INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

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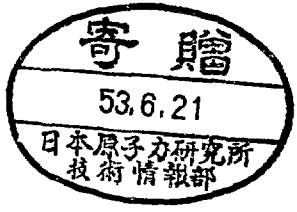
RESEARCH REPORT



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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan

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Abstract

A new eigenmode analysis is established in plasma with arbitrary density profile. Finite Larmor effect is taken into account even if $k_x\rho > 1$. Eigenfrequencies are determined through a compact "quantization condition".

The stability criterion of inhomogeneous plasma is given by the eigenmode analysis.[1-3] Usual treatments of drift wave in collisionless Vlasov plasmas are based on the expression of unknown function $\Psi(x')$ about $x'=x$, i.e.,

$$\Psi(x') = \Psi(x) + (x'-x)\frac{d\Psi}{dx} + \frac{1}{2}(x'-x)^2\frac{d^2\Psi}{dx^2} + \cdots \qquad (1)$$

where $\Psi(x)$ is a potential[1] or a component of electromagnetic field[4] of the eigenmode and x' is the x-component of unperturbed orbit. The expression (1) is applicable only when the Larmor radius ρ is small compared to the "wave length" in x-direction (i.e., $k\rho < 1$ where $k \sim \Psi'(x)/\Psi(x)$).

The effect of finite Larmor radius can be taken into account exactly if $\Psi(x)$ is expanded in the Fourier integral

$$\Psi(x) = \frac{1}{2\pi}\int \Psi(k)e^{ikx}\,dk \quad . \qquad (2)$$

The Fourier component $\Psi(k)$ is determined by an integral equation[5,6]

$$F(k)\Psi(k) = \int K(k, k')\Psi(k')dk' \quad . \qquad (3)$$

The expression (3) is also introduced from the general relation for electric field in an inhomogeneous medium

$$\Psi(x) = \int D(x, x')\Psi(x')dx' \quad . \qquad (4)$$

If we transform (4) into k-space, we can immediatly obtain the relation (3). In this case, $F(k)=1$ and $K(k, k')$ is given by

$$K(k, k') \equiv \frac{1}{2\pi}\int D(x, x')e^{-ikx+ik'x'}\,dxdx' \quad . \qquad (5)$$

Analytical treatments of Eq.(3) are done by several authors using an expansion of the kernel of Eq.(3).[7,8] They investigated the eigenmode profile in x-space. But their methods are not convenient and not satisfactory when the wavelength is smaller than the inhomogeneity scale length. Another treatment is given by Krall and Rosenbluth.[9] They derived first an approximate dispersion relation based on the assumption $\Psi(x)$=constant. Next they examined the validity of this approximation through the integral equation (3). They introduced a function g(k) defined by

$$\Psi(k) \equiv \exp[-i \int^{k} g(k')dk'] \qquad (6)$$

and obtained approximate form of g(k). They showed, however, only the validity of their approximate dispersion relation based on $\Psi(x)$=constant.

The present new treatment is also based on an analysis of Eq.(3) in terms of the function g(k) defined by Eq.(6). Physical meaning of the function g(k) is made clear from the potential profile in x-space $\Psi(x)$,

$$\Psi(x) = \int \Psi(k) e^{ikx} \frac{dk}{2\pi} \quad . \qquad (7)$$

If we substitute the expression (6) into Eq.(7) and evaluate the integration by the steepest descent method, we have

$$\Psi(x) \sim \frac{1}{2\pi} \sqrt{\frac{2\pi}{-ig'(k)}} \quad e^{ikx-i\int^{k} g(k')dk'} \qquad (8)$$

where k is the saddle point of integrand of Eq.(7) and determined by

$$x - g(k) = 0 \quad . \tag{9}$$

If we change the integration veriable in Eq.(8) from k' to x using the relation (9) and perform once a partial integration, we have a WKB solution[10]

$$\Psi(x) \sim \sqrt{\frac{i}{2\pi}\frac{dk(x)}{dx}} \quad e^{i\int^x k(x)dx} \tag{10}$$

where function k(x) is determined by Eq.(9). Then the value of g(k) indicates a position x and k is a wave number of WKB solution at this point x.

The function g(k) is determined by Eq.(3) and Eq.(6), i.e.,

$$F(k) = \int K(k,k') \exp[-i \int_k^{k'} g(k'')dk'']dk' \ , \tag{11}$$

The integral equation (3) can be regarded as a wave coupling equation, that is, the wave with wave number k is generated from a coupling between a wave with k' and the spatial plasma inhomogeneity, the wave number of which is of the order of L^{-1} (L is the spatial scale length of plasma inhomogeneity). Then the range of k' which gives main contributions in the integration of Eq.(11), is given by

$$|k' - k| \lesssim L^{-1} \quad . \tag{12}$$

The function g(k") (which is of the order of L) can be expanded about k such as

$$g(k") = g(k) + g'(k)(k"-k) + 0(L\cdot(kL)^{-2}). \tag{13}$$

To obtain an approximate solution of Eq.(11), we expand g(k) as follows

$$g(k) = g_0(k) + g_1(k) + 0(L(kL)^{-2}) \ , \qquad (14)$$

If we substitute Eqs.(13) and (14) into Eq.(11), we have

$$H(k, g_0) \equiv \int K(k,k')e^{-i(k'-k)g_0(k)}\, dk' - F(k) = 0 \ , \qquad (15)$$

$$g_1(k) = -\frac{i}{2}\frac{\partial^2 H}{\partial g_0^2}\left(\frac{\partial H}{\partial g_0}\right)^{-1} g_0'(k) \qquad (16)$$

from each order of $(kL)^{-1}$.

The functions $\frac{\partial H}{\partial g_0}$ and $\frac{\partial^2 H}{\partial g_0^2}$ in Eq.(16) are defined by

$$\frac{\partial H}{\partial g_0} \equiv -i\int (k'-k)K(k,k')e^{-i(k'-k)g_0(k)}\, dk' \ , \qquad (17)$$

$$\frac{\partial^2 H}{\partial g_0^2} \equiv -\int (k'-k)^2 K(k,k')e^{-i(k'-k)g_0(k)}\, dk' \qquad (18)$$

and these definitions are coincident with the derivatives of the function $H(k, g_0)$(15). The equation (15) determines the function $g_0(k)$ and Eq.(16) determines the first order correction term $g_1(k)$.

The wave propagation characteristics are determined by the structure of the function $g(k)$. A spatially localized mode is expressed by a loop of $g=g(k)$ as shown in Fig.1 if there are no external signal sources. The value x_{t1} and x_{t2} correspond to the x-coordinates of turning points. The localized mode is constructed by the superposition of plane waves with wave number k, the range of which is given by

$$k_{t1} < k < k_{t2} \ .$$

Next we examine the "quantization condition" which determines the eigenfrequency ω. If a localized solution is present, the solution $g_0(k)$ becomes multivalued as shown in Fig.1. In the neighbourhood of turning point k_t in k-space ($k_t=k_{t1}$ or k_{t2} in Fig.1), we can expand $H(k, g_0)$ such as

$$H(k, g_0) \simeq \frac{\partial H}{\partial k})_{k=k_t} (k-k_t) + \frac{1}{2} \frac{\partial^2 H}{\partial g_0^2})_{k=k_t} (g_0-g_t)^2 \qquad (19)$$

where we used the relation

$$\frac{\partial H}{\partial g_0})_{k=k_t} = 0 \quad .$$

Equation (19) gives the function $g_0(k)$ as

$$g_0(k) = g_t \pm [-(\frac{\partial H}{\partial k})_{k=k_t} / (\frac{\partial^2 H}{\partial g_0^2})_{k=k_t}]^{1/2} (k-k_t)^{1/2}. \qquad (20)$$

The function g(k) becomes a single valued function if we make branch cuts in k-space as shown in Fig.2 (where we assumed that turning points k_{t1} and k_{t2} are complex in general). A general asymptotic solution of Eq.(3) is expressed by

$$\Psi(k) = A_1 e^{-i\int_{c_1} g dk} + A_2 e^{-i\int_{c_2} g dk} \qquad (21)$$

where A_1 and A_2 are arbitrary constants. If k is in the range of "$k < k_{t1}$" or "$k > k_{t2}$" , the solution (21) is composed of functions which are increasing or decreasing with k. To obtain a localized solution in x-space, the decreasing solution in the region of "$k < k_{t1}$" must be selfconsistently connected with the decreasing solution in the range of "$k > k_{t2}$" . Using the same argument as usual connection formula for wave euqation[11],

we obtain a "quantization condition"

$$\oint_{\Gamma} g(k)dk = 2n\pi \qquad (n=0, \pm 1, \pm 2, \cdots\cdots) \qquad (22)$$

where the integration path Γ is illustrated in Fig.3. Substituting Eqs.(14), (16) and (19) into Eq.(22), the "quantization condition" reduces to

$$\oint_{\Gamma} g_0(k)dk = 2(n + \frac{1}{2})\pi \qquad (n = 0, \pm 1, \pm 2, \cdots) \ . \qquad (23)$$

The new treatment of eigenmode analysis is summarized as follows:

1. Calculate the kernel of integral Equation (3) from Vlasov equation.
2. Carry out inverse Fourier transformation of the kernel as shown in Eq.(15), and obtain the function $H(k, g_0)$.
3. Calculate the function $g_0(k)$ from $H(k, g_0)=0$.
4. Determine the eigenfrequency ω from the "quantization condition" (23).

If we apply the present method to a wave equation

$$\psi''(x) + V(x)\,\psi_{(x)} = 0 \ , \qquad (24)$$

the quantization condition (23) reduces to the well known result

$$\int_{x_1}^{x_2} \sqrt{V(x)}\ dx = (n + \tfrac{1}{2})\pi \qquad (n = 0, 1, \cdots)$$

where x_1 and x_2 are the turning points defined by $V(x) = 0$.

It will be worthwhile to represent $H(k, g_0)$ in the following form

$$H(k, g_0) = \int D(x, g_0(k))e^{-ik(x-g_0(k))}\, dx - 1$$

where we have used Eqs.(5) and (15).

Applications of the present method will be discussed elsewhere.

The present method is also applicable to some wave propagation problems in an inhomogeneous plasma such as the reflection, the transmission of waves and the linear mode conversion.

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10. If we substitute Eqs.(14) and (16) into Eq.(10), this form (10) reduces to

$$\Psi(x) \sim \sqrt{-2\pi i/(\partial H/\partial k)}\ \exp\left[i\int k(x)dx + \frac{1}{2}\int \frac{\partial}{\partial x}\ln(\partial H/\partial k)dx\right]$$

which is the same expression as the usual WKB solution

$$\Psi(x) = \frac{1}{\sqrt{k(x)}}\ \exp\left[i\int k(x)dx\right]$$

for the wave equation (24) because of $H(k, g_0)=V(g_0)-k^2$.

11. see for example; A.B. Migdal, Approximation methods in quantum mechanics (Benjamin Press (1969)) Chapter 3.

Figure Captions

Fig.1 Schematic plot of $g=g(k)$ for spatially localized mode

Fig.2 Branch cuts in k-space and two integration pathes C_1 and C_2 giving two independent solutions of $\Psi(k)$,

——— : path in above Riemann surface

– – – : path in under Riemann surface

Fig. 3 Contour of integration for the "quantization condition"

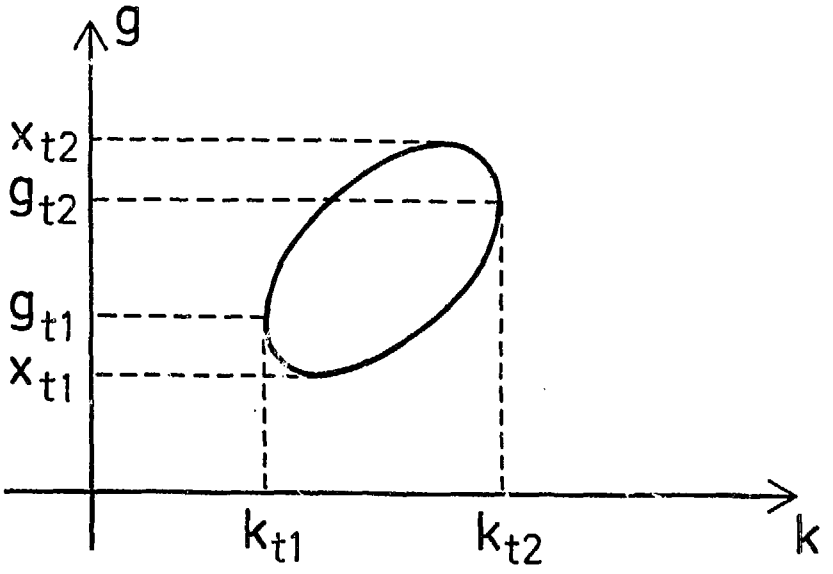

Fig. 1

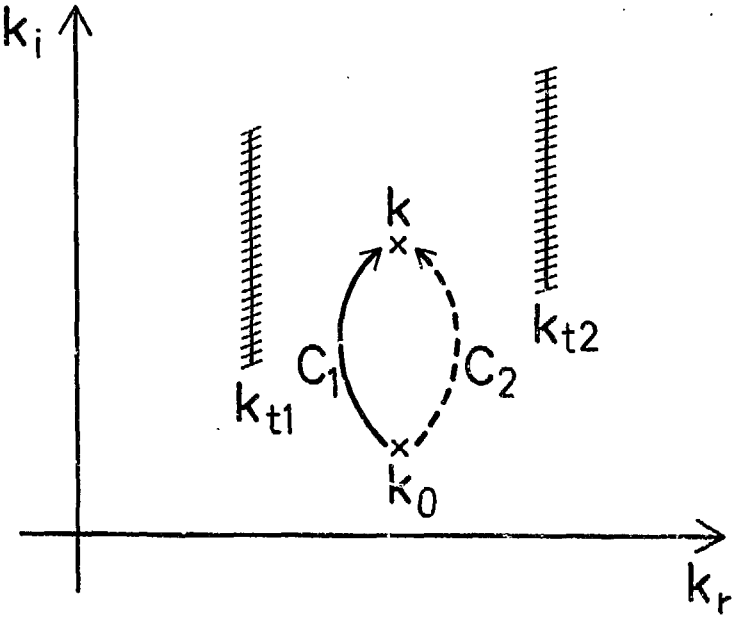

Fig. 2

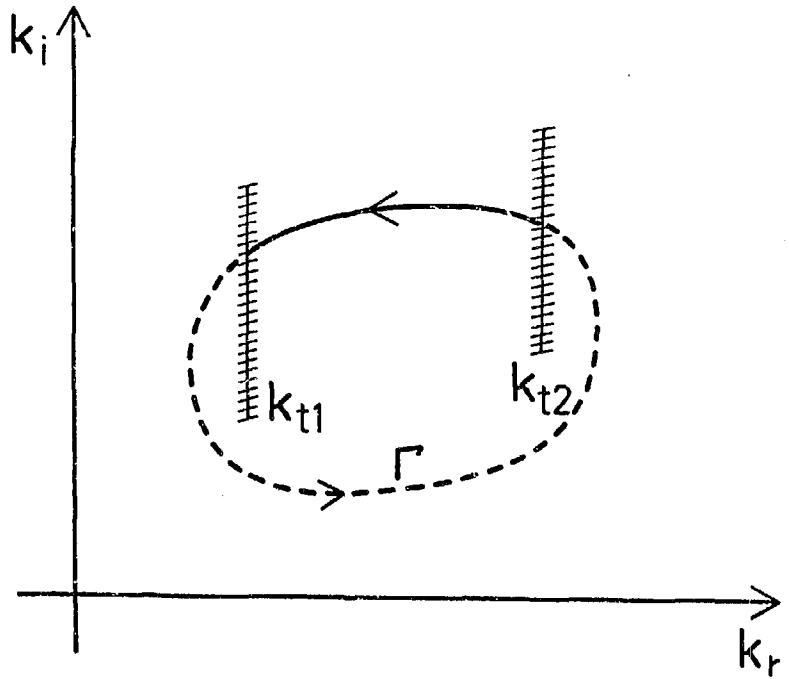

Fig. 3