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Statistical Precision of Delayed-Neutron Nondestructive Assay Techniques

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OAK RIDGE NATIONAL LABORATORY
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ORNL/TM-6569
Distribution
Category UC-77

Contract No. W-7405-eng-26

METALS AND CERAMICS DIVISION

HTGR Fuel Recycle Development Program
(189a OH045)

Fuel Refabrication — Task 500

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Date Published - February, 1979

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DEPARTMENT OF ENERGY

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STATISTICAL PRECISION OF DELAYED-NEUTRON NONDESTRUCTIVE ASSAY TECHNIQUES

C. K. Bayne* and S. R. McNeany†

ABSTRACT

The purpose of this paper is to present a theoretical analysis of the statistical precision of delayed-neutron nondestructive assay instruments. Such instruments measure the fissile content of nuclear fuel samples by neutron irradiation and delayed-neutron detection. The precision of these techniques is limited by the statistical nature of the nuclear decay process, but the precision can be optimized by proper selection of system operating parameters.

Our method is a three-part analysis. We first present differential-difference equations describing the fundamental physics of the measurements. We then derive and present complete analytical solutions to these equations. Final equations governing the expected number and variance of delayed-neutron counts were computer programmed to calculate the relative statistical precision of specific system operating parameters.

Our results show that Poisson statistics do not govern the number of counts accumulated in multiple irradiation-count cycles and that, in general, maximum count precision does not correspond with maximum count as first expected. Covariance between the counts of individual cycles must be considered in determining the optimum number of irradiation-count cycles and the optimum irradiation-to-count time ratio. For the assay system in use at ORNL, covariance effects are small, but for systems with short irradiation-to-count transition times, covariance effects force the optimum number of irradiation-count cycles to be half those giving maximum count.

We conclude that the equations governing the expected value and variance of delayed-neutron counts have been derived in closed form. These have been computerized and can be used to select optimum operating parameters for delayed-neutron assay devices.

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INTRODUCTION

Nondestructive assay, commonly abbreviated as NDA, is a title given to a class of measurement techniques used to determine the isotopic or elemental composition of nuclear fuel. As the name implies, the techniques do not destroy the sample being measured. Usually, nuclear radiation either inherently associated with the fuel or induced by an external source is detected, and its intensity is related to the quantity of the isotope or element being measured.

The NDA technique examined in this report is one in which fission events are induced in a fuel sample by neutron irradiation after which delayed-fission neutrons are detected and related to the sample's fissionable material content. The primary focus of this study is to determine the statistical precision of measurements made in a delayed-neutron type of assay device used at ORNL to determine the fissile content of HTGR fuel rods and particle samples. However, the analysis is sufficiently general to apply to all NDA devices based on delayed-neutron detection. The assay device at ORNL was developed by the HTGR Fuel Recycle Program and is described elsewhere.[1,2]

DESCRIPTION OF PROBLEM

The purpose of a delayed-neutron type of NDA device is to measure the fissile or fissionable material content in nuclear fuel materials. Typical applications examine items such as LWR fuel pellets, HTGR fuel rods and particle samples, samples of fuel solutions and powders, and fuel fabrication wastes. The results of these measurements usually indicate the total amount of fissile isotopes such as ^{233}U, ^{235}U, and ^{239}Pu, or fertile isotopes such as ^{232}Th and ^{238}U present in the fuel sample.

The technique used to make these measurements is simple. A sample is placed in a neutron flux to induce fissions. A thermal or epithermal flux is used to determine fissile isotopes and a fast-neutron flux is used to determine total fissionable isotopes, both fissile and fertile. After an

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appropriate irradiation period, the sample is removed from the neutron flux — by physical transport, shielding, or turning off the neutron source — and placed in a neutron detection environment where delayed-fission neutrons are counted. In general, the sample is then returned for further neutron irradiation, followed by delayed neutron counting. This cyclic operation continues for a specified time. The total number of counts accumulated is then translated into a fissile or total fissionable content through the use of a calibration curve. For the remainder of this report, only fissile assays will be considered, but the same results apply to measurements of total fissionable material.

A calibration curve is a mathematical relationship between the number of counts accumulated and the fissile content of the sample. It is determined by measuring the counts obtained from sample standards of well-known fissile content. An equation of the form in Eq. (1) has been found[3] to closely approximate the calibration data:

$$C = A(1 - e^{-BX}) , \qquad (1)$$

where C is the total number of counts accumulated, A and B are constants, and X is the amount of fissile material in the sample.

Since C is the measured quantity and X is the value to be determined, a more useful form of the calibration formula is given by Eq. (2).

$$X = \frac{1}{B} \ln\left(\frac{A}{A - C}\right) \qquad (2)$$

The constants A and B are determined by measuring standard fuel samples of known fissile content. Since these constants are determined infrequently, extra time is devoted to obtaining accurate and precise values for them. Consequently, for routine measurements, the statistical precision of the calculated fissile contents, X, depends primarily on the statistical precision of the measured counts, C. Propagation-of-error principles applied to Eq. (2) show the relative error in calculations of X to be governed by

$$\frac{\sigma_X}{X} \approx \frac{\sigma_C}{(A-C)\ln\left(\frac{A}{A-C}\right)} , \quad (3)$$

where σ_X and σ_C are the standard deviations of X and C, respectively. In most cases, calibration curves are reasonably linear (that is, A >> C). Thus, Eq. (3) can be reduced to

$$\frac{\sigma_X}{X} \approx \frac{\sigma_C}{C} . \quad (4)$$

In words, Eq. (4) expresses that the relative error of an assay for fissile content approximately equals the relative error in the number of counts measured. Assays of maximum precision require minimum relative error in the total number of counts accumulated.

The precision in the counts can result from two independent sources: (1) random variations in the system's operating times (that is, counting time, irradiation time, etc.) and (2) statistical precision due to the natural randomness of the nuclear decay process. The precision due to source (1) depends on the precision of the timing controls of the assay device, whereas that due to source (2) is a fundamental limiting factor and as such is the topic of study in this report. Contrary to other nuclear counting systems, minimizing statistical counting error does not necessarily correspond to maximizing the count. This is because covariance exists between the counts accumulated in individual cycles. This principle is applied to the following situation.

Consider a fuel sample physically shuttled between a thermal neutron-irradiation device and a high-efficiency neutron detector. Beginning with the sample in the irradiation position, one cycle of the transport system will consist of an irradiation period of duration t_I, a transport to the detector requiring a time $t_{I \to C}$, a counting period of length t_C, and finally, a transport back to the irradiation chamber of time $t_{C \to I}$. (See Fig. 1.)

During an irradiation period, the fission rate throughout the fuel sample is $\overline{\Sigma}_f \phi_{Th}$ fissions per cm^3/sec, where $\overline{\Sigma}_f$ is an average macroscopic

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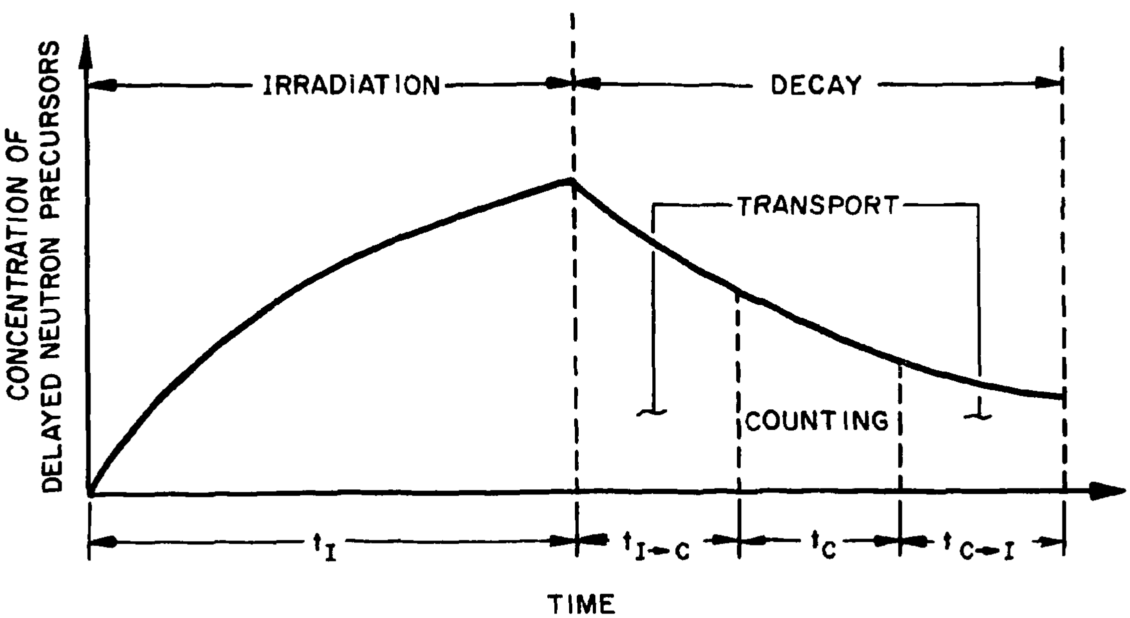

Fig. 1. The formation and decay of delayed-neutron precursors during the first cycle.

fission cross section and ϕ_{Th} is the thermal neutron flux. As a result, $\nu\bar{\Sigma}_f\phi_{Th}$ fission neutrons are produced, of which a fraction, β, are delayed, where ν is the average number of neutrons released per fission. Since one delayed neutron is emitted per precursor, $\beta\nu\bar{\Sigma}_t\phi_{Th}$ is the precursor formation rate. Noting that discrete groups of precursors are formed, $\beta_i\nu\bar{\Sigma}_f\phi_{Th}$ is the formation rate of the i^{th} precursor group. For simplicity, the distributional properties are derived for only one group of delayed-neutron precursors and the corresponding distributional properties for several groups of precursors can be easily derived on the basis of the independence of formation of two or more groups of precursors. Letting $N(t)$ represent the concentration (precursors/cm^3) of delayed neutron precursors at time t, the probability distributions of $N(t)$ for the irradiation segment and decay segment of a cycle will be derived in terms the decay constant, $\lambda(s^{-1})$, the rate of precursor formation, $\alpha = \beta\nu\bar{\Sigma}_f\phi_{Th}$; and the operating times, t_I, $t_{I\to C}$, t_C, and $t_{C\to I}$. From these derived distributions, the mean and variance of the total number of counts can be calculated.

Several properties of the Poisson distribution need to be considered before the distribution of the number of precursors can be derived. A discrete random variable X has a Poisson probability distribution with parameter $\mu > 0$, if the density function X is of the form:

$$\Pr(X = x) = \begin{cases} \frac{\mu^x}{x!} e^{-\mu} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{elsewhere} . \end{cases} \tag{5}$$

For the Poisson probability distribution, the expected value of X and the variance of X are $E(X) = \mu$ and $Var(X) = \mu$, respectively.

One method of deriving the distribution of a random variable is to show that its probability generating function is equal to a known probability generating function. The probability generating function for a discrete random variable X is unique and is defined by:[4]

$$G(s) = \sum_{x=0}^{\infty} s^x \Pr(X = x) \quad \text{for} \quad |s| \leq 1 . \tag{6}$$

For the Poisson probability distribution with parameter μ, the corresponding generating function is:

$$G(s) = exp[(s-1)\mu] . \tag{7}$$

In many counting statistic problems, the probability distribution of the sum of two Poisson random variables is needed. If X and Y are independent Poisson random variables with parameters μ_1 and μ_2, respectively, then the random variable $X + Y$ is also a Poisson random variable with parameter $\mu_1 + \mu_2$. However, Feller[5] shows that if the Poisson random variables X and Y are not independent, then the sum of the two random variables is not a Poisson distribution, and the mean and variance of the sum cannot be directly written down even if the individual means and variances are known. The correlation between random variables is at the heart of the problem of finding the mean and variance of the sum of the total number of counts. In this case, as in many cases, the assumption of independence is inappropriate.

PROBABILITY DISTRIBUTION MODELS

In this section, the probability distribution of the number of delayed-neutron precursors for a given precursor group is shown to be Poisson in both the irradiation and decay segments of each cycle. In addition, the number of counts accumulated in any individual cycle is also shown to be Poisson distributed with a parameter that is a function of the Poisson parameter for the number of delayed-neutron precursors present at the end of the irradiation segment of the first cycle. However, the accumulated counts over several cycles is shown not to be Poisson distributed.

Irradiation Segment of First Cycle

Let the random variable $N(t)$, $0 \leq t \leq t_I$, represent the number of delayed-neutron precursors at time t during the irradiation segment of the first cycle. The probability that n delayed-neutron precursors exist at time t, $\Pr[N(t) = n] = \Pr(n,t)$ can be written as a stochastic-difference equation in the time interval $(t, t + \Delta t)$:

$$\begin{aligned} \Pr(n,t+\Delta t) = {} & \Pr(n,t)\cdot\Pr(\text{No precursors form or decay in } \Delta t/N(t) = n) \\ & + \Pr(n-1,t)\cdot\Pr(\text{One precursor forms in } \Delta t/N(t) = n-1) \qquad (8) \\ & + \Pr(n+1,t)\cdot\Pr(\text{One precursor decays in } \Delta t/N(t) = n+1) \ . \end{aligned}$$

Here, we assume that in a sufficiently small interval, at most one event can occur; that is, events cannot happen simultaneously. The three conditional probabilities in Eq. (8) [that is, $\Pr(\cdot/\cdot)$, where "/" is read as "given the value of "] can be given in terms of the precursor formation rate, $\alpha = \beta\bar{\nu}\Sigma_f\phi_{Th}$ and the decay constant, λ. The conditional probability that one precursor decays, given there are $n + 1$ precursors at time t, can be expressed as a binomial probability distribution with $(n + 1)$ and $\lambda\Delta t$ as parameters. This distribution can be expanded in a binomial series in which the higher order terms in Δt can be neglected as Δt approaches 0. The conditional probability of one precursor forming, given that $n - 1$ precursors exist at time t, is independent of the number of existing precursors and depends only on the formation rate of a delayed-neutron precursor. The conditional probability that no precursors form or decay, given that n precursors exist at time t, is one minus the other two conditional probabilities, given the value of $N(t) = n$. Thus, the three conditional probabilities have the value:

$$\begin{aligned}
&\Pr(\text{No precursors form or decay in } \Delta t/N(t) = n) = 1 - \alpha\Delta t - n\lambda\Delta t + 0(\Delta t) \ , \\
&\Pr(\text{One precursor forms in } \Delta t/N(t) = n-1) = \alpha\Delta t \ , \text{ and} \qquad (9) \\
&\Pr(\text{One precursor decays in } \Delta t/N(t) = n+1) = (n + 1)\lambda\Delta t + 0(\Delta t) \ ,
\end{aligned}$$

where $0(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. Substituting the three values in Eq. (9) into Eq. (8) gives:

$$\begin{aligned}
\Pr(n,t+\Delta t) = {} & (1 - \alpha\Delta t - n\lambda\Delta t)\Pr(n,t) + \alpha\Delta t\Pr(n-1,t) \\
& + (n + 1)\lambda\Delta t\Pr(n+1,t) + 0(\Delta t) \ . \qquad (10)
\end{aligned}$$

By transposing $\Pr(n,t)$ to the left side, dividing by Δt, and taking the limit $\Delta t \to 0$, one obtains the differential-difference equation:[6,7]

$$\frac{\partial \Pr(n,t)}{\partial t} = -(\alpha + n\lambda)\Pr(n,t) + \alpha\Pr(n-1,t) + (n + 1)\lambda\Pr(n+1,t) \ . \qquad (11)$$

From Eq. (11) the time-dependent probability generating function of $N(t)$,

$$G(s,t) = \sum_{n=0}^{\infty} s^n \Pr(n,t) \quad \text{for} \quad |s| \leq 1, \quad \text{and} \quad 0 \leq t \leq t_I \ , \qquad (12)$$

can be shown to be a probability generating function of a Poisson distribution. Equation (11) is first substituted into $\partial G(s,t)/\partial t$ and the probability generating function is a solution to the initial value partial differential equation:

$$\frac{\partial G(s,t)}{\partial t} = \alpha(s - 1)G(s,t) - \lambda(s - 1)\frac{\partial G(s,t)}{\partial s} \tag{13}$$

with initial value:

$$G(s,0) = 1 \ . \tag{14}$$

The initial value, Eq. (14), indicates that no delayed-neutron precursors exist before irradiation begins. The solution to Eq. (13) subject to initial condition Eq. (14) is:

$$G(s,t) = exp[(s-1)(\frac{\alpha}{\lambda})(1-e^{-\lambda t})] \ , \tag{15}$$

which is the form of the Poisson generating function with the parameter:

$$\mu(t) = (\frac{\alpha}{\lambda})(1 - e^{-\lambda t}) \quad \text{for} \quad 0 \leqslant t \leqslant t_I \ . \tag{16}$$

Decay Segment

At the end of the irradiation segment, the fuel sample is shuttled to a neutron detector to be counted and then shuttled back to the irradiation device. During the time when the fuel sample is not in the irradiation device ($t_D = t_{I \to C} + t_C + t_{C \to I}$), the number of precursors are decaying. The initial-value, partial differential equation for the decay segment can be derived in a similar manner to Eq. (13) with the exception that the precursor formation rate is zero (that is, $\alpha = 0$), and the initial value condition depends on the expected number of delayed-neutron precursors at the end of the irradiation segment. Therefore, the initial value partial differential equation of the probability generating function that describes the distribution of the number of delayed-neutron precursors during the decay segment of the first cycle, $[N(t),\ t_I \leq t \leq t_I + t_D)]$ is:

$$\frac{\partial G(s,t)}{\partial t} = -\lambda(s - 1)\frac{\partial G(s,t)}{\partial s} \ , \tag{17}$$

with the initial value condition:

$$G(s,t_I) = exp[(s-1)\mu(t_I)] \ . \tag{18}$$

The function $\mu(t_I)$ is the expected value of the number of delayed-neutron precursors at the end of the first irradiation segment and it is found by evaluating Eq. (16) at $t = t_I$. The solution to Eq. (17) is the Poisson probability-generating function:

$$G(s,t) = exp[(s-1)\mu(t_I)e^{-\lambda(t-t_I)}] \quad \text{for} \quad t_I \leq t \leq t_I + t_D \tag{19}$$

with the Poisson parameter:

$$\mu^*(t) = \mu(t_I)e^{-\lambda(t-t_I)} \ . \tag{20}$$

The asterisk on the Poisson parameter, $\mu^*(t)$, indicates the decay segment of a cycle while the Poisson parameter, $\mu(t)$ without the asterisk, represents the irradiation segment of a cycle.

Equations (17) and (18) describe the probability-generating function for the decay segment for any cycle if the value of $\mu(t_I)$ represents the expected number of precursors at the end of the previous irradiation segment. If the fuel sample is cycled for H cycles, then the expected value of the number of delayed-neutron precursors during the h^{th} cycle is:

$$\mu^*(t) = \mu[(h-1)T+t_I]e^{-\lambda[t-(h-1)T-t_I]} \quad \text{for} \quad (h-1)T + t_I \leq t \leq hT \ , \tag{21}$$

where $T = t_I + t_D$ the total time of one cycle.

Before a similar formula [Eq. (21)] can be written for the irradiation segment, the distribution of the number of delayed-neutron precursors in an irradiation segment must be derived for the case when there is a nonzero number of delayed-neutron precursors remaining after the decay segment.

General Irradiation Segment

For any cycle the probability generating function of the number of delayed-neutron precursors during the irradiation segment is described by the initial-value partial differential equation Eq. (13). However, the initial-value condition of each cycle depends on the expected number of delayed-neutron precursors at the end of the previous decay segment. For the first cycle, no delayed neutron precursors existed before the irradiation segment began, so the initial value condition is given by Eq. (14). For the h^{th} cycle, h = 2, 3, ..., H, when the irradiation starts at time $(h-1)T$, the initial value condition is:

$$G[s,(h-1)T] = exp\left\{(s-1)\mu^*[(h-1)T]\right\} \tag{22}$$

The solution to partial differential equation Eq. (13) using initial value Eq. (22) is:

$$G(s,t) = exp\left\{(s-1)\left[(\tfrac{\alpha}{\lambda})\left(1-e^{-\lambda[t-(h-1)T]}\right) + \mu^*[(h-1)T]e^{-\lambda[t-(h-1)T]}\right]\right\},$$

$$\text{for} \quad (h-1)T \leqslant t \leqslant (h-1)T + t_I \ . \tag{23}$$

Equation (23) is the probability generating function of the Poisson distribution function for the irradiation segment of the h^{th} cycle with Poisson parameter:

$$\mu(t) = (\tfrac{\alpha}{\lambda})\left(1-e^{-\lambda[t-(h-1)T]}\right) + \mu^*[(h-1)T]e^{-\lambda[t-(h-1)T]}$$

$$\text{for} \quad (h-1)T \leqslant t \leqslant (h-1)T + t_I \ . \tag{24}$$

By substituting Eq. (21) into Eq. (24), a difference equation can be derived for the Poisson parameter at the end of the irradiation segment of the h^{th} cycle with the initial condition given by Eq. (16) at $t = t_I$:

$$\mu(t_I) = (\tfrac{\alpha}{\lambda})(1-e^{-\lambda t_I}) \quad \text{for} \quad h = 1, \text{ and} \tag{25}$$

$$\mu[(h-1)T+t_I] = \mu[(h-2)T+t_I]e^{-\lambda T} + \mu(t_I)$$

$$\text{for} \quad h = 2, 3, \ldots, H \; . \tag{26}$$

The solution to this difference equation is given for any h^{th} cycle as:[8]

$$\mu[(h-1)T+t_I] = \frac{1 - e^{-\lambda hT}}{1 - e^{-\lambda T}} \mu[t_I] \quad \text{for} \quad h = 1, 2, \ldots, H \; . \tag{27}$$

Equation (27) gives an easy way to calculate the expected value of the number of delayed-neutron precursors at the end of the h^{th} cycle by using the expected value at the end of the first irradiation segment. A similar relationship can be derived for the decay segment by substituting Eq. (27) into the recursive decay equation Eq. (21) and evaluating t at hT:

$$\mu^*(hT) = \left(\frac{1 - e^{-\lambda hT}}{1 - e^{-\lambda T}} \right) e^{-\lambda t_D} \mu(t_I) \; . \tag{28}$$

Equations (27) and (28) completely determine the expected value and variance of the number of delayed-neutron precursors at the end of the irradiation segment and decay segment, respectively, of any cycle, given the precursor formation rate, the precusor decay constant, the time of the irradiation segment, and the time of the decay segment.

Number of Counts

After the fuel sample is transported to the delayed-neutron detector, the number of delayed-neutron precursors that decay are counted. The number of counts depends on the number of delayed-neutron precursors at the end of the previous irradiation segment as well as on the efficiency of the detector.

Let: C_h = Number of counts in the h^{th} cycle at the end of counting time t_C,

p = Probability that a precursor existing at the end of an irradiation segment decays during the corresponding counting interval t_C,

$p = (1 - e^{-\lambda t_C})\ e^{-\lambda t_{I \to C}}$, and

ε = Probability of counting a decay or the efficiency of the delayed-neutron detector.

Then the conditional probability of the number of counts given the number of delayed-neutron precursors at the end of the irradiation segment of the h^{th} cycle is a binomial distribution:[9]

$$\Pr(C_h = c_h / N_h = n_h) = \binom{n_h}{c_h} (\varepsilon p)^{c_h} (1 - \varepsilon p)^{n_h - c_h} \tag{29}$$

where: $c_h = 1, 2, \ldots, n_h$. The stochastic random variables and parameters used to derive the distributional properties of the number of counts are evaluated at either the end of the irradiation segment or the end of the decay segment. Thus they will be indexed by only the corresponding cycle. The unconditional probability of the number of counts is found by multiplying the conditional probability Eq. (29) by the unconditional Poisson probability for the number of delayed-neutron precursors at the end of the irradiation segment and summing over all possible values of n_h.

$$\Pr(C_h = c_h) = \sum_{n_h = 0}^{\infty} \binom{n_h}{c_h} (\varepsilon p)^{c_h} (1 - \varepsilon p)^{n_h - c_h} \Pr(N_h = n_h) \ . \tag{30}$$

To find the form of Eq. (30) the probability generating function of $Pr(C_h = c_h)$ can be evaluated.

$$G(s,t) = \sum_{c_h=0}^{\infty} s^{c_h} Pr(C_h = c_h) \ , \tag{31}$$

$$G(s,t) = exp[(s-1)\varepsilon p \mu_h] \ . \tag{32}$$

Hence, the unconditional probability distribution of the number of counts is also a Poisson distribution with Poisson parameter $\varepsilon p \mu_h$. The expected value and variance of the number of counts in the h^{th} cycle follows from the properties of a Poisson distribution:

$$E(C_h) = \left(\frac{1 - e^{-\lambda h T}}{1 - e^{-\lambda T}}\right)\varepsilon p \mu_1 \ , \tag{33}$$

and

$$Var(C_h) = \left(\frac{1 - e^{-\lambda h T}}{1 - e^{-\lambda T}}\right)\varepsilon p \mu_1 \ . \tag{34}$$

To measure the fissile content of a fuel sample, the counts are summed over H cycles to get the total counts,

$$C. = \sum_{h=1}^{H} C_h \tag{35}$$

To find the expected value of the total counts, the sum of the individual expected values is evaluated.

$$E(C.) = \sum_{h=1}^{H} E(C_h) = \frac{\varepsilon p \mu_1}{1 - e^{-\lambda T}} \sum_{h=1}^{H} (1 - e^{-\lambda h T}) \; . \qquad (36)$$

The variance of the total counts is the sum of the variance of each count plus the sum of the covariance between each pair of counts:[10]

$$Var(C.) = \sum_{h=1}^{H} Var(C_h) + 2 \sum_{h=2}^{H} \sum_{k=1}^{h-1} Cov(C_h, C_{h-k}) \; . \qquad (37)$$

The covariance term for any two counts represents a measure of their dependence and is related to their correlation coefficient, $\rho_{h,h-k}$ by $\{Var(C_h)\ Var(C_{h-k})\}^{1/2}\ \rho_{h,h-k} = Cov(C_h, C_{h-k})$. If the two counts are independent, such as in the case when the decay segments are so long that all the delayed-neutron precursors decay to zero, then the value of all the covariance terms will be zero. The difficulty of evaluating the covariances arises from the fact that the sum of dependent Poisson random variables is not distributed as a Poisson random variable.[11] Therefore, the covariance terms have to be derived with the probability distributions found for $N_h(t)$ and C_h.

The special case of $Cov(C_h, C_{h-1})$ is derived first and then the general case follows. Because $E(C_h)$ and $E(C_{h-1})$ are known by Eq. (33) and

$$Cov(C_h, C_{h-1}) = E(C_h C_{h-1}) - E(C_h)\ E(C_{h-1}) \; , \qquad (38)$$

only the expected value of the product $E(C_h C_{h-1})$ needs to be determined. This expected value is defined by:

$$E(C_h C_{h-1}) = \sum_{c_h=0}^{\infty} \sum_{c_{h-1}=0}^{\infty} c_h c_{h-1} Pr(C_{h-1}=c_{h-1}) Pr(C_h=c_h / C_{h-1}=c_{h-1}) \; . \qquad (39)$$

Equation (39) could be evaluated if the conditional expected value:

$$E(C_h/C_{h-1}) = \sum_{c_h=0}^{\infty} c_h \Pr(C_h=c_h/C_{h-1}=c_{h-1}) \tag{40}$$

was known since $\Pr(C_{h-1}=c_{h-1})$ is a Poisson distribution with a Poisson parameter given by Eq. (33). The probability distribution of the counts depends on the expected number of delayed-neutron precursors in the previous irradiation segment. Therefore, the conditional probability in Eq. (40) is determined by first evaluating the expected number of delayed-neutron precursors in the h^{th} cycle at the end of the irradiation segment given that there were c_{h-1} counts in the $(h-1)^{th}$ cycle. To evaluate this expected value, the conditional distribution of the neutron precursors at the end of the $(h-1)^{th}$ irradiation cycle given c_{h-1} counts must be examined:

$$\Pr(N_{h-1}=n_{h-1}/C_{h-1}=c_{h-1}) = \frac{\Pr(N_{h-1}=n_{h-1})}{\Pr(C_{h-1}=c_{h-1})} \Pr(C_{h-1}=c_{h-1}/N_{h-1}=n_{h-1}) \tag{41}$$

The probabilities on the right side of Eq. (41) can be evaluated since the two unconditional probabilities are Poisson distributions and the conditional probability is a binomial distribution:

$$\Pr(N_{h-1}=n_{h-1}/C_{h-1}=c_{h-1}) = \frac{[(1-\varepsilon p)\mu_{h-1}]^{n_{h-1}-c_{h-1}}}{(n_{h-1}-c_{h-1})!} e^{-(1-\varepsilon p)\mu_{h-1}} , \tag{42}$$

where $n_{h-1} = c_{h-1}, c_{h-1} + 1, \ldots .$

The value of n_{h-1} must be as large as the given number of counts. If $\theta = e^{-\lambda t_D}$ is the probability of no decays during the decay cycle, then the conditional probability of the number of delayed-neutron precursors at the end of the decay segment of the $(h-1)^{th}$ cycle, given that both the number of delayed-neutron precursors at the end of the irradiation segment and the number of counts is a binomial distribution in θ. By multiplying this binomial distribution by the conditional probability Eq. (42) and summing over the possible number of delayed-neutron precursors at the end of the irradiation cycle, the conditional probability of the number of delayed-neutron precursors at the end of the decay segment (denoted by a *) of the $(h-1)^{th}$ cycle given just the number of counts, c_{h-1}, can be expressed as:

$$\Pr(N^*_{h-1}=n^*_{h-1}/C_{h-1}=c_{h-1}) = \sum_{x=c_{h-1}}^{\infty} \binom{x}{y} \theta^y (1-\theta)^{x-y} \Pr(N_{h-1}=x/C_{h-1}=c_{h-1}) \ , \quad (43)$$

where $x = n_{h-1}$ and $y = n^*_{h-1}$.

The probability generating function for the conditional probability Eq. (43) is found by summing the corresponding probability generating function over all possible values of $n^*_{h-1} = 0, 1, 2, \ldots, n_{h-1}$:

$$G[s,(h-1)T] = [1 + \theta(s-1)]^{c_{h-1}} \, exp[(s-1)\theta(1-\varepsilon p)\mu_{h-1}] \ . \quad (44)$$

Equation (44) is not a probability generating function of a Poisson distribution, but it does represent the initial condition for the probability generating function at the beginning of irradiation segment of the h^{th} cycle. The probability generating function of the irradiation segment satisfies the initial-value partial differential equation [Eq. (13)]. By imposing the initial condition [Eq. (44)], the solution of Eq. (13) represents the probability generating function of the conditional probability distribution function of the number of delayed-neutron precursors in the irradiation segment of the h^{th} cycle given the number of counts in the $(h-1)^{th}$ cycle.

$$G(s,t) = \left\{1 + (s-1)\theta e^{-\lambda[t-(h-1)T]}\right\}^{c_{h-1}} exp\left\{(s-1)\left([\theta(1-\epsilon p)\mu_{h-1} - \frac{\alpha}{\lambda}] e^{-\lambda[t-(h-1)T]} + \frac{\alpha}{\lambda}\right)\right\},$$

$$\text{for} \quad (h-1)T \leq t \leq (h-1)T + t_I . \qquad (45)$$

The probability generating function [Eq. (45)] is not a Poisson generating function, so the expected value of the number of delayed-neutron precursors conditioned on the number of counts in the previous cycle does not immediately follow. However, the expected value can be found from the definition of the probability generating function by evaluating $\partial G(s,t)/\partial s$ at $s = 1$. The expected value of the number of delayed-neutron precursors given the number of counts in the previous cycle for the end of the irradiation segment of the h^{th} cycle is:

$$E(N_h/C_{h-1}=c_{h-1}) = (1 - \epsilon p)e^{-\lambda T}\mu_{h-1} + \mu_1 + c_{h-1}e^{-\lambda T} . \qquad (46)$$

Now the unconditional expected value of the number of counts can be written as a function of the conditional expected value:

$$E(C_h) = E_N[E_C(C_h/N_h)] \text{ , or}$$

$$E(C_h) = \epsilon p E_N(N_h) . \qquad (47)$$

where the N and C subscripts indicate the probability distribution for the number of delayed-neutron precursors and the conditional probability distribution for the number of counts, respectively, used to evaluate the expected value. Thus, the conditional probability of the number of counts in the h^{th} cycle given the number of counts in the $(h-1)^{th}$ cycle is:

$$E(C_h/C_{h-1}) = \epsilon p E(N_h/C_{h-1}=c_{h-1}) . \qquad (48)$$

The conditional expected value [Eq. (48)] is the term needed to evaluate the expected value in Eq. (39) and consequently, the covariance in Eq. (38):

$$\mathrm{Cov}(C_h\ C_{h-1}) = \varepsilon^2 p^2 e^{-\lambda T}\ \mu_{h-1}\ . \tag{49}$$

To derive the general covariance term $\mathrm{Cov}(C_h, C_{h-k})$; $k = 1, 2, \ldots, h-1$, the recursive relation [Eq. (26)] for the Poisson parameter of the number of delayed-neutron precursors can be used. Suppose the number of counts on the $(h-k)^{th}$ cycle c_{h-k}, is known, then $E(N_{h-k+1}/C_{h-k}=c_{h-k})$ can be calculated by Eq. (46). By the recursive relation Eq. (26), the following conditional expected value can be derived:

$$E(N_h/C_{h-k}=c_{h-k}) = e^{-\lambda(k-1)T} E(N_{h-k+1}/C_{h-k}=c_{h-k}) + \sum_{j=0}^{k-2} e^{-j\lambda T}\ \mu_1\ . \tag{50}$$

From the expressions Eqs. (48) and (50), the general expected value of $E(C_h/C_{h-k})$ can easily be found and this conditional expectation can be used to find the general covariance term:

$$\mathrm{Cov}(C_h, C_{h-k}) = \varepsilon^2 p^2 e^{-\lambda kT}\ \mu_{h-k} \tag{51}$$

or

$$\mathrm{Cov}(C_h, C_{h-k}) = \varepsilon^2 p^2 \mu_1\ \frac{e^{-\lambda kT} - e^{-\lambda hT}}{1-e^{-\lambda T}} \tag{52}$$

Equation (52) can now be substituted into variance [Eq. (37)] to find the variance of the total number of counts for H cycles:

$$\mathrm{Var}(C.) = \frac{\varepsilon p \mu_1}{1-e^{-\lambda T}} \sum_{h=1}^{H} (1-e^{-\lambda hT}) + \frac{2\varepsilon^2 p^2 \mu_1}{1-e^{-\lambda T}} \sum_{h=2}^{H} \sum_{k=1}^{h-1} [e^{-\lambda kT} - e^{-\lambda hT}]\ . \tag{53}$$

The expected value [Eq. (36)] and the variance [Eq. (53)] of the total number of counts are derived for a particular group of precursors with the same formation rate and decay rate. If several groups of delayed-neutron precursors are counted simultaneously, each with different formation rates and decay rates, the expected value and variance of the total number of counts are the sum of the individual expected values [Eq. (36)] and the sum of the individual variances [Eq. (53)], respectively. The variance of the total counts assumes that the different precursor groups are independent. The expected value and variance of the counts for G different groups of delayed-neutron precursors after some algebraic simplification are:

$$E\left(\sum_{g=1}^{G}\sum_{h=1}^{H} C_{gh}\right) = \sum_{g=1}^{G} \frac{(\epsilon p \alpha)_g \left(1-e^{-\lambda_g t_I}\right)}{\lambda_g \left(1-e^{-\lambda_g T}\right)} \left(H - \frac{1-e^{-\lambda_g HT}}{e^{\lambda_g T} - 1}\right) \tag{54}$$

and

$$\begin{aligned} \mathrm{Var}\left(\sum_{g=1}^{G}\sum_{h=1}^{H} C_{gh}\right) = {} & \sum_{g=1}^{G} \frac{(\epsilon p \alpha)_g \left(1-e^{-\lambda_g t_I}\right)}{\lambda_g \left(1-e^{-\lambda_g T}\right)} \left(H - \frac{1-e^{-\lambda_g HT}}{e^{\lambda_g T} - 1}\right) \\ & + 2\sum_{g=1}^{G} \frac{(\epsilon p)_g^2 \alpha_g \left(1-e^{-\lambda_g t_I}\right)}{\lambda_g \left(1-e^{-\lambda_g T}\right)\left(e^{\lambda_g T}-1\right)^2} \left\{ (H-1)\left(e^{\lambda_g T} - e^{-H\lambda_g T}\right) - (H+1)\left(1-e^{-\lambda_g (H-1)T}\right) \right\}, \end{aligned} \tag{55}$$

where the subscript g indicates the parameters for the g^{th} group.

The statistical precision of delayed-neutron assays is best measured by the relative error of the total counts accumulated. This indicator is defined as the ratio of the standard deviation of the total counts [square root of the value of Eq. (55)] to the expected value of the total [Eq. (54)].

$$\text{Relative Error} = (\text{Variance})^{1/2}/(\text{Expected Value}) \tag{56}$$

or

$$\text{Relative Error} = \left[\frac{1}{(\text{Expected Value})} + \frac{\text{Covariance Term}}{(\text{Expected value})^2}\right]^{1/2} \tag{57}$$

The first term on the right hand side of Eq. (57) shows the usual inverse dependence on the square root of the total count typical of Poisson statistics found in many nuclear particle counting systems. The second term includes the covariance term that arises out of the interdependence among the counts accumulated in any cycle and those accumulated in all other cycles.

RESULTS AND CONCLUSIONS

To study the numerical behavior of the equations governing the expected value, variance, and relative error of the total counts [i.e., Eqs. (54), (55), and (57)] resulting from fissile assays, the equations were computer programmed to accept data and output results on an interactive basis. Data describing the nuclear properties (ν, β_g, λ_g, etc.) of fissile isotopes were taken from Lamarsh[12] and stored within the computer program. In this way, only data describing the operating parameters of a delayed-neutron assay system and the isotope being measured need to be supplied to the computer model. Four input operating parameters were required for the program: (1) the total measurement time, T; (2) the total number of cycles, H; (3) the transport times, $t_{I\to C}$ and $t_{C\to I}$; and (4) the ratio of irradiation time to counting time, $R = t_I/t_C$. This data is used by the program to calculate the expected number of counts accumulated during the measurement, the statistical variance of the accumulated counts, and the relative error for the specified fissile nuclide. These results are used to compare the statistical precision of one set of operating parameters relative to another. Parameters that affect the actual magnitude of the expected value and the variance, such as neutron flux level, total fissile content, neutron self-shielding, and detector efficiency were arbitrarily selected. The value of these parameters will depend on each NDA system but the relative results can be used by all NDA systems to determine the operating parameters that will yield the best statistical precision.

This computer study analyzed the effects of the operating parameters on the expected number of accumulated counts and statistical precision and yielded the following results:

1) The expected total accumulated count is maximum at an irradiation-to-count-time ratio of one (R = 1) if all other parameters are constant. In addition, the same expected total count is achieved whether the ratio is R or 1/R. This result is illustrated in Fig. 2, where the expected total counts for three different values of total number of cycles are plotted against the irradiation-to-count-time ratio. The graph is for a total measurement time of 10 min on ^{233}U with transport times of approximately 4 s ($t_{I\rightarrow C} + t_{C\rightarrow I} = 4$ s). The curves for the three cycles are symmetrical about R = 1/R = 1. Analytically this result can be proven by rewriting Eq. (54) in terms of total measurement time and irradiation to count time ratio, R, and showing that the same expression is obtained when 1/R is substituted for R.

2) In most cases the statistical precision associated with fixed total time measurements does not vary rapidly with the number of cycles; however, in many cases a lower relative error can be obtained with fewer cycles than those required to maximize the count. In some cases the same number of cycles is required to minimize the relative error as that required to maximize the count, but in no case is the relative error minimized with more cycles than required to maximize the count. Also, the minimum relative error usually occurs at R > 1. As an illustration of this effect, consider a system with a total transport time per assay cycle of 0.01 s and assume that assays are to be made on a sample of ^{235}U. Table 1 lists the operating parameters required to maximize the count, along with those yielding the minimum relative error for various fixed-time measurements. As seen in the table, improved statistical

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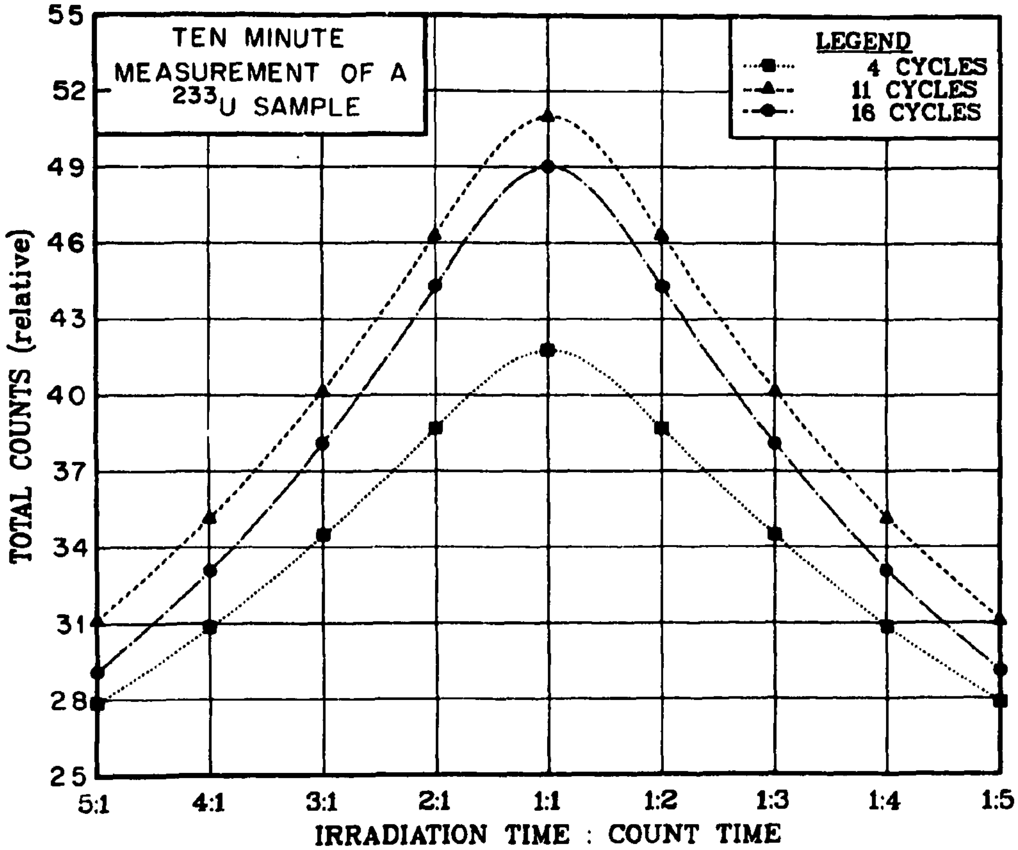

Fig. 2. Expected value of counts accumulated by ORNL's assay device. This value is a function of the irradiation-time-to-count-time ratio and number of irradiation-count cycles for a fixed total measurement time.

precision can be achieved with less than one-half the number of assay cycles required to maximize the count. This improvement means less wear on the physical and electrical components of an assay system and consequently, longer life.

3) The operating parameters that yield the minimum relative error depend strongly on the time that elapses between the neutron irradiation and delayed-neutron detection segments of each assay cycle. Figure 3 presents the optimum operating parameters as a function of total measurement time for three hypothetical assay systems. In each system the time

Table 1. Operating Parameters to Maximize Count and Minimize Relative Error in an Assay System with a Total Transport Time of 0.01 s/cycle

Total Measurement Time (s)	Operating Parameters					
	Maximum Count			Minimum Relative Error		
	Cycle(s)	R[a]	Relative Error	Cycle(s)	R[a]	Relative Error
10	2	1.000	2.523	1	1.000	2.518
20	12	1.000	1.706	5	1.041	1.689
30	22	1.000	1.348	9	1.064	1.337
40	32	1.000	1.143	14	1.079	1.134
50	42	1.000	1.008	20	1.089	1.000

[a]R is the ratio of irradiation time (t_I) to count time (t_C).

lapse between the end of irradiation and the start of counting was taken to be equal to the time between the end of counting and the start of irradiation. The total time lapse (total transport time) on a per cycle basis is listed for each system in the legend of the figure.

As an example of how Fig. 3 is used, consider a system with a total transport time of 1.0 s/cycle and a desire to make an assay measurement of maximum statistical precision using 100 s of time. The figure shows that nine irradiation-count cycles should be used with R = 1.04. The relative error of this measurement would be 3.9 times larger than the relative error obtained measuring the same sample over 1000 s on an identical system with a 0.01-s transport time.

4) The operating parameters yielding minimum relative error in assays using ORNL's delayed-neutron assay system are presented in Table 2. In all cases, the tabulated parameters are within 10% of the parameters required to maximize the count. The reason for this is that the total transport time for the ORNL device is approximately 5 s, causing most of the statistical precision to depend on the total counts with only slight dependence on the covariance term.

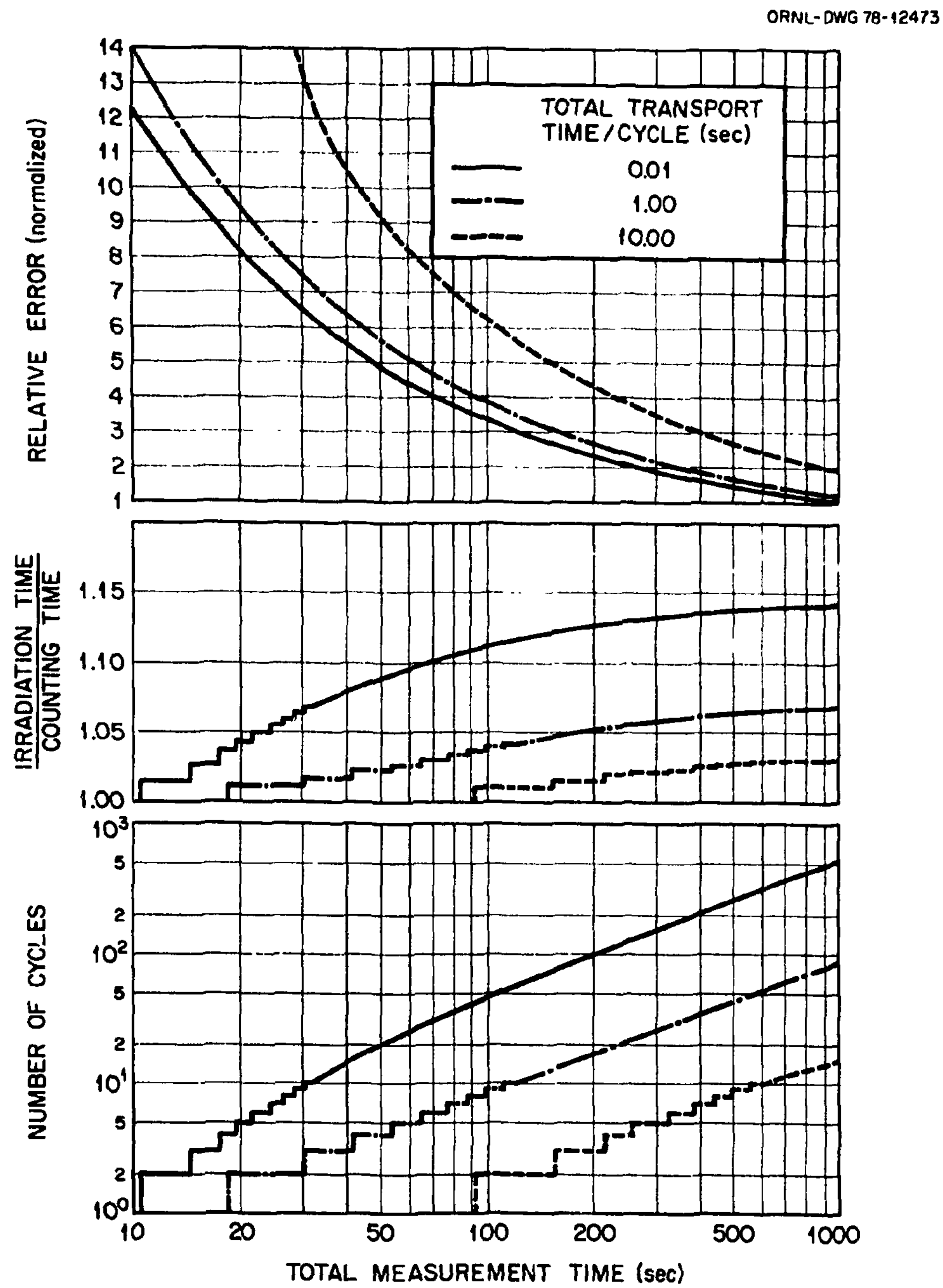

Fig. 3. Operating parameters that yield the minimum relative error in assays of ^{235}U.

Table 2. Operating Parameters Yielding Minimum Relative Error in Assays Made on ORNL's Delayed-Neutron Assay Device

Total Measurement Time (min)	Operating Parameters					
	^{235}U Samples			^{233}U Samples		
	Cycle(s)	R^a	Relative Error	Cycle(s)	R^a	Relative Error
1	1	1.000	18.67	1	1.000	27.07
10	15	1.040	5.525	12	1.041	7.997
30	45	1.045	3.171	38	1.048	4.583
60	90	1.045	2.239	76	1.048	3.234
300	451	1.045	1.000	383	1.048	1.444

[a] R is the ratio of irradiation time (t_I) to count time (t_C).

ACKNOWLEDGMENTS

The authors wish to express their appreciation to D. G. Wilson of the Mathematical and Statistical Research Department (MSRD) for his assistance with the solutions of the partial differential equations, to L. R. Shenton of the University of Georgia for his suggestions on how to find the covariance of the total number counts, and to J. D. Jenkins of the Engineering Technology Division and K. O. Bowman of MSRD for their assistance in reviewing the draft report. Thanks go to H. R. Livesey for illustrations and to Sandi Lyttle for preparing the manuscript for publication.

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