THEORY OF DRIFT AND TRAPPED-ELECTRON INSTABILITIES

PLASMA PHYSICS LABORATORY

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Theory of Drift and Trapped-Electron Instabilities


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ABSTRACT

This paper deals with the theoretical investigation of low-frequency drift and trapped-particle instabilities in systems with magnetic shear, by analytic and numerical procedures and by computer simulations. In particular, results are presented for calculations which demonstrate: (1) the stability of both collisionless and dissipative drift eigenmodes at long radial wavelengths \( k R < 1 \) in a sheared slab (one-dimensional) geometry; (2) the presence and structure of drift and trapped-electron eigenmodes in an axisymmetric toroidal (two-dimensional) geometry; and (3) the nonlinear evolution and resultant anomalous transport from trapped-electron instabilities.

Drift Instabilities in a Sheared-Slab Geometry

The influence of radially nonlocal effects, such as magnetic shear, on the presence of absolutely unstable drift modes in a confined plasma is a fundamental problem that has been actively investigated over the years. Most of the theoretical studies in this area have been carried out for a slab geometry with shear and have focused in particular on drift waves with long radial wavelengths. This approach leads to a one-dimensional (radial) differential equation of the form

\[
\left[ \frac{d^2}{dx^2} + Q(x, \omega) \right] \phi(x) = 0
\]

(1)

with

\[
Q(x, \omega) = \lambda + x^2/4 + (\mu \delta / |x|)^2 (\delta / |x|)
\]

(2)

for the collisionless ("universal") case, and with

\[
Q(x, \omega) = \lambda + x^2/4 + (\mu \delta / |x|)^2 (\delta / |x|) \]
for the collisional (dissipative) case. In Eq. (2), \( \lambda \) is a dimensionless variable [1], \( z \) is the familiar plasma dispersion function,

\[
\lambda = \frac{1}{2\pi} \left( 1 - \frac{k^2}{\omega^2} \right) \left( \frac{\omega}{c_s} \right) c_s \cdot \left( \frac{1}{\Omega_i} \right) \frac{1}{2},
\]

where \( c_s \) and \( \Omega_i \) are respectively the shear length and the equilibrium density scale length, and \( \omega \) is the collision frequency. In Eq. (3), \( x \) is another dimensionless variable [2],

\[
x^2 = \frac{1}{\nu_e c_s^2} k_{\parallel}^2 v_e^2 c_s^2, \quad \lambda \equiv \frac{1}{\nu_e c_s^2} k_{\parallel}^2 (\omega / c_s)^2,
\]

Recent numerical solutions of Eqs. (1) - (3) have indicated that neither collisionless [1] nor collisional [2] electrostatic drift wave eigenmodes with long radial wavelengths \( k_{\parallel} < 1 \) are ever unstable in a sheared slab geometry. This, of course, is in contradiction to previous analytical calculations [3,4] which predicted the presence of unstable normal modes if the shear is sufficiently weak. For the collisionless problem the error in earlier work [3] can be traced to the approximation of the complete electron Z-function term in Eq. (2) by just its resonant part and/or to the use of improper perturbation methods. In the case of collisional drift waves, previous calculations have generally been qualitative or heuristic in character [4] and have not rigorously investigated the normal mode problem. In the present work, detailed analytical studies are presented which support the conclusions of the recent numerical calculations and provide a clearer understanding of those results.

To solve the collisionless eigenmode problem, governed by Eqs. (1) and (2), the basic approach here is to apply the WKBJ method [5]. The eigenvalue is then determined by

\[
\left| \frac{dt}{\tau} \right|^{1/2} = n + 1/2
\]

where \( \tau \) is given in Eq. (1), \( \tau = x + iy \). \( \omega \) is the radial mode number, and the integration is to be carried out between the appropriate turning points. Before proceeding, it should be noted that the analytic continuation of Eq. (1) into the complex plane, \( \tau = x + iy \), is determined by

\[
|x| + |y| \frac{1}{2}
\]

The Riemann structure then consists of two cuts which originate at \( \tau = 0 \) and which can be taken along the positive and negative imaginary axes. The two sheets here will be referred to as the physical
and the non-physical sheet for $(L/L_n)^{1/2} = 0$. In

continuing the procedure it is very important to recognize that there
are two pairs of turning points which are critical to the eigencmode

and located at \( t = \frac{1}{2} \) and another pair induced by the electron dynamics (via

for the equation) and located at \( t = \frac{1}{2} \leq \frac{1}{2} \).

It should be remembered that the lowest shift of the non-encyency

required for instability in the local theory is governed by

possible \( k_x^2 \) in the non-linear coordinate, a turning

points located on the physical sheet while the other located on the

linear sheet. Applying the usual arguments we can determine

the value \( \tau \) then determines the value of \( t \) as the appropriate turning

points to be used in Eq. (4). It should be remembered that

to be damped for $k_x^2 < k_{\text{crit}}^2$ and when the value of the

value of the eigenmode parameter \( \tau \) is further increased ($k_x^2 > k_{\text{crit}}^2$), the pairs of turning

points remain on the imaginary axis with the amplitude of the $y$-function being very

small. As \( \tau \) was increased, the turning point was damped for $k_x^2 < k_{\text{crit}}^2$ and

and the conclusion that the eigencmodes are marginally stable for $k_x^2 > k_{\text{crit}}^2$ and

and $(L_s/L_n)_{\text{c}} < (L_s/L_n)$ confirm the numerical results [1], [8]. It should

also be noted here that the location of the turning points on the $y$-axis

(which leads to marginal stability) is a direct result of the use of the

full $Z$-function representation of the electron dynamics. For the case of strong shear, $(L_s/L_n) < (L_s/L_n)_{\text{c}}$, the usual conclusion that the
eigencmodes are damped for all values of $k_x^2$ is recovered. Here the pairs of turning points never coalesce and $t = \frac{1}{2}$ is again the appropriate choice to be used in Eq. (4). Calculating $(L_s/L_n)$ as a function of the

mass ratio leads to the scaling, $(L_s/L_n)_{\text{c}} = \frac{3N_y}{M_y}\epsilon^{1/4}$.

To solve the collisional eigencmode problem, governed by Eq.'s (1)

and (3), it is convenient to employ a matched asymptotic treatment.

provided $\frac{2}{\kappa} 2(\nu_{\text{cr}} L_s)(\omega_{\text{cr}} M_y L_n) \ll 1$. In the outer region
\(|x|_{\nu}^{2} \gg |x|_{K}^{2}\), \(\xi = \Phi_{0}(x)\) can be expressed in terms of Fermi's Confluent Hypergeometric Function. For the inner region \(|x| \ll |x|_{K}^{1/2}\), \(\xi = \Phi_{1}(x)\) can be expressed in terms of the Associated Legendre functions. Matching \(\Phi_{1}\) and \(\Phi_{0}\) then yields the following eigenvalue condition:

\[
\frac{c v(t)}{c v(t+a)} \begin{bmatrix}
1 - i/2 \\
0
\end{bmatrix} = \begin{bmatrix}
1 + b_1/2 & 0 \\
0 & 1 + b_1/2
\end{bmatrix} \begin{bmatrix}
1 - i/2 \\
0
\end{bmatrix}
\]

where

\[
\begin{pmatrix}
b_1 \\ a_1
\end{pmatrix} \begin{pmatrix}
1 + i/2 & 0 \\
0 & 1 + i/2
\end{pmatrix} \begin{pmatrix}
a \cos(\pi/2) & 0 \\
0 & a \sin(\pi/2)
\end{pmatrix}
\]

\(a = \frac{b-1}{2}, b = i + a, a = (1 + 4b)^{1/2}, z = i(x + 1/2),\)

\(L = i(x_K)^2, \ \frac{z}{L} = \frac{1}{2} - i(x_K/x_S^2), \ \text{and} \ \eta = i(x/2)(\frac{z}{L} + k_{y}^2/4).\)

This equation reduces to considerably simpler forms in both the weakly collisional \((\eta < 1)\) and strongly collisional \((\eta > 1)\) limits. Solutions to the corresponding dispersion relations indicate that there are no absolutely unstable eigenmodes. Collisionality, in fact, is found to further enhance the shear damping. Application of the WKBJ method to this problem has provided solutions in the other regimes, i.e., \(\xi |_{\nu}^{1/2} = 1\). In addition, perturbative methods (valid in the weakly collisional limit) have been used to study the influence of ambient temperature gradients and electron temperature fluctuations on these modes. All of these calculations indicate that the basic stability properties described remain unchanged. Finally, it should be noted that since the absence of absolutely unstable eigenmodes in the SLOB model is due to the presence of shear damping, any mechanism which suppresses this effect (e.g., sharp variations in the equilibrium density profile [7] or strong toroidal coupling [8]) should reintroduce these instabilities. In fact, unstable drift eigenmodes with growth rates increasing with collisionality have been recovered for cases where the shear damping is nullified [2].

At shorter radial wavelengths \((k_p \rho_i \geq 1\) with \(\rho_i\) being the ion gyroradius), the usual second-order differential eigenmode equation (Eq. (1)) ceases to be valid. To treat this more general problem, the perturbed ion density response can be expressed as:
In order to properly assess the danger of low-frequency drift-type instabilities for tokamak confinement, it is essential to take into account the fully two-dimensional nature of the axisymmetric toroidal geometry. At the simplest level, toroidal effects can be introduced into the two-dimensional analysis by considering the long W-drift terms. In earlier work [8] it was pointed out that in certain limits, these toroidal terms can produce a ballooning of the mode structure along the field line and cause the otherwise sheared-stabilized universal eigenmodes to become unstable. However, the complete electron response (full L-function) was not taken into account. Including this term and then following the procedure introduced by Taylor [8], the eigenmode equation in the long radial wavelength limit can again be cast in the form of Eq. (1), i.e., \[ \mathbf{Q}(\nu, \omega) \delta = 0 \]
\( \Omega(y, \omega) = \left[ \frac{y}{q} + \frac{q^2}{|y|} \right] \left[ \frac{1}{|y|} \right] / n, \)

where \( A = 1 - \left( \frac{L}{R_0} \right) \left( \frac{v_\|}{v_n} \right)^2 \), \( y \) is a dimensionless variable,

\[ \lambda = \left( \frac{1}{\alpha} - 1 - \frac{\kappa}{v_n} \right) \left( \frac{L}{R_0} \right) \left( \frac{v_\|}{v_n} \right), \]

\( \alpha \) and \( \kappa \) are defined following Eqs. (1) - (3), \( \alpha = L/R, R \) is the major radius, \( \kappa = \alpha q / \delta, \delta = \mu_{J}/\mu_{T}. \)

and \( q \) is the usual safety factor. It should be noted here that as in Taylor's calculation [9], the strong toroidal coupling was made to arrive at the equation, which primarily represents a structure along the field line. Solutions obtained by this method of the KKB procedure [5] indicate that the proper inclusion of the full electron response does not alter the basic conclusions of the earlier work [8]. Specifically, provided the factor \( q' \) in Eq. (8) is negative

\( \frac{1}{2} \leq q' \leq 1 \), the governing potential \( \Omega(y, \omega) \) in the differential equation is the form of a "well" instead of an "anti-well." Hence, the toroidal terms due to the ion VB drifts can lead to the presence of unstable, non-propagating drift eigenmodes which are insensitive to shear.

It is important to recall that in addition to their well-known destabilizing influence, the presence of magnetically trapped particles can also contribute to strong toroidal coupling [10]. The influence of magnetic shear on unstable eigenmodes in a toroidal geometry should then be accordingly affected. These effects, together with all VB-drifts, have been included in a comprehensive numerical analysis of the basic eigenmode equation which is valid for arbitrary \( k_p \). The general approach here involves converting this integral equation into a matrix equation by expanding the potential in complete sets of appropriate radial and poloidal basis functions; i.e.,

\[ \psi(S, 0) = \exp[-i\alpha t] \exp(i(m\theta - n\phi)] \]

with

\[ \psi(S, 0) = \sum_{j=0}^{\infty} \exp(i\alpha t) \sum_{\ell=0}^{\infty} \hat{\delta}_{j,\ell} h_{\ell}(S + j) \]  

where \( S = m - nq(r) \) and the form of the Hermite functions, \( h_{\ell} \), is given in Eq. (7). Unlike earlier calculations of this type [10], the slow radial variation in equilibrium quantities and their gradients is now taken into account. This new feature enables us to realistically study the large-scale radial localization of the eigenmodes.

The stabilizing effect of shear on the drift and trapped-electron eigenmodes was investigated in different collisionality regimes. On Fig. 1, growth rates are plotted versus the dimensionless shear parameter, \( q'r/q \), which generally falls between 0 and 2 in tokamaks. For the
In studying the global structure of drift and trapped electron eigenmodes, it is necessary to allow for the variation in phase with the variation in the equilibrium profiles of density, temperature, and current (and hence q). As shown on Fig. 2, the poloidal mode structure is characterized by a "ballooning" around the magnetic axis and the radial structure by a broad localization centered on the equilibrium variations. This case was run for idealized parameters that correspond to those in a particle simulation study, which will be described later in this paper. For the interpretation given here the toroidal mode number n = 2, the poloidal harmonics, and the neutral surfaces in the plasma and k = 0.6, 0.6, 0.6 in the neutral surface. However, for realistic plasma configurations, for example, the IHL and AUGATOR tokamaks, $A_{\parallel}$ appear to be present in the plasma, so that the toroidal mode numbers of interest are not integer. Since the computations, which accordingly involve retaining a much larger number of poloidal harmonics, are considerably more time-consuming than this idealized case, these calculations are currently in progress.

As a final point it should be noted that since $A_{\parallel}$ (plasma pressure-magnetic pressure) in tokamak systems generally falls in the range $m/\bar{n_p} < 0.1$, it is important to determine the influence of electromagnetic effects on the electrostatic drift-type modes. This requires one to deal with the perturbed parallel vector potential, $A_{\parallel}$, as well as $E_\parallel$ and involves solving the parallel current equation together with the quasineutrality equation. These features have not been included in the numerical code to study finite-$B$ effects on drift and trapped electron eigenmodes in a toroidal geometry. Preliminary results indicate that in the absence of equilibrium variations, the finite-$B$ effects tend to be very slightly destabilizing up to $B_n$ (poloidal beta) of order unity. For $B_n > 1$, a stabilizing trend due to the finite-$B$ effects appears. In other work in progress, the strong coupling, Taylor-type analytic calculation for ballooning drift instabilities has been generalized to the finite-$B$ case.
Particle Simulation of Nonlinear Evolution of Trapped-Electron Modes.

Since the aforementioned low-frequency toroidal modes are likely to be present in toroidal systems, it is expected to study their nonlinear evolution together with the associated turbulence and anomalous transport. A three-dimensional (3D) electron-particle code [11] has been developed to investigate this problem. In this toroidal model, the external magnetic field is in uniform form, \( B = B(t, \theta) \), with \( B_0 \) and \( B_0 \) being the density profiles of the form, \( n(r) = n_0 \exp(-r^2/\sigma^2) \), with \( \sigma \) being an adjustable parameter. The temperature is taken to be spatially uniform for the ions but is allowed to vary as \( T_0 \exp(-r^2/\sigma^2) \) for the electrons. Here, \( \sigma \) is an adjustable parameter, and \( \Delta n = 2/\sigma \) = constant everywhere. The aspect ratio is chosen to be \( R/a = 3 \), and the current profile, which is assumed parabolic, leads to a q-profile varying as \( q = 1 + 2.5 r^2/\sigma^2 \). Other simulation parameters, which specify the average density and temperature are: \( \sigma_c/\sigma = 5 \), \( T_e/T_i = 4 \), and \( \omega_{pi}/c = 3 \), with \( \omega_{pi} \) being the electron plasma frequency at the magnetic axis, \( \sigma \) designates spatial average, \( \lambda_e \) is the electron Debye length, \( \Delta \) is the spatial grid size, and \( \omega_{pe} \) is the electron plasma frequency. To account for the electron-ion pitch-angle scattering essential to a proper description of the trapped-electron modes, a Monte-Carlo collision model with \( v_{ei} = v_c \) is adopted. Since \( v_{ei} \) varies roughly from 0.5 to 10 across the plasma for the parameters chosen, the conditions here are appropriate for simulating phenomena in the banana and plateau regimes.

Results from a typical simulation [12] (with \( n_e = 1 \)) are displayed in Fig. 3. On this figure are shown the time evolution of (a) the electric field energy, \( U(m,n) \), for different poloidal \( (m) \) harmonics with toroidal mode number \( n = 2 \); (b) the radial mode structure; and (c) the poloidal mode structure. The growth rate, eigenfrequency, and mode structure, measured during the linear phase of the simulation, are in reasonable agreement with results from the 5D eigenmode calculations described earlier in this paper. Since the aspect ratio is small \( (R/a = 3) \), the resultant strong toroidal coupling for the various \( m \) harmonics causes them to grow at about the same growth rate. This is
turbulence. The drift of the parallel wave and the drift of the perpendicular wave (\( m \times \mathbf{B}_\perp \times \mathbf{v}_{\perp} \)) is due in part to the electron gyro-motion. The electron density distribution to an extent is also the cause of the growth of these waves.

The growth of these instabilities, the coupling between the parallel and perpendicular motion has not yet been observed.
Regarding another characteristic of trapped-electron modes, it should be noted that, in accordance with predictions from linear theory [10], the instability was found to be weaker when the initial electron temperature profile $T_e$ was taken to be zero or negative. For $T_e = 0$, the observed growth rates were roughly a factor of 3 smaller than the case for $T_e = 1$. For $T_e = -1$, no instabilities were detected. Finally, in order to study the influence of electromagnetic effects, a 2D sheared slab model was developed to simulate sheared Alfvén waves as well as finite- $eta$ tilted drift waves.

References:

**Figures**

**Fig. 1.** Growth rates (normalized to the thermal electron rate) versus the dimensionless shear parameter, $a/\nu$, for drift and trapped-electron eigenmodes. (PRL 78, 2213)

**Fig. 2.** Absolute magnitude of the perturbed electrostatic potential versus $a$ for the trapped-electron mode plotted as a function of radius and $a/\nu$. Solid line with $a = 0$ at the magnetic field line. (PRL 78, 2213)

**Fig. 3.** Simulation results for trapped-electron instability displaying (a) the evolution of the electric field energy for different $a$, and (b) the radial mode structure at $t/a = 2$. (PRL 78, 2213)

**Fig. 4.** (a) Time evolution of electric field energy recovered over time for initial angle, as a function of $a$; (b) trapped-electron simulations, (a) $a = 0$, and (b) $a = 0.27$. (PRL 78, 2213)
Fig. 1. Growth rates (normalized to the thermal electron bounce frequency) versus the dimensionless shear parameter, $q'r/q$, for drift and trapped-electron eigenmodes.  

(PPL 782213)
Fig. 2. Absolute magnitude of the perturbed electrostatic potential eigenfunction for the trapped-electron mode plotted as a function of radius and poloidal angle (with $\theta = 0$ at the magnetic field minimum). (PPL 77:7232)
Fig. 3. Simulation results for trapped-electron instability displaying temporal evolution of (a) electric field energy for different m harmonics with n = 2; (b) radial mode structure for (m, n) = (4, 2) and (3, 2); and (c) poloidal mode structure at r/a = 0.47 and n = 2. (PPL 78:149)
Fig. 4. (a) Time evolution of electric field energy (averaged over radius and toroidal angle) as a function of $n$; (b) frequency spectrum for $(m, n) = (4, 2)$ and $(6, 2)$ at $r/a = 0.47$. (PPL 782214)

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