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Using Field Theory in Hadron Physics*†

HENRY D.I. ABARBANEL
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

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†I would like to dedicate these lectures to the memory of Benjamin W. Lee: colleague, teacher, friend.

We would like to dedicate these lectures to the memory of Benjamin W. Lee, friend and colleague of us all. The news of Ben's death arrived shortly before this Topical Seminar began. Ben contributed key ideas to every subject touched on by every speaker. His contributions to the further development and understanding of these exciting issues can never be replaced. We may hope, however, he would have found the substitutes we must create as lively and stimulating as we hope they shall be.

I. INTRODUCTION

This decade has seen a marvelous return to quantum field theory among theorists in high energy physics. The compelling beauty of non-Abelian gauge theories, the striking empirical support for spontaneous symmetry breaking in weak interaction theories, the deep connection between asymptotic freedom and the approximate scaling in inelastic lepton scattering on hadrons, the entrancing suggestion of quark confinement–all this and more has drawn our focus once again on quantum field theories. We have learned to view field theory as providing us with fundamental degrees of freedom rather than thinking that each new particle or resonance as requiring a new field for its description. Indeed the idea that some small set of degrees of freedom (quarks and gluons) provides the basis for all observed mesons and baryons seems to be a concrete realization of the ideas of "nuclear democracy" advocates of the last decade. [Increasing numbers of them have been seen with path integrals and Lagrangians lately.] It makes explicit the concept that all hadrons are composites; not of each other, though through unitarity all the hadrons can become the other hadrons within the restrictions of conservation of charge, baryon number, isospin, etc. It is a deeper way: they are all composites of quarks and gluons.

These lectures have no pretentions to cover all possible topics in the connection of field theory and hadron physics. Rather the goal is much more modest: I hope to touch on a number of tantalizing questions which will serve to some extent as an introduction for the student as well as the research person curious for more than a peek. Several "old" topics will be treated–the renormalization group and the infrared and ultraviolet limits of field theory, choosing Quantum Chromodynamics (QCD) from among all theories, various thoughts on spontaneous mass generation. Some newer ones are discussed here too–ideas on color confinement, instantons and the vacuum state in QCD, and related topics.

As general background material I recommend the article on "Gauge Theories" by E.S. Abers and B.W. Lee, Physics Reports 9C, 1 (1973); the lectures by S. Coleman at the 1975 Erice Summer School; the book by J.C. Taylor, Gauge Theories of Weak Interactions (Cambridge U. Press, 1976); and the review article by R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

II. THE RENORMALIZATION GROUP AND SOME CONSEQUENCES

Quantum field theories of relevance to particle physics all have divergences when one calculates in perturbation theory about the free theory characterized by propagators

$$(\text{Spin Numerator})/(m^2 - p^2 - i\epsilon) \quad . \tag{1}$$

Such theories are not defined by the classical Lagrange density; one has to give a prescription for making the theory finite in every order of perturbation theory before a calculational procedure of recognizable validity emerges. The generally accepted manner for doing this is to define the theory by giving the value of a few basic quantities (mass, coupling constant, ...) at some point in momentum space. So one takes the original theory defined by the classical Lagrangian and renormalizes, so the divergences are absorbed in scales of wave functions (or field operators) and other physically harmless locations. Since there is an enormous arbitrariness in how, precisely, one renormalizes, we can anticipate an invariance of physical quantities on changing the point in momentum space where that renormalization is done. The expression of that invariance is the renormalization group. The behavior of classes of quantum field theories under this group allows one to select those with controllable infrared or ultraviolet behavior and thus on the basis of the behavior of experiments which probe long or short wavelength phenomena to choose acceptable field theories.

To illustrate this in action let's look at a scalar field in D dimensions of space-time with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_0(x))^2 - \lambda_0(\phi_0(x))^N \quad . \tag{2}$$

When λ_0 *is dimensionless, namely when* $N = 2D/(D-2)$, *all the divergences of* quantities expanded in a perturbation series in λ_0 are logarithmic and the theory can be renormalized, i.e. made finite, by redefining the field

$$\phi(x) = Z^{-\frac{1}{2}}\phi_0(x) \tag{3}$$

and coupling

$$\lambda = Z_\lambda Z^{N/2}\lambda_0 \quad , \tag{4}$$

where the dimensionless (infinite) factors Z and Z_λ are constructed so all Green functions in the theory are finite. These renormalization factors are defined by giving the value of certain Green functions at some point $p^2 = -\mu^2$, $\mu^2 > 0$, in momentum space. These Green functions are given by

$$G_0^{(n)}(p_1, \dots p_n, \lambda_0)\delta^D(\sum_{j=1}^n p_j) =$$

$$\int dx_1 \dots dx_n e^{-i\sum_{j=1}^n p_j \cdot x_j} \langle 0 | T(\phi_0(x_1) \dots \phi_0(x_n)) | 0 \rangle \quad . \tag{5}$$

To lowest order in λ_0 we have for $G^{(2)}$ and $G^{(N)}$

$$G^{(2)}(p^2, \lambda_0) = i/(p^2 + i\epsilon) \quad , \tag{6}$$

and

$$G^{(N)}(p_j, \lambda_0) = -i\,\lambda_0/(2\pi)^{D/2\,(N-2)} \quad . \tag{7}$$

We define renormalized Green functions by

$$G^{(n)}(p_1, \ldots p_n, \lambda, \mu) = Z^{n/2} G_0{}^{(n)}(p_1, \ldots p_n, \lambda_0) \quad , \tag{8}$$

and require (this is the renormalization)

$$\frac{\partial}{\partial p^2}\, i\, G^{(2)}(p^2, \lambda, \mu)^{-1}\Big|_{p^2=-\mu^2} = 1 = Z \frac{\partial}{\partial p^2}\, i\, G_0{}^{(2)}(p^2, \lambda_0)^{-1}\Big|_{p^2=-\mu^2} \quad , \tag{9}$$

and

$$G^{(N)}(p_1, \ldots p_N, \lambda, \mu)\Big|_{p^2=-\mu^2} = \frac{-i\lambda}{(2\pi)^{D/2\,(N-2)}} \tag{10}$$

$$= Z^{N/2} G_0{}^{(N)}(p_j, \lambda_0)\Big|_{p^2=-\mu^2} \quad . \tag{11}$$

These determine Z and then Z_λ once one has calculated $G_0{}^{(2)}$ and $G_0{}^{(N)}$ to whatever accuracy desired using perturbation theory in λ_0 and some method of cutting of the divergence integrations. After rescaling by Z and Z_λ via (3) and (4) the cutoff is sent to infinity with ϕ, λ, and μ held fixed. What is remarkable, then, is that the resulting theory is then finite to all orders in λ.

But what are we to make of μ? We began with a Lagrangian with a field of dimension (D–2)/2 and a dimensionless coupling. Now we also seem to have a mass scale μ. Since μ is arbitrary, the consequences of the theory should be independent of it. We can insure that by noting that since $G_0^{(n)}(p_j, \lambda_0)$ never heard of μ

$$\mu \frac{\partial}{\partial \mu} (G_0^{(n)}(p_j, \lambda_0)) = 0 \quad . \tag{12}$$

The appearance of μ will then be only apparent in this sense: only one real physical parameter enters this problem, namely the coupling λ. Since λ is defined by a Green function evaluated at $p_j^2 = -\mu^2$, λ may be traded off for μ. Physical masses, for example, must be of the form

$$M_{\text{Physical}} = \mu F(\lambda) \quad , \tag{13}$$

but $F(\lambda)$ must depend on $\lambda(\mu)$ in such a way that

$$\frac{\partial}{\partial \mu} M_{\text{Physical}} = 0 \quad . \tag{14}$$

From (12) and the definition of $G^{(n)}$ we learn

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \frac{n}{2} \gamma(\lambda)\right] G^{(n)}(p_1, \dots p_n, \lambda, \mu) = 0 \quad , \tag{15}$$

where

$$\beta(\lambda) = \mu \frac{\partial}{\partial \mu} \lambda \Big|_{\lambda_0 \text{ fixed}} \quad , \tag{16}$$

and

$$\gamma(\lambda) = \mu \frac{\partial}{\partial \mu} \log Z \Big|_{\lambda_0 \text{ fixed}} \quad . \tag{17}$$

This differential equation provides restrictions on the form $G^{(n)}$ can take, and, even more important, knowing the functions $\beta(\lambda)$ and $\gamma(\lambda)$ in perturbation theory can frequently give information on $G^{(n)}$ not possible to see in perturbation theory. We'll come to some examples of this.

Suppose the dimensions of $G^{(n)}$ are $\mathscr{D}$, then the solution of (15) is

$$G^{(n)}(\xi p_i, \lambda, \xi) = G^{(n)}(p_i, \bar{\lambda}(-\log \xi), \mu) \times$$

$$\times \exp + \int_{-\log \xi}^{0} [\mathscr{D} + \frac{n}{2} \gamma(\bar{\lambda}(t))]\, dt \tag{18}$$

where

$$\frac{d\bar{\lambda}(t)}{dt} = -\beta(\bar{\lambda}(t)) \;, \; \bar{\lambda}(0) = \lambda \quad , \tag{19}$$

is the running coupling constant. It tells us that at a scale determined by ξ the strength of the non-linearity in $\mathscr{L}$ may vary. Clearly that is because we do not do perturbation theory in λ but really in the operator $\lambda\phi^N$ which does depend on x.

As ξ varies, the argument of $\bar{\lambda}(-\log \xi)$ moves about and approaches zeroes of the function $\beta(\lambda)$. Clearly when λ is near λ_I where $\beta(\lambda_I) = 0$, $\dot{\lambda}(t) = 0$ and the effective coupling in $G^{(n)}$ is not λ, the renormalized coupling, but λ_I. Suppose near λ_I

$$\beta(\lambda) = \beta_1(\lambda - \lambda_1) \quad , \tag{20}$$

then

$$\tilde{\lambda}(-\log\xi) = \lambda_1 + \xi^{\beta_1}(\lambda - \lambda_1) \quad . \tag{21}$$

If $\beta_1 > 0$, then as $\xi \to 0$, $\tilde{\lambda}(-\log\xi) \to \lambda_1$. If $\beta_1 < 0$, then as $\xi \to \infty$, $\tilde{\lambda}(-\log\xi) \to \lambda_1$. Zeroes of β with positive slope govern the infrared behavior of the theory ($\xi p_i \to 0$); zeroes with negative slope, the ultraviolet behavior ($\xi p_i \to \infty$).

In QED in lowest order perturbation theory

$$\beta_{QED}(e) = +\frac{b}{2}e^3 \ , \ b > 0 \quad , \tag{22}$$

so the effective charge is

$$\tilde{e}^2(-\log\xi) = \frac{e^2}{1 - be^2\log\xi} \quad , \tag{23}$$

so perturbation theory for $G^{(n)}(p_i, \tilde{e}(-\log\xi), \mu)$ is good for $\xi \to 0$ or really for $\xi << \exp 1/(be)^2$ with b order unity, and $e^2 \simeq 1/137$. The infrared properties of QED are then calculable in perturbation theory; e.g. the electron or muon magnetic moment. The short distance behavior is unknown and the formula for $\tilde{e}^2(-\log\xi)$ clearly breaks down for ξ large.

In a pure Yang-Mills theory with the Lagrange density

$$\mathscr{L}_{YM} = -\frac{1}{4g^2}F_{\mu\nu}{}^aF_{\mu\nu}{}^a, \quad a = \text{gauge group index} \tag{24}$$

$$F_{\mu\nu}{}^{a} = \partial_\mu A_\nu{}^{a} - \partial_\nu A_\mu{}^{a} + f_{abc} A_\mu{}^{b} A_\nu{}^{c} \quad , \tag{25}$$

the function $\beta(g)$ for small g is

$$\beta(g) = -\frac{B}{2} g^3 + O(g^5) \ , \ B > 0 \tag{26}$$

and

$$\tilde{g}(-\log \xi)^2 = \frac{g^2}{1 + Bg^2 \log \xi} \quad . \tag{27}$$

so the ultraviolet or short distance behavior of the theory is given by perturbation theory in the small coupling $\tilde{g}(-\log \xi)$. When we add fermions to the theory the total Lagrangian is

$$\mathscr{L} = \mathscr{L}_{YM} + \bar{\psi}[\, i \not{\partial} + T^a \not{A}^a - m_0]\psi \quad , \tag{28}$$

with T^a a representation matrix for the representation of the gauge group in which the fermions sit. If the gauge group is SU(3) and the fermions (quarks) are members of the fundamental triplet representation then B in (20) is positive if the number of such fermions (number of flavors) is ≤ 16. The phenomenon of $B > 0$ and predictable ultraviolet behavior is called asymptotic freedom.

Another consequence of the renormalization group follows from looking at the renormalized propagator $G^{(2)}(p^2, \lambda, \mu)$ when there is no mass term in the original Lagrangian. By dimensional reasoning

$$G^{(2)}(p^2, \lambda, \mu) = \mu^{-2} F\left(\frac{p^2}{\mu^2}, \lambda\right) \quad , \tag{29}$$

where F is dimensionless. Now the renormalization group says

$$F\left(\frac{p^2}{\mu^2},\ \lambda\right) = F\left(1,\ \bar{\lambda}(-\log\frac{p^2}{\mu^2})\right)\left(\frac{\mu^2}{p^2}\right)\exp + \int_{-\log p^2/\mu^2}^{0} \gamma(\bar{\lambda}(t))dt \quad . \tag{30}$$

Define $\chi(\lambda)$ by

$$\frac{d\log\chi(\lambda)}{d\lambda} = -\frac{1}{\beta(\lambda)} \quad , \tag{31}$$

then

$$\chi(\bar{\lambda}(-\log p^2/\mu^2)) = \frac{p^2}{\mu^2}\chi(\lambda) \quad , \tag{32}$$

and

$$F(1,\bar{\lambda}(-\log p^2/\mu^2)) = \mathscr{F}\left(\frac{p^2}{\mu^2}\chi(\lambda)\right) \quad , \tag{33}$$

and

$$G^{(2)}(p^2,\lambda,\mu) = \frac{1}{p^2}\mathscr{F}\left(\frac{p^2}{\mu^2}\chi(\lambda)\right)\exp\int_{-\log p^2/\mu^2}^{0} dt\,\gamma(\bar{\lambda}(t)) \quad . \tag{34}$$

If $G^{(2)}$ has a pole at $p^2 = M^2(\lambda)$ away from $p^2 = 0$, it must take the form

$$\begin{aligned} M^2(\lambda) &= \mu^2\chi(\lambda)^{-1} \times \text{constant} \\ &= \mu^2 \exp - \int^{\lambda} \frac{dx}{\beta(x)} \end{aligned} \quad . \tag{35}$$

Suppose

$$\beta(x) = \frac{b}{2}x^3 \qquad , \tag{36}$$

then

$$M(\lambda)^2 = \mu^2 \exp 1/b\lambda^2 \tag{37}$$

and we see that in a massless theory mass can be generated dynamically only outside perturbation theory; indeed $M(\lambda)^2$ is zero to every order of expansion in λ.

III. SELECTING A FIELD THEORY

With the renormalization group as a tool we can begin the selection of a field theory. {Of course, purists will wonder whether it's a sufficient tool. The wondering should cease, however; the answer is no, but it's about all we have of a non-perturbative nature in field theory.} Years of deep inelastic lepton scattering[1] (Figure 1) indicates that for $|q^2| \geq 1\ (\text{GeV/c})^2$ the constituents (quarks or partons) of a proton carrying charge or isospin or hypercharge or charm are essentially point-like and free.

Boldly this is assumed to indicate the field theory of hadrons has asymptotic freedom which necessitates a non-Abelian gauge theory plus fermions carrying gauge charge plus ordinary Q, I, Y, C, ... Which gauge theory shall we choose? The answer is connected with hadron spectroscopy; that is, the character of bound states of the hadron degrees of freedom. Following the spirit of the old Fermi-Yang proposal that mesons are $N\bar{N}$ bound states one assumes that mesons are quark-antiquark bound states with three (or more) kinds of quarks carrying isospin, strangeness, charm, etc. for all the internal quantum numbers observed for mesons. Baryons are then made out of qqq states. The properties of the well-established quarks are in Figure 2. More quarks will be called for by the discovery of states like the upsilon, if, indeed, it is made out of $q\bar{q}$ as we now presume the $\pi, \rho, \psi, \ldots$ to be.

Some examples of the construction of observed states out of q's are as follows:

$$|\pi^+\rangle = |u\bar{d}\rangle \quad |K^+\rangle = |u\bar{s}\rangle \quad |\psi\rangle = |c\bar{c}\rangle$$

$$|p\rangle = |uud\rangle \quad |\Lambda\rangle = |uds\rangle \quad |\Omega^-\rangle = |sss\rangle \quad , \tag{38}$$

etc. The only flaw in this picture comes when one assumes the quarks in baryons are in relative L = 0 states. This requires $|qqq\rangle$ to be <u>symmetric</u>. Since L = 0 is very likely to be the ground state, we require another label in which to antisymmetrize the quark wave function. We want quarks to form triplets under the group of this additional degree of freedom, which is called <u>color</u>.[2] The natural color groups are SU(2) and SU(3). In either case a baryon will be

$$|B\rangle = |\epsilon_{\alpha\beta\gamma}\, q_\alpha q_\beta q_\gamma\rangle \quad \alpha, \beta, \gamma = 1, 2, 3 \quad . \tag{39}$$

Since no quantum number associated with color has been observed, we presume that baryons and mesons are singlets under color transformations. If we choose SU(2) for the color group, then singlets can be made as $|q_\alpha \bar{q}_\alpha\rangle$, $|q_\alpha q_\beta q_\gamma \epsilon_{\alpha\beta\gamma}\rangle$ as desired, but also $|q_\alpha q_\alpha\rangle$, $|\bar{q}_\alpha q_\beta q_\gamma \epsilon_{\alpha\beta\gamma}\rangle$ and other unwanted states are singlets. So choose SU(3). Then we can make $|q_\alpha \bar{q}_\alpha\rangle$ and $|\epsilon_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma\rangle$ into singlets using only two or three quarks. Of course $|q_\alpha \bar{q}_\alpha\, q_\beta \bar{q}_\beta\rangle$ is a singlet, but it comes up as a problem (or virtue) for heavier bound states than (probably) yet observed.

Now we must have a binding agent to hold the quarks together in the bound states called hadrons. This glue must be free of the Q, Y, I, C,... flavor quantum numbers listed above since it does not interact with the electromagnetic or weak current coupled to leptons. It does carry about 50% of the proton momentum, but not its quantum numbers. The choice made these days is to identify the glue with the gauge bosons of an SU(3) non-Abelian gauge theory, $A_\mu^a(x)$, a = 1,...8, and have them interact with the color of quarks.

So we have the following quarks and gluons:

$q_{\alpha f}(x)$	α=1,2,3	Local SU(3) gauge symmetry
	f=u,d,s,c,...	Global SU(N) flavor symmetry
$A_\mu^\alpha(x)$	α=1,...8	Local SU(3) gauge boson flavor singlet.

For these fields we write the Lagrangian density

$$\mathscr{L}_{QCD} = -\frac{1}{4g^2} F_{\mu\nu}^\alpha F_{\mu\nu}^\alpha + \bar{q}_{\alpha f}\Big[i \not\partial \delta_{\alpha\alpha'}\delta_{ff'} + \left(\frac{\lambda}{2}^\beta\right)_{\alpha\alpha'} \gamma^\mu A_\mu^\beta \delta_{ff'} - (M_0)_{ff'}\delta_{\alpha\alpha'}\Big] q_{\alpha' f'} \qquad , \tag{40}$$

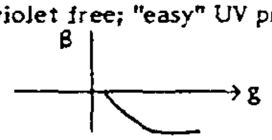
this defines the theory now popularly known as Quantum Chromodynamics (QCD). In it λ^β are the usual SU(3) 3 × 3 representation matrices. M_0 is a matrix in flavor space that contains mass splittings; so we can accommodate in this term the phenomenological fact that $m_u \approx m_d < m_s < m_c$ corresponding to $m_\pi < m_K < m_\psi$. The renormalization group function $\beta(g)$ for this theory in lowest order is

$$\beta(g) = -\frac{g^3}{48\pi^2}(33 - 2F) \tag{41}$$

when the flavor index f = 1,...F.

It is instructive to make a side by side list of comparisons between QCD and the more familiar QED:

QED	QCD
Photon $A_\mu(x)$	Gluon $A_\mu^\alpha(x)$
"Matter" e, μ $\psi_e(x)$, $\psi_\mu(x)$	"Matter" quarks $q_{\alpha f}(x)$ colors and flavors
Spectrum of States: γ, e, μ-read from Lagrangian	Spectrum of States: Hadrons
Bound states: positronium, H-atom	Bound states: π, K, ψ, p, Λ,...
Interaction among bound states: molecular forces, Van der Waals force	Interaction among bound states: hadron scattering
Infrared free; "easy" IR properties	Ultraviolet free; "easy" UV properties



QED	QCD
Photons have no charge	Gluons carry color charge
Charge is observed. Charge is not confined	Charge (color) is not seen; no massless bosons; no quarks. Confinement

Unfortunately we are not going to provide the full solution of the theory defined by $\mathscr{L}_{QCD}$. Indeed, the properties of the ground state are still under study. Let's discuss it a bit, anyway. First of all, in $\mathscr{L}$ are massless vector bosons which carry color. That's bad, but not terrible since these bosons interact among themselves via the cubic and quartic terms in $(F_{\mu\nu})^2$, so the real vector boson could be massive, if it is even permitted as an asymptotic state. Second it has quarks of unknown mass carrying quantum numbers (color) probably never seen. Third, it is asymptotically free[3]; so its UV behavior is calculable and as far as one can tell in detailed comparison with inelastic muon scattering data, compatible with short distance behavior seen experimentally. Its infrared behavior is totally unknown and not inferable from $\mathscr{L}_{QCD}$ by just looking at it. This is just the opposite of QED where we read $\mathscr{L}_{QED}$ like a book; namely, there are photons and electrons and small, $O(\alpha)$, corrections.

So what evidence do we have that $\mathscr{L}_{QCD}$ is connected with the world of real hadrons? Certainly the charmonium spectroscopy of the states in the mass range ~ 3 to ~ 4 GeV/c^2 strongly indicates that for heavy quarks, c quarks, various qualitative features of QCD are in operation. In deep inelastic lepton scattering, in e^+e^- annihilation, in the qualitative features of hadron spectroscopy–in all these places we sense QCD at work.

A very nice set of calculations in this regard has been done over the past few years by the ITEP, Moscow, group of Shifman, Vainstein, Voloshin, Zakharov, Novikov and Okun.[4] They study the physics of currents formed from massive quarks. In particular the charmed quark. For example, they consider the current-current matrix element of heavy quarks

$$D_{\mu\nu}{}^{c}(q) = \int d^4x e^{-iq\cdot x} \langle 0 | T(J_\mu{}^c(x) J_\mu^c(0)) | 0 \rangle \quad , \tag{42}$$

with

$$J_\lambda{}^c(x) = \bar{q}_{\alpha c}(x)\gamma_\lambda q_{\alpha c}(x) \tag{43}$$

which is the $c\bar{c}$ contribution to the electromagnetic current.

In the q^2 plane for this matrix element one has for large $-q^2$ asymptotic freedom where one may calculate $D_{\mu\nu}{}^c$. Near $q^2 = 0$ not much is known. For positive q^2 there are known resonances: ψ, ψ', ψ'',... to which $c\bar{c}$ couple. See Figure 3. In perturbation theory they are able to calculate graphs like the ones in Figure 4. The ITEP group argues that the distances in this perturbation theory which are essential in the calculation are $x \lesssim m_c^{-1}$. If $m_c \approx 1.5$ GeV/c^2, as indicated by the ψ mass and the charmed meson masses, then they argue further that even at $q^2 = 0$, asymptotic freedom has set in for these heavy quark matrix elements. They view the world as in Figure 5.

Now using dispersion relations in q^2, they relate the values of $D_{\mu\nu}{}^c$ at $q^2 = 0$ they calculate from perturbation using $\mathscr{L}_{QCD}$ to q^2 in the resonance region. Given the masses of the resonances they calculate quite a few numbers of which $\Gamma(\psi \to e^+e^-) = 5$ keV is representative.

To study even charge conjugation states they look at forward light by light scattering via the electromagnetic coupling to charmed quarks. From these calculations using the ITEP Freedom assumption and dispersion relations again, they are able to calculate a variety of quantities related to the C-even X states.

This approach seems very attractive to me. It maintains gauge invariance and relativistic co-variance at every stage. It is necessary, of course, to use the dispersion relations with care and good sense so one doesn't demand information about too many q^2 derivatives of $D_{\mu\nu}{}^c$ at $q^2 = 0$, for then one is reconstructing $D_{\mu\nu}{}^c$ via a Taylor series, and somewhere that is bound to fail.

In any case a significant number of phenomena about heavy quarks are derivable in this fashion. A blind application to strange quarks works "OK," and to light quarks doesn't do well at all. That is as it should be from the distance arguments given before, but leaves a lot of work to be done on the QCD aspects of light quarks.

IV. DYNAMICAL MASS GENERATION

The next topic I'll take up is the intriguing question of generation of mass in non-Abelian gauge theory by dynamical means. This is a situation where there appears no mass scale in the Lagrangian density (so in $\mathscr{L}_{QCD}$ above, $M_0 = 0$), but since a mass is introduced via the renormalization process, it is possible for the renormalized theory to have mass. We saw how this might operate by the study of the renormalization group in Section II; now we'll put a bit of flesh on these bones.

We really want to study QCD. So let's imagine $M_0 = 0$ in $\mathscr{L}_{QCD}$. It is then

$$\mathscr{L}_{QCD} = -\frac{1}{4g^2} F_{\mu\nu}{}^a F_{\mu\nu}{}^a + \bar{q} i \not{D} q \quad , \tag{44}$$

with

$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} + i(T^c)_{ab} A_\mu{}^c \quad . \tag{45}$$

The symmetry in this is $SU(3)_{color} \otimes G$ where

$$G = U(F)_{vector} \otimes U(F)_{Axial} \tag{46}$$

and F is the number of flavors. Associated with the global symmetry G are the conserved currents

$$\mathscr{V}_\lambda = \sum_{f=1}^{F} \bar{q}_f \gamma_\lambda q_f \quad , \quad \mathscr{A}_\lambda = \sum_{f=1}^{F} \bar{q}_f \gamma_\lambda \gamma_5 q_f$$

$$\mathscr{V}_\lambda^\alpha = \sum_{f',f=1}^{F} \bar{q}_f \gamma_\lambda T_{ff'}{}^\alpha q_{f'} \quad , \quad \mathscr{A}_\lambda^\alpha = \sum_{f'',f=1}^{F} \bar{q}_f \gamma_\lambda \gamma_5 T_{ff'}{}^\alpha q_{f'} \tag{47}$$

with $\alpha = 1,\ldots F^2-1$. The T^α are $F \times F$ representation matrices of SU(F). These currents, except for $\mathscr{A}_\lambda$, remain conserved under interaction. The axial current has an "anomaly" coming from the short-distance structure of the theory, and its divergence is not zero, but

$$\partial^\lambda \mathscr{A}_\lambda \propto g^2 F_{\mu\nu}{}^\alpha \tilde{F}_{\mu\nu}{}^\alpha \quad , \tag{48}$$

where $\tilde{F}_{\mu\nu}{}^\alpha = \frac{1}{2}\epsilon_{\mu\nu\tau\sigma} F_{\tau\sigma}{}^\alpha$ is the dual gauge field.

Now the symmetry of the hadron <u>states</u> is not as large as G. It is only $U_{Vector}(1) \times SU(F)$ with F^2-1 pseudoscalar mesons which are more or less certainly massless Goldstone bosons. Even the SU(F) flavor symmetry is approximate, and the massless bosons pick up a mass possibly from the weak interactions.

To generate these Goldstone bosons we must break the chiral $(U_{Axial}(F) \otimes U_{Vector}(F))$ symmetry of the bare vacuum (as defined by $\mathscr{L}_{QCD}$). Because of the anomaly it is possible (below we'll even see it's likely) that there is no massless boson associated with the $U_{Axial}(1)$ symmetry. This is fine, actually, since there appears to be no such massless (read small mass in practice) object.

We can generate a mass scale in a Lagrangian theory by two presently acceptable methods: (1)let a scalar meson field, ϕ, coupled to quarks as $\phi\, q\bar{q}$ develop a vacuum expectation value $\langle\phi\rangle \neq 0$. This is a "spontaneous symmetry breaking" and is essentially kinematic in origin; that is, it is put in by hand. (2)We can have composite fields like $q\bar{q}$ itself develop a non-zero vacuum expectation value. This is dynamical and by the same renormalization group arguments used before must take the form

$$< \bar{q}(0)q(0) > = \mu^3 \exp - \int^g dx/\beta(x) \quad . \tag{49}$$

How can we investigate this possibility in QCD? Let's look at the Ward identity satisfied by the proper quark-quark-axial vector current vertex $\Gamma_\lambda{}^\alpha$

$$q^\lambda \Gamma_\lambda{}^\alpha(p, p+q) = -\{\gamma_5 T^\alpha S^{-1}(p+q) + S^{-1}(p) T^\alpha \gamma_5\} \tag{50}$$

where S(p) is the quark propagator and T^α is a flavor matrix. Let's look at this as $q_\lambda \to 0$. The right hand side approaches $\{\gamma_5 T^\alpha, S^{-1}(p)\}$. If this anti-commutator is non-zero, then $\Gamma_\lambda{}^\alpha$ must have the property

$$\Gamma_\lambda{}^\alpha(p, p+q) \underset{q\to 0}{\simeq} F_\pi \mathscr{P}(p, p) \frac{q_\lambda}{q^2} T^\alpha \gamma_5 \quad , \tag{51}$$

where F_π is a constant and $\mathscr{P}$ is the wave function to find a $q\bar{q}$ pair in the pseudoscalar state called pion (Figure 6). In every order of perturbation theory $S^{-1}(p)$ is proportional to $A(p^2)\not{p}$, which anti-commutes with γ_5, so $\{\gamma_5 T^\alpha, S^{-1}(p)\}$ is zero. To generate a pion pole at $q^2 = 0$ we need a non-perturbative effect whereby

$$S^{-1}(p) = A(p^2)\not{p} - B(p^2) \quad , \tag{52}$$

then the ward identity tells us

$$2B(p^2) = F_\pi \mathscr{P}(p, p) \quad . \tag{53}$$

So proving that a non-zero anomalous part to $S^{-1}(p)$ exists would be equivalent to showing there is a massless pion. This is not more than a statement of Goldstone's theorem, since $B(p^2)$ has dimensions of mass it introduces into the problem a scale thus breaking the chiral symmetry, thus giving rise to a massless boson.

To investigate the presence of B one looks at the Bethe-Salpeter equation for $\mathscr{P}$ and concludes

$$B = \int d^4p \frac{B}{p^2A^2(p^2) - B^2(p^2)} K \tag{54}$$

where K is the irreducible kernel for $q\bar{q}$ scattering in QCD. Now Lane[5] has argued that only in an asymptotically free theory can we investigate this equation in some convincing way. Namely, we can look at the $p^2 \to \infty$ behavior of B and conclude that the equation may have a solution. Now since S^{-1} is a gauge dependent object, the conclusion still leaves one uneasy. Of course, the gauge dependence comes from the fact that

$$S(p) = \int d^4x e^{-ip\cdot x} \langle 0 | T(\bar{q}(x)q(0)) | 0 \rangle \quad , \tag{55}$$

and $\bar{q}(x)q(0)$ is gauge dependent except at $x = 0$. Since $p^2 \to \infty$ "takes" us to $x = 0$, the conclusion stated may be OK.

So we see that this standard Nambu-Jona-Lasinio approach is tricky in gauge theories, because gauge invariance (a crucial aspect of the theories) is treated cavalierly. What we need, for example, is a compact way to calculate quantities like $\langle 0 | \bar{q}(0)q(0) | 0 \rangle$ directly. { You can see from (55) that it is zero if $B(p^2)$ is zero. } This is zero in every order of perturbation theory and evaluating it outside of perturbation theory has proven more or less intractable without going full circle to equations like (54) for B.

We can, however, get a feeling for the issues by looking at a model introduced by Ansel'm[6] in 1959 and analyzed in 1961 by Vaks and Larkin[7]. It exists in one space and one time dimension and has N fermions $\psi_1 \ldots \psi_N$ coupled via

$$\mathscr{L} = \bar{\psi} i \not\partial \psi + g^2/2\,(\bar{\psi}\psi)^2 \qquad , \tag{56}$$

with

$$(\bar{\psi}\psi) = \sum_{i=1}^{N} \bar{\psi}_i \psi_i \qquad . \tag{57}$$

The generating functional for this theory is

$$e^{iW[J]} = \int d\psi d\bar{\psi}\, e^{i\int[\mathscr{L} + \bar{J}\psi + \bar{\psi}J]} \tag{58}$$

which is equivalent to[8]

$$e^{iW[J]} = \int d\psi d\bar{\psi} d\sigma\, e^{i\int[\bar{\psi} i \not\partial \psi - \sigma^2/2 - g\sigma\bar{\psi}\psi + \bar{J}\psi + \bar{\psi}J]} \qquad , \tag{59}$$

which can be seen directly by integrating over the auxiliary field σ. σ is not a dynamical variable, $\frac{\partial\sigma}{\partial t}$ does not appear in $\mathscr{L}$, but simply plays the role of separating the $(\bar{\psi}\psi)^2$ in the interaction and allowing us to deal with the more familiar form for $\mathscr{L}$ in (53). If $\langle\sigma\rangle \neq 0$, then the fermion acquires a mass $m_F = g\langle\sigma\rangle$; so σ is "substituting" for $\bar{\psi}\psi$.

To investigate whether $\langle\sigma\rangle \neq 0$, let's set $\sigma(x) = v + \chi(x)$ and study the energy density of the resulting theory as v varies. If we make this substitution and introduce a source S for the χ,field we have

$$e^{iW[S]} = \int d\psi d\bar{\psi} d\chi\, e^{i\int[\bar{\psi} i\not\partial\psi - \chi^2/2 - \chi v - gv\bar{\psi}\psi - g\chi\bar{\psi}\psi + S\chi]} \qquad , \tag{60}$$

which becomes on integrating over ψ and $\bar{\psi}$

$$e^{iW[S]} = \int d\chi \exp i \int \left[-\frac{\chi^2}{2} - v\chi - i \operatorname{tr} \log [1 - \frac{1}{i\partial - gv} g\chi] + S\chi \right] , \quad (61)$$

$$= \int d\chi \, e^{i\int [\hat{\mathscr{L}} + S\chi]} \quad . \quad (62)$$

This $\hat{\mathscr{L}}$ is non-local, non-Hermitean, and non-finite. It is non-local because it contains all the processes in Figure 7 where the dashed line represents a χ and the solid line a propagator $(\not{p} - gv)^{-1}$. It is non-Hermitean because it contains particle production. It is non-finite because the first two graphs in Figure 7 are infinite and are the wave function and mass renormalization of the original theory, which, by power counting, is just renormalizable at D = 2.

After renormalization, the vacuum, or ground state is defined by $\langle\chi\rangle = 0$ or equivalently $\frac{\partial \hat{\mathscr{L}}_R}{\partial \chi}\Big|_{\chi=0} = 0$. In D space-time dimensions this means

$$v \left\{ 1 + \frac{2\lambda^2 \operatorname{tr} \underset{\sim}{1} \, \pi^{D/2} \Gamma(2 - \frac{D}{2})}{(2\pi)^D (2 - D)} \left[(D-1) \int_0^1 dy [F(\lambda)^2 + y(1-y)]^{\frac{D}{2} - 1} - F(\lambda)^{D-2} \right] \right\} = S \quad (63)$$

with the dimensionless coupling

$$\lambda = \sqrt{N} \, g(\mu)^{D/2 - 1} \quad (64)$$

defined in terms of the normalization point μ, and

$$F(\lambda) = gv/\mu \quad . \quad (65)$$

In $D = 2 + \varepsilon$ dimensions we can study (63) as $S \to 0$. Requiring the vacuum to be stable which means $\partial^2(-\hat{\mathscr{L}}_R)/\partial \chi^2|_{\chi=0} > 0$, we find

$$S \to 0 \quad \begin{cases} v = 0 & \lambda^2 \le \pi\varepsilon \\ v \ne 0 & \lambda^2 > \pi\varepsilon \end{cases} \quad . \tag{66}$$

At D = 2 exactly we can carry out the integral at S = 0 to find

$$\sqrt{1 + 4F(\lambda)^2}\ \log\left[\frac{\sqrt{1+4F(\lambda)^2} - 1}{\sqrt{1+4F(\lambda)^2} + 1}\right] = -\frac{2\pi}{\lambda^2} \quad , \tag{67}$$

and discover $F(\lambda)$ behaves as shown in Figure 8. From the renormalization group we know

$$\frac{d \log F(\lambda)}{d\lambda} = -\frac{1}{2\beta(\lambda)} \quad , \tag{68}$$

so we deduce that $\beta(\lambda)$ behaves as in Figure 9. Near $\lambda = 0$ $\beta(\lambda)$ behaves as

$$\beta(\lambda) = -\frac{\lambda^3}{4\pi} + \lambda\, e^{-2\pi/\lambda^2} + \ldots \quad , \tag{69}$$

so the non-analytic structure in $\beta(\lambda)$, coming from the non-analytic behavior of $F(\lambda)$ (or m_F if you like), shows up in the non-perturbative requirement of vacuum stability. The infrared stable behavior near $\lambda^2 = \pi$ is a direct result of the non-analytic terms in (69) or more precisely (68).

The hope in treating this unrealistic model in detail[9] is to learn what must be the ingredients for a study of the more complicated vacuum of non-Abelian gauge theories. To date that has not been a realized hope.

V. QUARK CONFINEMENT

We have talked at length in these lectures about quarks (with quantum numbers given in Figure 2) and the gauge bosons which are supposed to glue them together. To date no one has reported a convincing sighting of either a quark or a gluon. Perhaps we will find them. Even more intriguing: perhaps we will never find them. This, taken at its face, means $\mathscr{L}_{QCD}$ does not represent the spectrum of QCD but only the degrees of freedom: $A_\mu^{\alpha}(x)$ and $q_{\beta f}(x)$. The idea that these degrees of freedom can never be observed by themselves but only in the bound state combinations of color singlets we call hadrons goes by the label of color or quark confinement.

Clearly this question of observing color co-ordinates is an infrared or long distance problem: only if we can separate quarks can we study them individually. At short distances we may sense their existence by local probes of the weak current. With reference to the infrared behavior of QCD let's look at the effective coupling

$$\bar{g}_{QCD}^{\,2}(p^2/\mu^2) = g^2/(1 + bg^2 \log p^2/\mu^2) \quad , \quad b > 0 \quad . \tag{70}$$

For $p^2 \to \infty$, it goes to zero; that's just asymptotic freedom. As $p^2 \to 0$, $\bar{g}^2$ grows and in the approximation which leads to (70) blows up. So the long distance behavior of QCD is a strong coupling problem. In contrast, QED has $b < 0$ in (70) and the infrared behavior has $\bar{g}_{QED}^{\,2} \to 0$. This ties in nicely with the fact that the spectrum of QED can be read directly from the Lagrangian.

In (70) there is the arbitrary renormalization scale μ which tells us when $\bar{g}^2$ is small ($p^2 \gg \mu^2$) or large ($p^2 \lesssim \mu^2$). Presumably it is when $\bar{g}^2$ is order of unity that strong coupling and binding effects come into play. So we can tentatively identify

$\mu \approx 1$ GeV as the hadronic scale and expect that when $p^2 > 1\ (\mathrm{GeV})^2$, $\tilde{g}_{QCD}{}^2$ is small. Similarly when distances $> 1/\mu$, we can expect strong forces to come into play.

To investigate the forces between colors we develop now a picture originally due to Ken Wilson.[10] Picture two color charges at a fixed time separated by a distance R (Figure 10). As R grows, at some point the energy in the field between the charges becomes of order m_{hadron}. At that distance it becomes energetically favorable to put that energy into the creation of two color charges which then shield the color force between the original charges (Figure 11). Before this creation of pairs from the "vacuum", we can ask about the forces between the charges.

To do this in a relativistic fashion we need a space-time picture. So imagine that at $t = 0$ the pair is created and separates a distance R in a short time; then moves along for a long time ($\approx$T) at distance R and finally annihilate (Figure 12). What we have pictured here is a large charge loop of area $\approx$ RT located in the vacuum. Suppose we calculate the energy per unit volume of this loop

$$e^{-iE(R)T} = \langle e^{i\int J^{\mu a}_{charge}(x) A_\mu{}^a(x) d^4x} \rangle \qquad , \qquad (71)$$

where the average is taken over configurations of the gauge field $A_\mu{}^a(x)$. For a single charge moving along the path $z_\mu(\tau)$ the current is

$$J^\mu_{charge}(x) = \int d\tau\, g \frac{dz^\mu(\tau)}{d\tau} \delta^4(x - z(\tau)) \qquad , \qquad (72)$$

so

$$\exp - iE(R)T = \langle \exp ig \int_{Loop} dz_\mu A^\mu(z(\tau)) \rangle \qquad . \qquad (73)$$

To evaluate E(R), the energy stored in the loop as a function of R, we must evaluate the vacuum average of the loop integral. When that integral behaves as $\exp -iR^{\alpha}T$ with $\alpha > 0$, we can say the binding energy is confining since it rises with separation. In QED, $E(R) \xrightarrow[R\to\infty]{}$ finite number. In QCD it is conjectured that $E(R) \simeq R$ for large R; equivalently the loop integral behaves as exp -i(Area of loop). Wilson has studied this loop integral in QED on a lattice and concludes that for e^2 large enough, the loop integral behaves as the area.

To address the question of confinement it seems we ought to look at the vacuum structure of QCD to be sure we have it in hand. Later we can come to the question of evaluating the energy of a loop in the vacuum.

To look at vacuua in quantum theory we must first go back to classical physics. Recall the elementary problem of the harmonic oscillator with hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad . \tag{74}$$

The classical ground state has $x = 0$. The quantum ground state is characterized by the wave function

$$\psi_0 \sim e^{-x^2} \tag{75}$$

which represents fluctuations about the classical ground state at $x = 0$.

In QED our co-ordinate is the value of $A_\mu(x)$ at every point in space-time. The classical ground state has $F_{\mu\nu} = 0$ and the energy momentum tensor $\theta_{\mu\nu} = 0$. Indeed this allows us to conclude that $A_\mu = 0$ is a fine classical ground state. The quantum mechanical ground state is represented by a wave functional $\Psi_0[A_\mu(x)]$ something like

$$\Psi_0[A_\mu] = \exp - \int d^4x\, A_\mu(x)A^\mu(x) \quad . \tag{76}$$

In QCD we expect that in the classical ground state $F_{\mu\nu}{}^a(x) = 0$ again and also $\theta_{\mu\nu} = 0$. This does not mean, however, $A_\mu{}^a = 0$, and that's they key to the subtlety of the quantum vacuum in QCD. Let's go over to a matrix formulation of the theory by introducing a set of matrices T^α satisfying

$$[T^\alpha, T^\beta] = if_{\alpha\beta\gamma}T^\gamma \tag{77}$$

with $f_{\alpha\beta\gamma}$ the structure functions of our gauge group. Now let

$$F_{\mu\nu}(x) = -i \sum_\alpha T^\alpha F_{\mu\nu}{}^\alpha(x) \quad , \tag{78}$$

and

$$A_\mu(x) = -i \sum_\alpha T^\alpha A_\mu{}^\alpha(x) \quad , \tag{79}$$

so

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu, A_\mu] \quad . \tag{80}$$

If $F_{\mu\nu}(x) = 0$ it implies

$$A_\mu(x) = g^{-1}(x)\partial_\mu g(x); \quad g(x)^+ = g^{-1}(x) \quad , \tag{81}$$

and $g(x)$ is in the gauge group. So $A_\mu(x)$ is a pure gauge field. The representation of $A_\mu(x)$ in terms of $g(x)$ may not be unique. We must ask how many <u>different</u> mappings there are from $(\vec{x}, t)$ to the gauge group. Each such inequivalent mapping will give a different classical vacuum with $F_{\mu\nu}(x) = 0$. Mathematicians have been kind enough to prove for us that for SU(N), ($N \geq 2$), as the gauge group, there are an infinite number of inequivalent mappings labeled by an integer $q = 0, \pm 1, \pm 2, \ldots$ Essentially this represents the number of times (and "direction") that the $(\vec{x}, t)$ manifold is traced out on the SU(N) manifold.

A visualizable example is the mapping of the real line onto a circle. In the language above we consider elements g(x) in 0(2) and ask how many times the manifold of g(x), the circle, is covered as $-\infty < x < \infty$. Use the usual stereographic projection of the real line onto the unit circle (Figure 13) with θ the angle of the line joining the north pole to the real line. As x goes from $-\infty$ to $+\infty$ the angle θ runs from $-\pi/2$ to $+\pi/2$. Let Θ the angle on the 0(2) circle by $\Theta = \pm n(2\theta)$ then we cover the 0(2) manifold $\pm n$ times as x goes over its manifold.

From this observation on the classical vacuua in QCD which is due to Belavin, et al.,[11] we conclude there are an infinite number of inequivalent classical ground states. Which one of the $|\pm q\rangle$, $q = 0, 1, 2, \ldots$ shall we choose for the quantum theory?

Let's go back to an example to guide us. Consider the hamiltonian

$$H = p^2 + g(1 - \cos \chi) \quad . \tag{82}$$

The classical theory has an infinite number of equilibrium states (ground states) at $\chi = \pm 2\pi q$, $q = 0, 1, 2, \ldots$ Which one shall we choose for the quantum ground state? In good old quantum mechanics we don't have to think twice about this since these classical ground states are connected by <u>tunneling</u> so a linear superposition

$$|\,\text{Vacuum}\rangle = \sum_{q=-\infty}^{+\infty} a_q |q\rangle \quad , \tag{83}$$

with $|q\rangle$ a state centered about $\chi = 2\pi q$, is needed.

In field theory we must be a bit more careful. Consider the ϕ^4 theory

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{\mu^2}{2}\phi^2 - \lambda^2 \phi^4 \tag{84}$$

so the potential is

$$V(\phi) = -\frac{\mu^2 \phi^2}{2} + \lambda^2 \phi^4 \tag{85}$$

and has two equal energy minima at

$$\phi_{min} = \pm \mu/2\lambda \quad . \tag{86}$$

If the number of dimensions of space-time is greater than two, there is no tunneling between these classical ground states and either one forms the lowest state, in the sense,

$$\Psi_0[\phi] \propto e^{-\int d^D x [\phi(x) - \phi_{min}]^2} \quad , \tag{87}$$

on which the hilbert space of states may be built. We are to choose one or the other of the classical vacuua, (86), and the physics should be much the same.

So we want to inquire if there is tunneling between the infinity of classical ground states in QCD. To look at this it is convenient to go over to imaginary time, {the lectures of S.J. Chang at this seminar indicate how to do all this directly in Minkowski space}, or Euclidian space. In quantum theory going under the barrier is an operation that can be described as a motion in imaginary time for a real co-ordinate since

$$\frac{dx}{dt} = \sqrt{2m(E - V(x))} = i\sqrt{2m(V(x) - E)} \quad , \tag{88}$$

or

$$\frac{dx}{d(it)} = \sqrt{2m(V(x) - E)} \quad . \tag{89}$$

We want to consider only solutions to the classical QCD equations, which are

$$D_\mu F^{\mu\nu} = 0 \tag{90}$$

where

$$D_\mu \psi = \partial_\mu \psi + [A_\mu, \psi] \quad , \tag{91}$$

which have zero $\theta_{\mu\nu}$. This solution can take us from one classical vacuum to another. Now

$$\theta_{\mu\nu} = \frac{1}{8}\left[(F_{\mu\lambda}{}^{\alpha} - \tilde{F}_{\mu\lambda}{}^{\alpha})(F_{\lambda\nu}{}^{\alpha} + \tilde{F}_{\lambda\nu}{}^{\alpha}) + \mu \leftrightarrow \nu \right] \quad , \tag{92}$$

with

$$\tilde{F}_{\mu\nu}{}^{\alpha} = ½\, \epsilon_{\mu\nu\sigma\tau} F_{\sigma\tau}{}^{\alpha} \quad . \tag{93}$$

So $\theta_{\mu\nu} = 0$ requires

$$F_{\mu\nu}{}^{\alpha} = \pm \tilde{F}_{\mu\nu}{}^{\alpha} \quad . \tag{94}$$

Take, say, $F_{\mu\nu} = \tilde{F}_{\mu\nu}$. This means

$$F_{12} = F_{34} \ , \ F_{13} = -F_{24} \ , \text{ and } F_{14} = F_{23} \qquad . \tag{95}$$

Suppose we introduce (Yang,[12] Belavin and Zakhorov[13])

$$y = (x_1 + ix_2)/\sqrt{2} \ , \ z = (x_3 - ix_4)/\sqrt{2} \qquad , \tag{96}$$

then the condition $F_{\mu\nu} = \bar{F}_{\mu\nu}$ means

$$F_{yz} = 0 \longrightarrow A_y = M^{-1}\partial y M \ , \ \det M = 1 \tag{97}$$

and

$$F_{y\bar{y}} + F_{z\bar{z}} = 0 \qquad . \tag{98}$$

This last equation may be written in terms of the hermitean matrix

$$H = MM^{+}$$

as

$$-M^{+}\left\{ \sum_{j=y,z} \partial_{\bar{j}}(H^{-1}\partial_j H) \right\} (M^{-1})^{+} = 0 \tag{99}$$

Under a gauge transformation $M \to Mg$, $g^{+} = g^{-1}$, so H is gauge invariant. The general self-duality condition reads

$$\sum_{j=y,z} \partial_{\bar{j}}(H^{-1}\partial_j H) = 0 \qquad . \tag{100}$$

The general solution of this is not known. Particular solutions are known for SU(2). The first is due to Belavin, et al.[11] and reads

$$A_\mu(x) = \frac{x^2}{x^2+1} g^{-1}(x)\partial_\mu g(x) \quad , \qquad (101)$$

with

$$g(x) = (x_4 + i\vec{\sigma}\cdot\vec{x})/\sqrt{x_4^{\,2}+\vec{x}^2} \quad , \qquad (102)$$

an element of SU(2). This solution has q = 1 where we characterize solutions by the topological charge

$$q = \frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}^{\alpha}\tilde{F}_{\mu\nu}^{\alpha} \quad , \qquad (103)$$

and can connect classical ground states with label q to states with q ±1. These classical field configurations with non-trivial topological charge are called instantons.

In SU(2) a set of solutions with q = N have been given by 't Hooft[14]; Corrigan and Fairlie[15]; and Jackiw, Nohl, and Rebbi.[16] They are

$$A_\mu^{a}(x) = -\eta_{\mu\nu}^{a}\partial_\nu \log \phi(x) \qquad (104)$$

where $\eta_{\mu\nu}^{a}$ can be read off from (101) and (102) when N = 1. $\phi(x)$ satisfies

$$\partial^2\phi(x) = 0 \qquad (105)$$

and for q = N has the form

$$\phi(x) = \sum_{j=1}^{N+1} \lambda_j/(x-x_j)^2 \quad . \qquad (106)$$

The existence of these tunneling solutions give a classical path with real coordinate A_μ^{a} and imaginary time which takes us from classical vacuum $|q>$ to

$|q \pm N>$, N = 1, 2,... The transition probability from $|q>$ to $|q+N>$ is finite and approximately of order

$$\exp -N(8\pi^2/g^2) \quad , \tag{107}$$

with g the QCD coupling constant.

This all means that the quantum mechanical ground state in QCD is a superposition of classical vacuua

$$|\Omega> = \sum_{n=-\infty}^{\infty} c_n |n> \quad , \tag{108}$$

whereas in QED $|\Omega> = |0>$. We go from $|n>$ to $|n+1>$ by a gauge transformation since in state $|n>$ we have $F_{\mu\nu} = 0$ and

$$A_\mu^{(n)}(x) = g_n^{-1}\partial_\mu g_n \tag{109}$$

where the gauge transformation g_1 takes us from $|n>$ to $|n+1>$. This gives rise to a periodicity similar to that in crystals and Bloch's theorem tells us

$$|\Omega> = \sum_{n=-\infty}^{\infty} e^{in\theta}|n> \quad . \tag{110}$$

Some immediate physics may be extracted from this vacuum structure:

1. Perturbation theory about $A_\mu^a = 0$ is not strictly correct since there are tunneling amplitudes of order $\exp -(g^{-2})$ to other field configurations. Presumably one wants to perturb about $|\Omega>$.

2. The so-called $U_A(1)$ problem is solved by this. We'll come to this.

3. It may be connected with quark confinement.

Let us look now at the $U_A(1)$ problem.[17] In massless QCD with F flavors the symmetry of the theory is the gauge group and

$$G_{flavor} = SU(F)_A \otimes SU(F)_V \otimes U_V(1) \otimes U_A(1) \quad . \tag{111}$$

The current associated with $U_A(1)$ is

$$\mathscr{A}_\lambda(x) = \sum_{f=1}^{F} \bar{q}_f \gamma_\lambda \gamma_5 q_f \quad . \tag{112}$$

The other currents are conserved to all orders in g, while the axial current has an anomaly so

$$\partial^\lambda \mathscr{A}_\lambda(x) = cg^2 \sum_\alpha F_{\mu\nu}{}^\alpha \tilde{F}_{\mu\nu}{}^\alpha \quad , \tag{113}$$

$$= cg^2 \partial^\lambda x_\lambda \quad , \tag{114}$$

where we note that $F_{\mu\nu}{}^\alpha \tilde{F}_{\mu\nu}{}^\alpha$ is a total divergence. Now it looks like

$$Q_5 = \int d^3x \, \mathscr{A}_0(\vec{x}, t) \tag{115}$$

is conserved because of the total divergence structure of $F_{\mu\nu}{}^\alpha \tilde{F}_{\mu\nu}{}^\alpha$, but because of the instanton fields with $q \neq 0$ there are matrix elements of $\partial \cdot \mathscr{A}$ of the form

$$\langle\Omega|(\bar{q}q \; \partial \cdot \mathscr{A})|\Omega\rangle \;\; \alpha \;\; e^{-8\pi^2/g^2} \tag{116}$$

which vanish in all orders of perturbation theory. Chiral symmetry breaking which leads to massless pions would, we usually expect, lead to a ninth Goldstone boson with quantum numbers of $\partial \cdot \mathscr{A}$. But since $\mathscr{A}_\lambda$ is not conserved really, the problem is absent.

So we see that $|\Omega\rangle$ is not chirally invariant. Indeed because of the anomaly

$$[\, g_1, Q_5 \,] = -2F \, g_1 \tag{117}$$

and the real invariance of QCD is

$$U_{Vector}(1) \otimes SU_V(F) \otimes SU_A(F) \quad . \qquad (118)$$

How $SU_A(F)$ is broken to produce massless pions is still not agreed upon in detail.

We began our excursion into instantonology because we wanted to learn about the QCD vacuum for confinement purposes. Polyakov[18] has given convincing arguments how instantons give rise to confinement in two space and one time dimension, but in 3 space, one time a similar hope has not yet been realized.

Indeed the various approaches to confinement have tried to populate the vacuum with various objects. Mandelstam[19] has considered filling the vacuum with magnetic monopoles which will crowd electric field lines into "strings" (much in the way Cooper pairs allow Abrikosov flux lines in type II superconductivity) which will then bind color. So far these attacks have been very illuminating, but not yet conclusive. No doubt at the next Topical Seminar held here in Tubingen we will hear conclusive progress in this matter. I know the generous hospitality and excellent organization of our hosts makes all of us plan on returning for that seminar before the university here passes its second 500 years.

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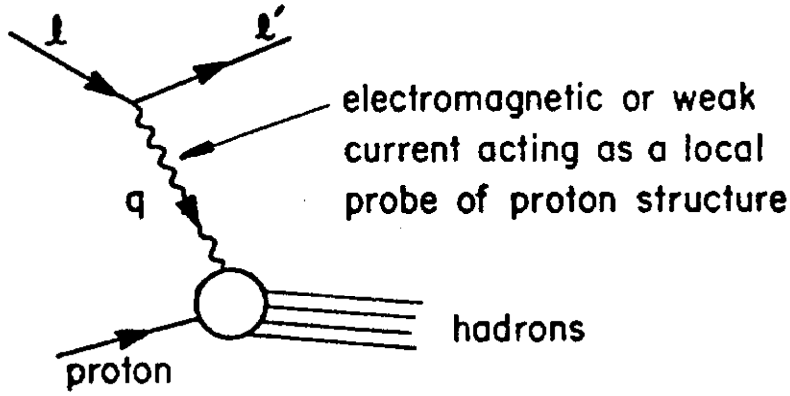

Fig. 1

	Q	I_3	Y	B	C
u	$2/3$	$+1/2$	$1/3$	$1/3$	0
d	$-1/3$	$-1/2$	$1/3$	$1/3$	0
s	$-1/3$	0	$-2/3$	$1/3$	0
c	$2/3$	0	$-2/3$	$1/3$	1

$$Q = I_3 + Y/2 + C$$

Fig. 2

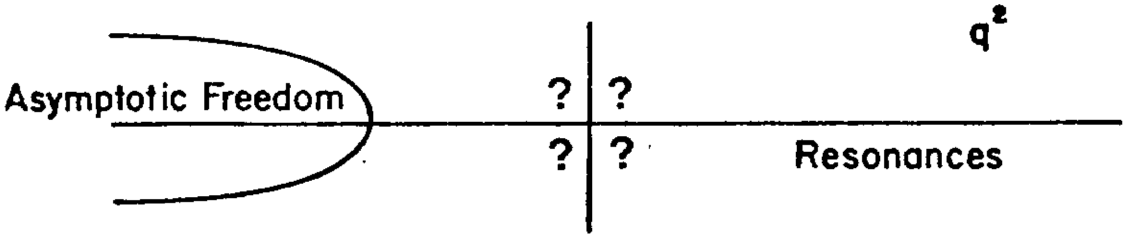

Fig. 3

Fig. 4

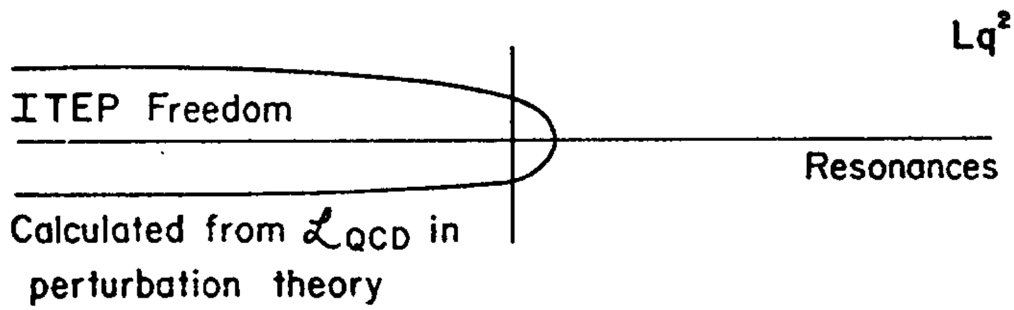

Fig. 5

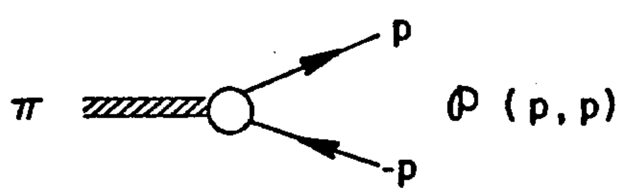

Fig. 6

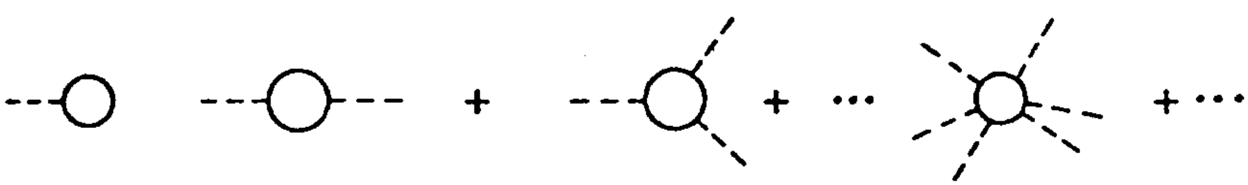

Fig. 7

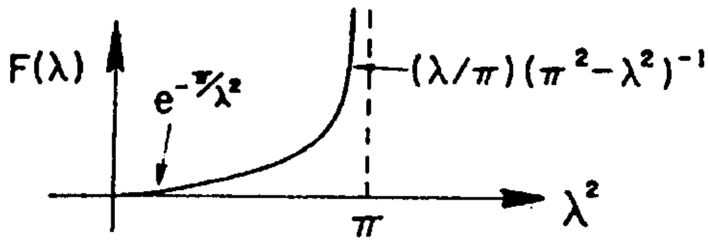

Fig. 8

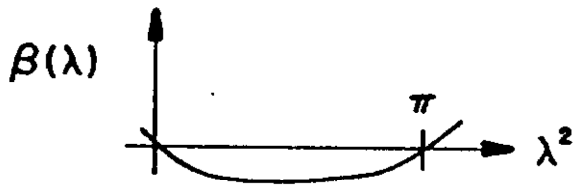

Fig. 9

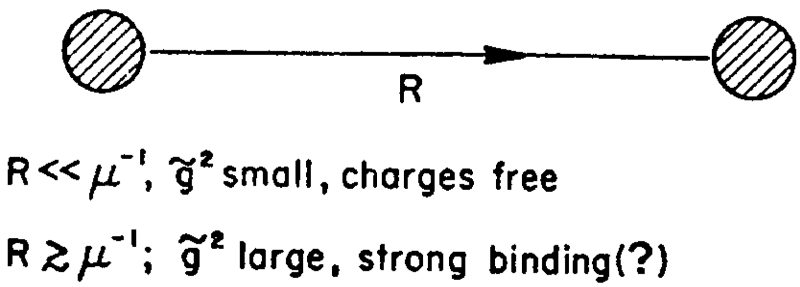

Fig. 10

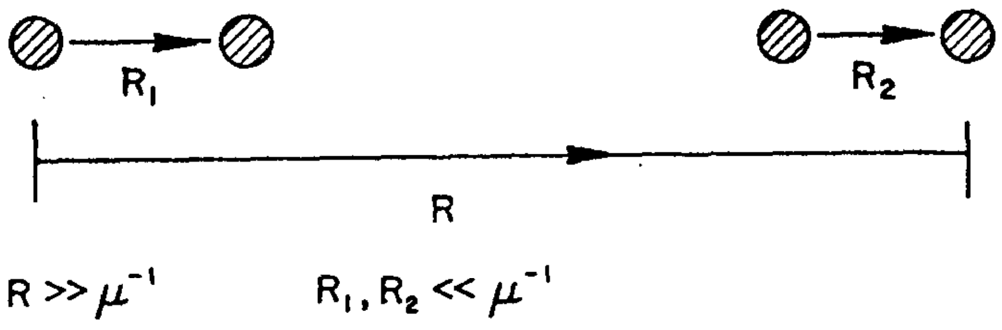

Fig. 11

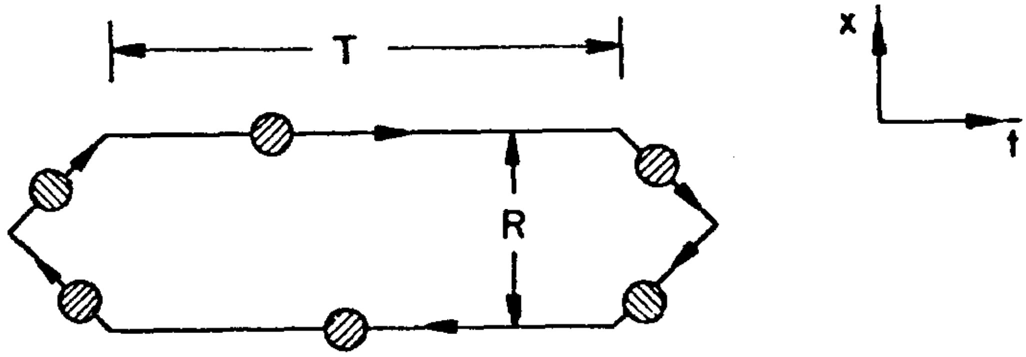

Fig. 12

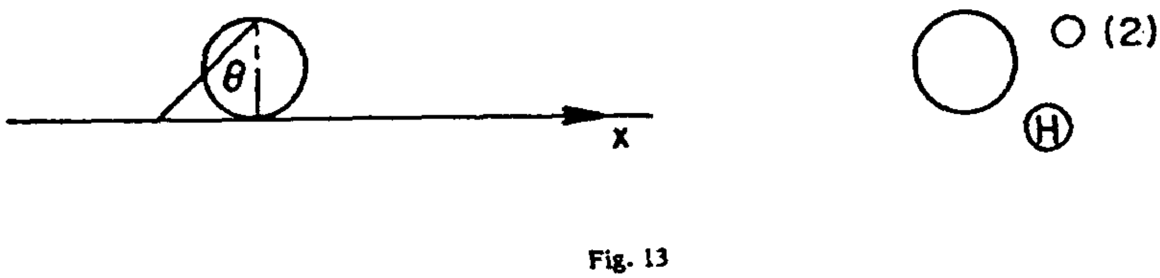

Fig. 13