findings are as follows. 1) When $B_z B_z'$ is expressed through q (flux conservation) and p' through the entropy equation, the equilibrium equation for $V(\psi)$ becomes linear, if the adiabatic index $\gamma = 5/3$ is taken to be 2, for convenience. Since the difference is not large this indicates a relatively weak nonlinearity of the actual adiabatic problem. 2) The coupling between the equilibrium equation and the entropy equation is of $O(\beta) << 1$. The zeroth order equilibrium is determined by the q-profile and initial adiabate. Their choice can be crucial for stability. Particularly, only if $q(\psi) = 1 + k\psi^n$ with $n \geq 3$ will the axial current density vanish at the plasma edge. 3) The conservation of azimuthal symmetry in the circular cylindrical geometry considered here prevents the flux conserving pressure driven axial current rise to reach a O(1) magnitude, as is the case for the toroidal geometry. 4) A sharply localized beam heating source destroys local flux conservation according to $\frac{\partial\psi}{\partial t} \propto \eta Q'(\psi)$. 5) For the beam heating case, the broadening of the pressure profile is determined by the convolution $\int d\psi q Q$. For a "core heating" profile $Q(\psi)$ a diamagnetic $B_z$ profile results, and for an "edge heating" Q, a paramagnetic one. 6) For the adiabatic compression case, the axial magnetic field is displaced to the plasma edge. The ensuing discontinuity $\Delta B_z$ can be controlled not only by (the well known) volume expansion but also by a proper choice of profiles of the initial equilibrium.

ACKNOWLEDGMENTS

This research was sponsored by the Office of Fusion Energy (ETM), U.S. Department of Energy under contract W-7405-eng-26 with the Union Carbide Corporation and under contract No. ERDA (40-1)-5260 at the College of William and Mary. We thank Drs. Nelson and Bateman for critical discussions.

APPENDIX

To quantify the general conclusions derived in Sec. V we specify the input profiles in the following representative way:

$$\alpha_i = 1 - \psi^2, \qquad q = 1 + 2\psi^3, \tag{A.1}$$

and

$$Q = \begin{cases} 1 - Q_1\psi \;\ldots\; \text{for core heating} \\ Q_1\psi \;\ldots\; \text{for edge heating} \end{cases} \tag{A.2}$$

Then one obtains the following results:

$$\int_0^1 d\psi q = \frac{3}{2} \tag{A.3}$$

$$\int_0^\psi d\psi \frac{\alpha_i}{q} = \frac{A}{6}\ln\frac{(A+\psi)^2}{A^2 - A\psi + \psi^2} + \frac{A}{\sqrt{3}}\arctan\frac{\sqrt{3}\,\psi}{2A - \psi} - \frac{1}{6}\ln(1 + 2\psi^3) \tag{A.4}$$

where $A = 2^{-1/3} = .7937$, so that

$\int_0^1 d\psi \frac{\alpha_i}{q} = .5653$, and from (27a), there results

$$\alpha^{(0)} = 1 - \psi^2 + \frac{4}{9} Qt(1 + 2\psi^3)^2 \tag{A.5}$$

Further, one obtains

$$\frac{3}{2} S = \int_0^1 d\psi qQ = \begin{cases} \frac{3}{2} - \frac{9}{10} Q_1 \;\ldots\; \text{core heating} \\ \frac{9}{10} Q_1 \;\ldots\; \text{edge heating} \end{cases} \tag{A.6a}$$

so that

$$S - Q = \begin{cases} Q_1 \left(- \frac{3}{5} + \psi\right) \ldots \text{ core heating} \\ Q_1 \left(\frac{3}{5} - \psi\right) \ldots \text{ edge heating} \end{cases} \qquad (A.6b)$$

which confirms the claims made for Eqs. (38) and (39) for the beam heating case. For the case of adiabatic heating, Eq. (40) requires

$$\Gamma_1 - \frac{\alpha_i}{\varepsilon^2} = .3768 - \frac{1 - \psi^2}{(1 + 2\psi^3)^2}$$

indeed a monotonically increasing function. Finally, Eqs. (42, 43) can be verified easily using the result Eq. (A.4).

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# FIGURE CAPTIONS

Figure 1 The safety factor profile $q(\psi)$, and the pressure profile $p(\psi)$ for the case of adiabatic heating.

Figure 2 The toroidal magnetic field $B_z(\psi)$ for the adiabatic heating case. It is constant for low $\beta$ while for high $\beta$ equilibria it becomes diamagnetic. Since toroidal flux is conserved there must be an outward shift of the flux surfaces, as is evident from this diagram.

Figure 3 The pressure profile $p(\psi)$ for the case of nonadiabatic heating: (a) core heating, (b) edge heating.

Figure 4 The toroidal magnetic field $B_z(\psi)$ corresponding to Fig. 3. For low $\beta$ equilibria $B_z$ is a constant.

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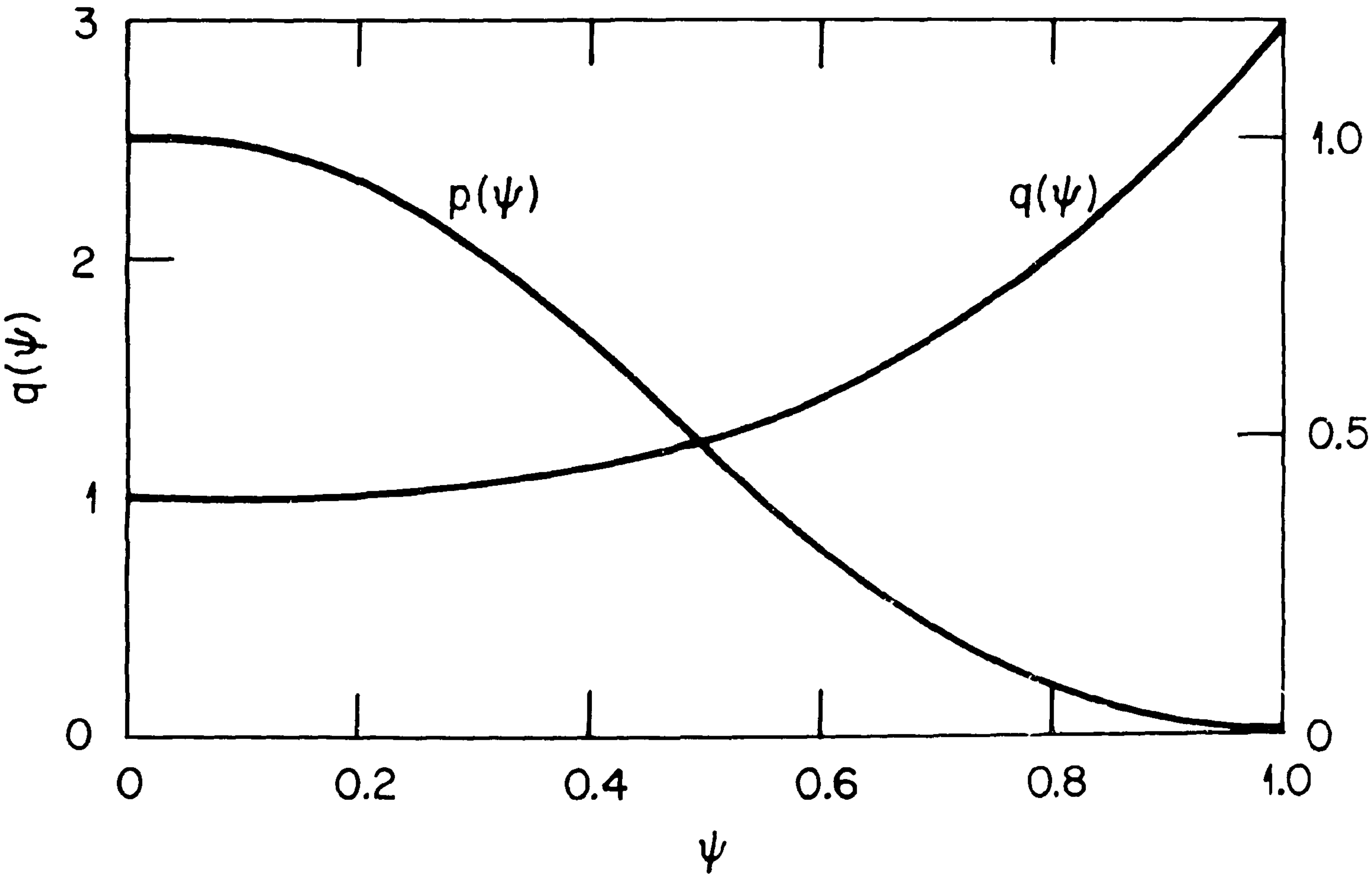


Fig. 1.

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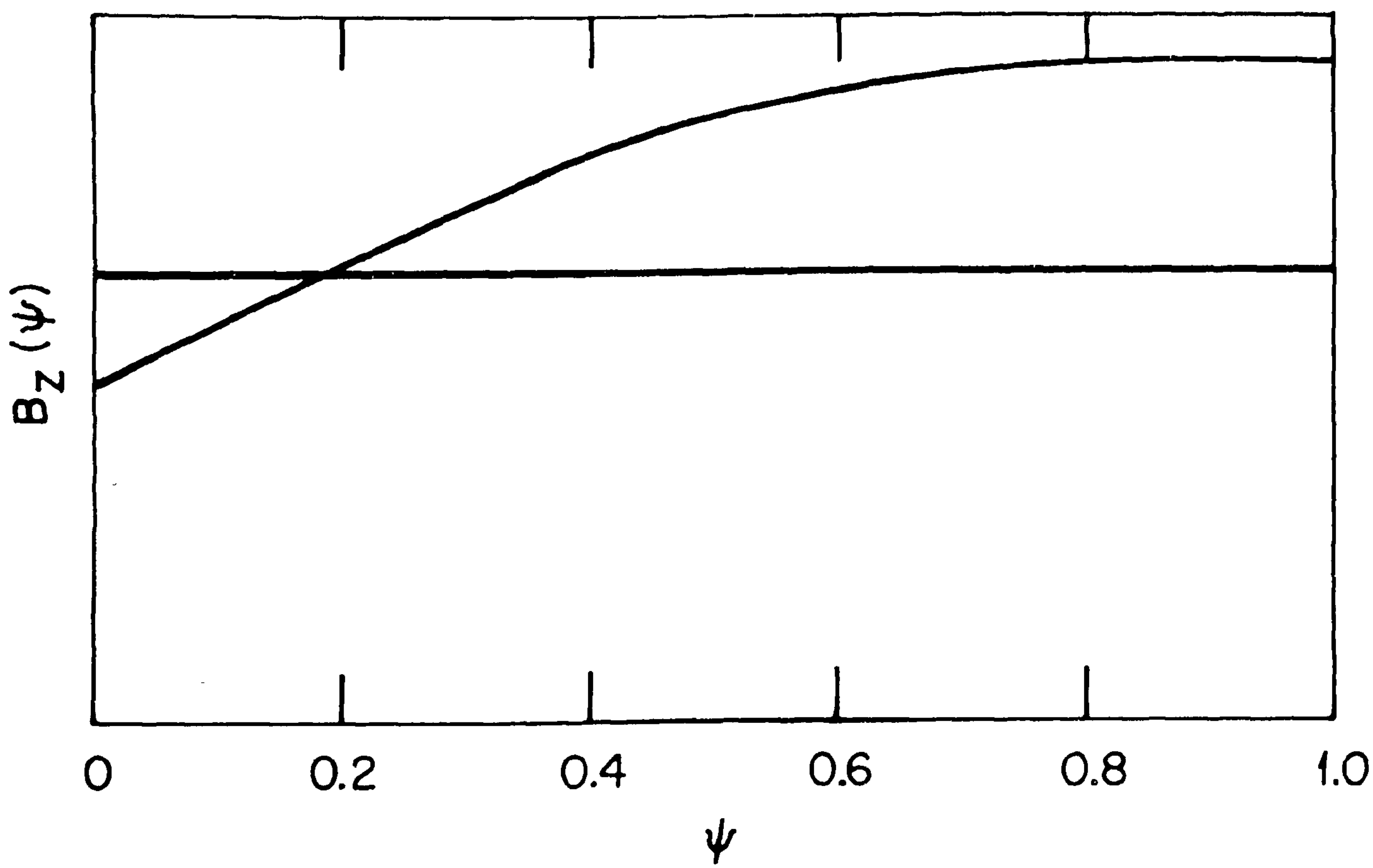


Fig. 2.

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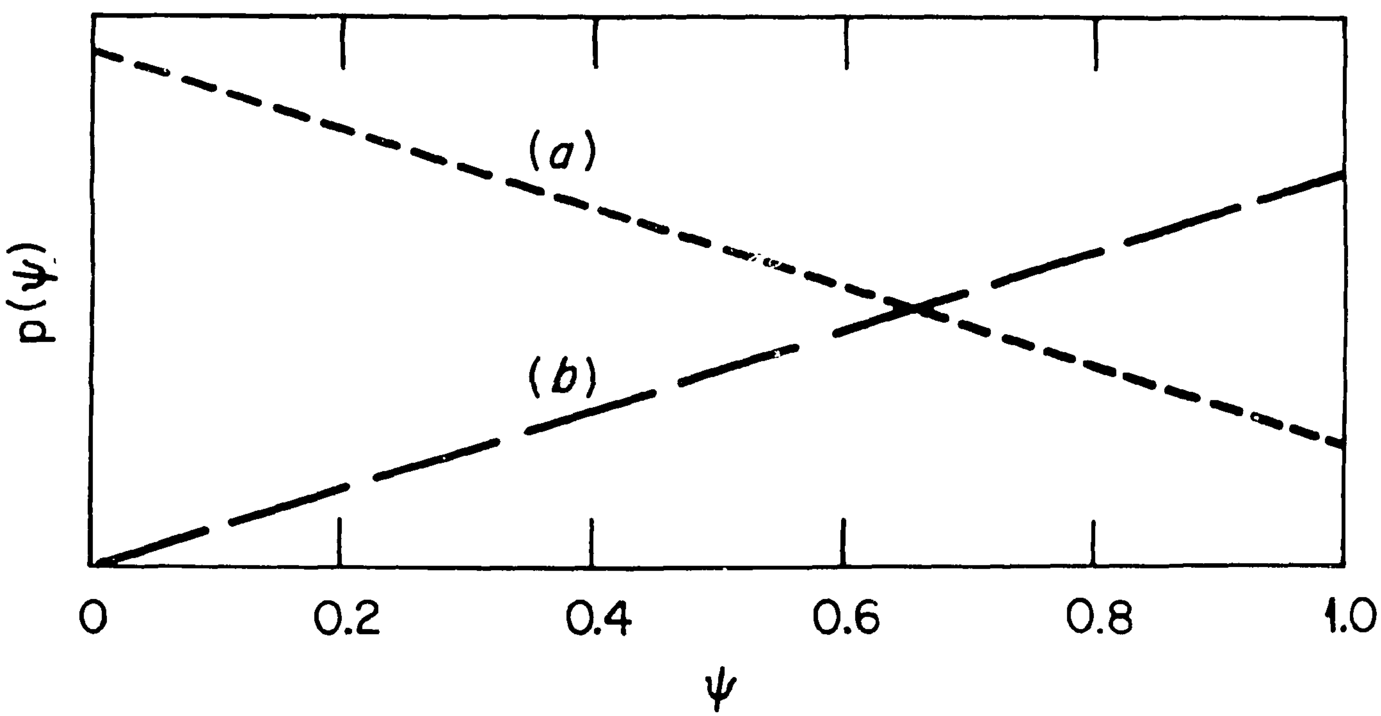


Fig. 3.

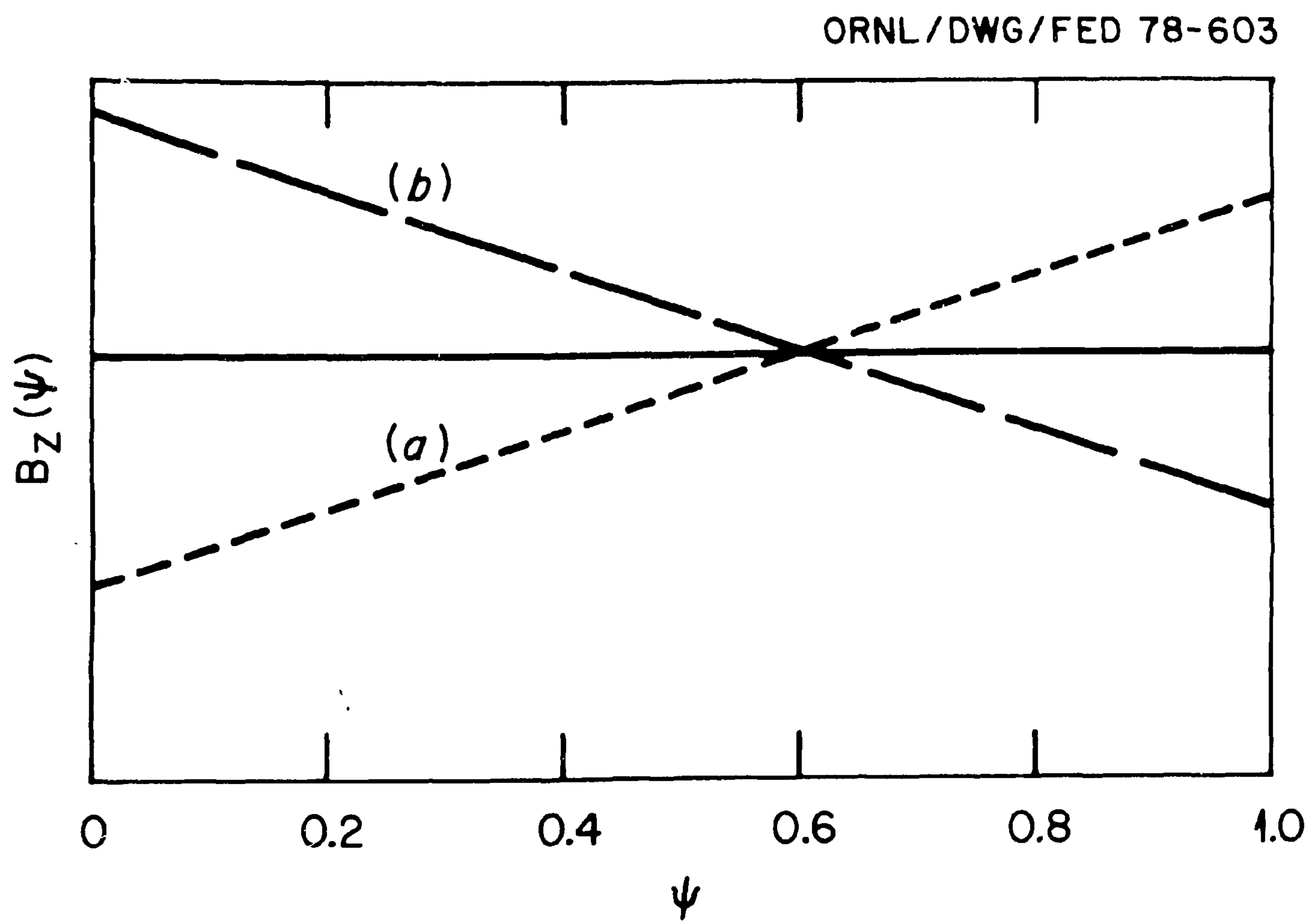


Fig. 4.

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