INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

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by a Frequency near the Ion Cyclotron Frequency

Kazuo Minami* and Kazunori Sato*

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RESEARCH REPORT

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan

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* Permanent Address; Department of Electric and Electronic Department, Tokyo Institute of Technology, Tokyo, 152 Japan

Synopsis

The dynamic shear stabilization of the hydrcmagnetic instability in low-beta plasmas by an axial RF current whose frequency is not much smaller than the ion cyclotron frequency ω_{ci} is analyzed in some detail. We adopt the simple model of a uniform plasma column with infinite conductivity. Attention is limited to the case of the $m = 1$ kink mode with long wave lengths. The Mathieu equation, in which the effect of the ion cyclotron motion is taken into account, is derived. It is shown that the dynamic shear stabilization is still effective, even if the frequency of the applied RF current is of the order of ω_{ci}, which is considerably higher than the frequencies believed to be available in the previous analyses.

It is well known that the longitudinal currents in pinches and tokamaks are restricted because of destabilization of hydromagnetic kink modes. Therefore, the dynamic stabilization[1-4] of such instabilities is believed to be an important problem in fusion research. One simple way of the dynamic stabilization is to superpose longitudinal RF current on the static and/or quasi-static current[5]. The RF current produces a poloidal RF magnetic field near the periphery of the plasma column. Dynamic shear stabilization is possible by that poloidal RF magnetic field. In literature , dynamic shear stabilization was analyzed in the past by means of Mathieu equation which was based on the linear dispersion relation of the kink modes obtained from one-fluid MHD equations. As is well known, the MHD equations are applicable in the case that the characteristic frequency, ω, of the phenomena is much smaller than the ion cyclotron frequency ω_{ci} of applied static magnetic field. However, the statement $\omega \ll \omega_{ci}$ is not necessarily satisfied in the real experiments. For example, consider a $z-\theta$ pinch plasma of helium with axial magnetic field of 1 (T), where $\omega_{ci}/2\pi \simeq 4$ (MHz). The growth rates of the kink instabilities in such a plasma are often of the order of several hundreds of KHz. Then, the frequency of RF magnetic field for stabilization might not be less than 1 MHz. In such a case, the assumption $\omega \ll \omega_{ci}$ is no longer valid.

It is the purpose of this paper to analyze the dynamic shear stabilization with a frequency not much less than ω_{ci}.

We consider an infinitely long straight plasma column with radius a and with uniform mass density ρ. The conductivity of the plasma is assumed to be infinite. Then, the current flows on the surface of the column. We assume a uniform magnetic field B_z inside the plasma, whereas outside the plasma B_θ and B_z in azimuthal and axial directions, respectively, are assumed. We confine our attention to the incompressible perturbation bacause it may be the most dangerous among various perturbations.[6] We assume that the perturbation is proportional to $\exp i(m\theta + kz)$.

We start from the linearized equations of Magneto-Ion Dynamics[7] given by

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_z \ , \tag{1}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_z) - \frac{M}{e} \frac{\partial}{\partial t} \nabla \times \mathbf{v} \ . \tag{2}$$

The displacement vector $\boldsymbol{\xi}$ given by

$$\mathbf{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{3}$$

is introduced for convenience. Then, eq. (2) is integrated in time as

$$\mathbf{B}_1 = ik B_z \boldsymbol{\xi} - \frac{M}{e} \frac{\partial}{\partial t} \nabla \times \boldsymbol{\xi} \ . \tag{4}$$

Using eqs. (3) and (4), eq. (1) is written as follows,

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla \times p^* + \frac{ikB_z}{\mu_0} \left(ikB_z \boldsymbol{\xi} - \frac{M}{e} \frac{\partial}{\partial t} \nabla \times \boldsymbol{\xi} \right) , \tag{5}$$

where

$$p^* = p + \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{B}_z \ .$$

We limit our analysis to the case $ak \ll 1$, then we may put $\xi_z = 0$. Assuming that $\boldsymbol{\xi} \propto \exp(i\omega t)$, eq. (5) is decomposed to r and θ components, respectively, as

$$(C_A^2k^2-\omega^2)\,\xi_r - \frac{i\,C_A^2k^2\omega}{\omega_{ci}}\,\xi_\theta = -\frac{d}{dr}\frac{p^*}{\rho}\,, \tag{6}$$

$$\frac{i\,C_A^2k^2\omega}{\omega_{ci}}\,\xi_r + (C_A^2k^2-\omega^2)\,\xi_\theta = -\frac{im}{r}\frac{p^*}{\rho}\,. \tag{7}$$

From eq. (5) one directly obtains $\nabla^2 p^* = 0$, because the incompressible displacement, i.e., $\nabla\cdot\boldsymbol{\xi} = 0$, is assumed.[6)] Then $p^*(r)$ is expressed as

$$p^*(r) = p^*(a)\,\frac{I_m(rk)}{I_m(ak)}\,e^{i(m\theta+kz)}\,. \tag{8}$$

From eqs. (6) $\sim$ (8), $\xi_r(a)$ is solved as

$$\xi_r(a) = \frac{p^*(a)}{\rho}\cdot\frac{k\,\frac{I_m'(ak)}{I_m(ak)}\,(\omega^2-C_A^2k^2)+\frac{m}{a}\frac{\omega}{\omega_{ci}}\,C_A^2k^2}{(\omega^2-C_A^2k^2)^2-\frac{\omega^2}{\omega_{ci}^2}\,C_A^4k^4}\,. \tag{9}$$

The perturbation of the magnetic field outside the column is given by $\mathbf{B}_{1e} = \nabla\psi$ and $\nabla^2\psi = 0$, which result in

$$B_{re} = \frac{\partial\psi}{\partial r} = C\,\frac{k\,K_m'(rk)}{K_m(ak)}\,, \tag{10}$$

where C is a constant. The boundary condition that the pressure at the plasma-vacuum interface should be continuous is expressed

$$p^*(a) = \frac{i}{\mu_0\rho}\left(kB_z+\frac{m}{a}B_{\theta a}\right)C - \frac{B_{\theta a}^2}{\mu_0\rho a}\,\xi_r(a)\,, \tag{11}$$

where $B_{\theta a}$ is the external poloidal magnetic field at $r = a$. Another boundary condition is the continuity of the magnetic field perpendicular to the surface given by

$$Ck\,\frac{K_m'(ak)}{K_m(ak)} = i\left(kB_z+\frac{m}{a}B_{\theta a}\right)\xi_r(a)\,, \tag{12}$$

where eq. (10) has been used. Eliminating C, $\xi_r(a)$ and $p^*(a)$ from eqs. (9), (11) and (12), the following relation is derived after some calculation,

$$(\omega^2 - C_A^2 k^2)^2 - \frac{\omega^2}{\omega_{ci}^2} C_A^4 k^4 + \left\{ -\frac{k I_m'(ak)}{I_m(ak)} (\omega^2 - C_A^2 k^2) \right.$$

$$\left. + \frac{m}{a}\frac{\omega}{\omega_{ci}} C_A^2 k^2 \right\} \times \left\{ \frac{1}{\mu_0 \rho}\left(kB_z + \frac{m}{a}B_{\theta a}\right)^2 \frac{K_m(ak)}{k K_m'(ak)} + \frac{B_{\theta a}^2}{\mu_0 \rho a} \right\} = 0. \quad (13)$$

If we limit our attention to the case $ak \ll 1$, eq. (13) is simplified, by using the formula $I_m'(ak)/I_m(ak) \simeq m/ak$, $K_m'(ak)/K_m(ak) \simeq -m/ak$, as

$$\left(\omega^2 - C_A^2 k^2 + \frac{\omega}{\omega_{ci}} C_A^2 k^2\right) \left\{ \omega^2 - C_A^2 k^2 - \frac{\omega}{\omega_{ci}} C_A^2 k^2 \right.$$

$$\left. - \frac{1}{\mu_0 \rho}\left(kB_z + \frac{m}{a} B_{\theta a}\right)^2 + \frac{m B_{\theta a}^2}{\mu_0 \rho a^2} \right\} = 0 , \quad (14)$$

where $C_A = B_z/(\mu_0 \rho)^{\frac{1}{2}}$. Therefore, the following dispersion relations are obtained

$$\omega^2 - C_A^2 k^2 + \frac{\omega}{\omega_{ci}} C_A^2 k^2 = 0 , \quad (15)$$

(ion cyclotron waves)

$$\omega^2 - C_A^2 k^2 - \frac{\omega}{\omega_{ci}} C_A^2 k^2 - C_A^2 \left\{ k^2 + \frac{2mk}{a}\frac{B_{\theta a}}{B_z} + \frac{m(m-1)}{a^2}\frac{B_{\theta a}^2}{B_z^2} \right\} = 0. \quad (16)$$

(kink modes)

Equation (16) includes an additional term, i.e., the third term, to the usual dispersion relation of kink modes. We investigate eq. (16), in some detail, for the case of the $m = 1$ mode, which is, as is well known, the most dangerous among various m modes. Solving eq. (16) for ω, one finds that the Kruskal-Shafranov condition[6] (K-S condition, hereafter), where $Im(\omega) = 0$, is modified to

$$\left(\frac{c}{\omega_{pi}}\right)^2 k^3 + 8\left(k + \frac{1}{a}\frac{B_{\theta a}}{B_\theta}\right) = 0\,, \tag{17}$$

where ω_{pi} is the ion plasma frequency and c is the velocity of light in vacuum. The first term is the correction by the ion cyclotron motion to the usual K-S condition. It is shown in eq. (17) that the K-S condition moves toward smaller k's due to the ion cyclotron correction. The correction, however, is small, since we have assumed that $ak \ll 1$. In other words, the ion cyclotron correction is not important near K-S condition, since $|\omega| \ll \omega_{ci}$. On the other hand, the correction can be significant where the growth rate of the kink instability is the same order as ω_{ci}. From eq. (16), the presence of the correction of the ion cyclotron motion no longer permits a purely growing solution of kink modes, but introduces a small real part to the frequency ω. This raises the interesting possibility of the nonlinear ion cyclotron damping of kink modes when their phase velocities in axial direction are comparable to the ion thermal speed. When the correction is small, eq. (16) is solved approximately as,

$$\omega \simeq \frac{1}{2}\left[\frac{C_A^2 k^2}{\omega_{ci}} \pm i\left\{2C_A\sqrt{-2k\left(k+\frac{1}{a}\frac{B_{\theta a}}{B_z}\right)} - \frac{1}{4C_A\sqrt{-2k\left(k+\frac{1}{a}\frac{B_{\theta a}}{B_z}\right)}}\,\frac{C_A^4 k^4}{\omega_{ci}^2}\right\}\right]. \tag{18}$$

As is shown in eq. (18), the kink mode not only acquires a small real part but also has a decreased growth rate.

Since eq. (16) is a linear dispersion relation, the original equation of motion for ξ_r may be obtained by putting $\omega = i d/dt$, resulting in

$$\frac{d^2\xi_r}{d\tau^2} \pm i\,\frac{C_A^2k^2}{\omega_{ci}}\frac{d\xi_r}{d\tau} + 2C_A^2k\left(k+\frac{1}{a}\frac{B_{\theta a}}{B_z}\right)\xi_r = 0\,. \qquad (19)$$

Replacing $B_{\theta a}$ by $B_{\theta a} + \tilde{B}_{\theta a}\cos\omega_0 t$, and putting $\tau = \omega_0 t/2$, one finds the following equation,

$$\frac{d^2\xi_r}{d\tau^2} \pm i\delta\,\frac{d\xi_r}{d\tau} + (\theta_0 + 2\theta_1\cos 2\tau)\xi_r = 0\,, \qquad (20)$$

where

$$\left.\begin{aligned}
\delta &= \pm\,\frac{2C_A^2k^2}{\omega_{ci}\,\omega_0}\,,\\
\theta_0 &= -\,\frac{8C_A^2k^2}{\omega_0^2}\left(\frac{1}{q}-1\right),\\
\theta_1 &= \frac{4C_A^2k}{\omega_0^2 a}\,\frac{\tilde{B}_{\theta a}}{B_z}\,,\\
q &= -\,\frac{akB_z}{B_{\theta a}}\,,
\end{aligned}\right\} \qquad (21)$$

and $\tilde{B}_{\theta a}$ is the poloidal RF magnetic field at the plasma surface caused by the axial RF current with frequency $\omega_0/2\pi$. Equation (20) is transformed, by putting $\xi_r = \eta\exp(\kappa\tau)$, as

$$\frac{d^2\eta}{d\tau^2} + (2\kappa \pm i\delta)\frac{d\eta}{d\tau} + (\kappa^2 \pm i\kappa\delta + \theta_0 + 2\theta_1\cos 2\tau)\eta = 0\,. \qquad (22)$$

If one puts $2\kappa \pm i\delta = 0$ in eq. (22), one obtains the well-known Mathieu equation given by[8]

$$\frac{d^2\eta}{d\tau^2} + (\theta_0' + 2\theta_1\cos 2\tau)\,\eta = 0\,, \qquad (23)$$

$$\theta_0' = \frac{\delta^2}{4} + \theta_0\,, \qquad (24)$$

$$\xi_r = \eta\exp(\pm i\delta\tau/4)\,. \qquad (25)$$

The growth rate, ν, of the kink instability is obtained by solving eq. (23). Since $\delta^2 > 0$ in eq. (24), the ion cyclotron motion enhances the effect of dynamic shear stabilization. As is shown in eq. (25), the unstable kink deformation propagates at a small velocity $\pm\omega_0\delta/4k$ in the axial direction. Since the coefficient of the second term in eq. (23) is a periodic function of τ, there exists a Floquet solution of the form[8)]

$$\eta = A\,P(\tau)\exp(i\gamma\tau) + B\,P(-\tau)\exp(-i\gamma\tau)\ , \quad AB = 0\ , \qquad (26)$$

where γ is the characteristic exponent which depends on δ, θ_0 and θ_1 in eq. (21), $P(\pm\tau)$ is a periodic function of the period π. Since we are interested in an unstable solution of eq. (23), we assume that γ in eq. (26) is pure imaginary. Hence, the growth rate of the instability is given by $\nu = i\gamma\omega_0/2$. It is known that the characteristic exponent, γ, is given by[8)]

$$\theta_0' = \gamma^2 + \frac{\theta_1^{\,2}}{2(\gamma^2-1)} + \frac{(5\gamma^2+7)\,\theta_1^{\,4}}{32(\gamma^2-1)^3(\gamma^2-4)} + \cdots\ , \qquad (27)$$

if $|\theta_1| \ll 1$. Eliminating k from θ_0' and θ_1 in eqs. (21) and (24), one obtains a relation between θ_0' and θ_1. Substituting the relation into eq. (27), we eliminate θ_0'. Then, one can calculate the growth rate ν as a function of θ_1 , which is proportional to the applied axial RF current. Finally, we obtain ν vs k, when other parameters such as B_z, $B_{\theta a}$, a and ω_0 are given.

An example of numerical calculation is shown in Fig. 1. The density is assumed to be 10^{14} (cm^{-3}) of helium plasma, $B_z = 1$ (T), $B_{\theta a} = 0.1$ (T), $a = 1$ (cm), $\omega_0/2\pi = 2\times10^6$ (Hz) are used. The dashed line is the result of usual Mathieu equation for $\tilde{B}_{\theta a}/B_{\theta a} = 0.3$. The thick solid line is the result of the present analysis at $\tilde{B}_{\theta a}/B_{\theta a} = 0.3$ where the effect of ω_{ci} is taken into account.

In conclusion, the dynamic shear stabilization can be effective even at the frequency near ω_{ci}. The effect of ω_{ci} enhances the stabilization of kink instabilities in low-beta plasmas. It should be noted that the inclusion of the cyclotron motion results in the propagation of the kink instabilities in the axial direction. This fact suggests the new possibility for stabilizing the instability by the nonlinear ion cyclotron damping.

References

1) G. Berge, Nuclear Fusion 12 (1972) 99.

2) O. Gruber, Z. Physik 251 (1972) 333.

3) G.G. Lister, F.S. Troyon and E.S. Weibel, Phys. of Fluids 16 (1973) 297.

4) S.M. Osovets, Sov. Phys. USPEKHI 17 (1974) 239.

5) M.G. Nikulin, Sov. Phys.-Tech. Phys. 14 (1970) 1618.

6) B.B. Kadomtsev, in "Review of Plasma Physics" edited by M.A. Leontovich (Consultants Bureau, New York, 1966) Vol. 2, pp 153.

7) T. Kakutani, T. Kawahara and T. Taniuti, J. Phys. Soc. Japan 23 (1967) 1138.

8) See for example, "Handbook of Mathematical Functions" edited by M. Abramowitz and I.A. Segun (Dover Publications Inc., New York, 1968).

Figure Caption

Fig. 1 The decreased growth rates vs the wavenumber k. The dashed line is the usual Mathieu equation. The thick solid line is the present analysis. $B_z = 1$ (T), $B_{\theta a} = 0.1$ (T), $\tilde{B}_{\theta a}/B_{\theta a} = 0.3$, $a = 1$ (cm), $\rho/M = 10^{16}$ (cm^{-3}), $\omega_0/2\pi = 2 \times 10^6$ (Hz).

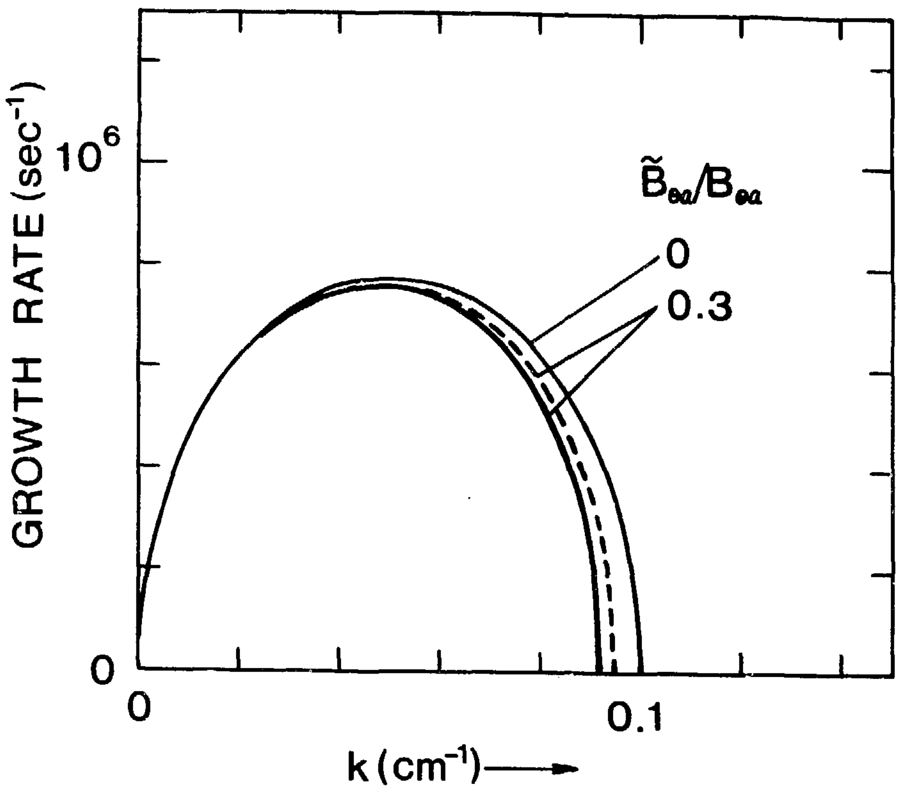

Fig.1