JAMES COOK UNIVERSITY OF NORTH QUEENSLAND

POINT COULOMB SOLUTIONS OF THE DIRAC EQUATION: ANALYTICAL RESULTS REQUIRED FOR THE EVALUATION OF THE BOUND ELECTRON PROPAGATOR IN QUANTUM ELECTRODYNAMICS.

by

I.B.WHITTINGHAM

NATURAL PHILOSOPHY RESEARCH REPORT NO. 48

Physics Department,
James Cook University of NOrth Queensland,
Townsville.

JCU/NP--48

December,1977

POINT COULOMB SOLUTIONS OF THE DIRAC EQUATION: ANALYTICAL RESULTS REQUIRED FOR THE EVALUATION OF THE BOUND ELECTRON PROPAGATOR IN QUANTUM ELECTRODYNAMICS.

by

I.B.WHITTINGHAM
Physics Department
James Cook University

Natural Philosophy Research Report No. 48

December, 1977

POINT COULOMB SOLUTIONS OF THE DIRAC EQUATION: ANALYTICAL RESULTS REQUIRED FOR THE EVALUATION OF THE BOUND ELECTRON PROPAGATOR IN QUANTUM ELECTRODYNAMICS

by

I.B.WHITTINGHAM
Physics Department,
James Cook University.

Abstract

ABSTRACT

The bound electron propagator in quantum electrodynamics is reviewed and the Brown and Schaefer angular momentum representation of the propagator discussed. Regular and irregular solutions of the radial Dirac equations for both $|E| < mc^2$ and $|E| \geq mc^2$ are required for the computation of the propagator. Analytical expressions for these solutions, and their corresponding Wronskians, are obtained for a point Coulomb potential. Some computational aspects are discussed in an appendix.

1. The bound electron propagator in quantum electrodynamics

In applying Dyson-Feynman covariant perturbation techniques to quantum electrodynamical processes such as Rayleigh scattering, bound Compton scattering and the Lamb shift of atomic energy levels, the Furry or Bound Interaction Picture (Jauch and Rohrlich 1955, p.306; Schweber 1964, p.566) must be used if the external, non-quantized, nuclear field is to be included exactly.

In the Bound Interaction Picture the initial and final electron wave functions $\psi(x)$ obey the external field Dirac equation

$$[\gamma_\mu \{\partial_\mu - \frac{ie}{\hbar} A_\mu^{(e)}(x)\} + \frac{mc}{\hbar}]\, \psi(x) = 0 \qquad (1.1)$$

and the virtual electron-positron intermediate states are represented by the bound electron propagator $S_F^{(e)}(x,x')$ which satisfies (Schweber 1964, p.570)

$$[\gamma_\mu \{\partial_\mu - \frac{ie}{\hbar} A_\mu^{(e)}(x)\} + \frac{mc}{\hbar}]\, S_F^{(e)}(x,x') = 2i\delta^{(4)}(x-x') \qquad (1.2)$$

subject to the boundary condition that $S_F^{(e)}(x,x')$ is asymptotic to the free electron propagator $S_F(x-x')$ in the limit of zero external field. In these equations $\gamma_k = -i\beta\alpha_k$ and $\gamma_4 = \beta$ are the Dirac 4 x 4 matrices,

$$x_\mu = (\underset{\sim}{x}, ict),\ \partial_\mu = \frac{\partial}{\partial x_\mu} = (\underset{\sim}{\nabla}, \frac{\partial}{\partial ict}),$$

$A^{(e)} = (\underset{\sim}{A}^{(e)}, i\phi^{(e)}/c)$ is the external electromagnetic potential (SI units). Greek subscripts range over 1,2,3,4; Latin subscripts over 1,2,3 and the repeated index summation convention is used.

To obtain an expression for the bound electron propagator we rewrite (1.1) as

$$i\hbar \frac{\partial\psi}{\partial t} = \hat{H}_D\, \psi \qquad (1.3)$$

where

$$\hat{H}_D = c\alpha_k \{-i\hbar\partial_k - eA_k^{(e)}(x)\} + \beta mc^2 + e\phi^{(e)}(x) \qquad (1.4)$$

and observe that for a time independent external field the Dirac wave functions have the form

$$\psi_{E\pm}(x) = \psi_{E\pm}(\underset{\sim}{x})\, e^{\mp iEt/\hbar} \qquad (1.5)$$

$\psi_E^{(\pm)}(\underset{\sim}{x})$ are the positive and negative energy eigenfunctions of $\hat{H}_D$, that is

$$\hat{H}_D\, \psi_{E\pm}(\underset{\sim}{x}) = \pm E\, \psi_{E\pm}(\underset{\sim}{x}). \qquad (1.6)$$

In terms of these positive and negative energy solutions the bound electron propagator is (Akheizer and Berestetskii 1965, p.485; Schweber 1964, p.596)

$$-\frac{1}{2} S_F^{(e)}(x,x') = \begin{cases} \sum_{a,E+} \psi_{a,E+}(x)\, \bar{\psi}_{a,E+}(x') & t > t' \\ -\sum_{a,E-} \psi_{a,E-}(x)\, \bar{\psi}_{a,E-}(x') & t < t' \end{cases} \qquad (1.7)$$

where $\bar{\psi}(x) \equiv \psi^{\dagger}(x)\,\gamma_4$ and the label a represents the other quantum numbers needed to specify the complete set $\{\psi_{a,E+}(x),\ \psi_{a,E-}(x)\}$. With the relation

$$\partial_t\, \theta(t-t') = \delta(t-t')$$

for the step function

$$\theta(y) = \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases} ,$$

it is easily shown that (1.7) satisfies (1.2). Also we note that for a time independent external field

$$S_F^{(e)}(x,x') \rightarrow S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',\ t-t').$$

For quantum electrodynamical processes involving the inner electrons of heavy atoms we have a scalar external potential which can be assumed spherically symmetric. The Dirac Hamiltonian is then

$$\hat{H}_D^{CF} = -i\hbar c\, \alpha_k \partial_k + \beta mc^2 + V(r) \qquad (1.8)$$

where $\underset{\sim}{x} = (r,\theta,\phi) = (r,\Omega)$ and $V(r) = e\phi^{(e)}(r)$. The wave functions $\psi_{a,E}(\underset{\sim}{x})$ are then the simultaneous eigenfunctions of $\hat{H}_D^{CF}$, $\hat{\underset{\sim}{j}}^2$, $\hat{j}_z$ and $\hat{K} = \beta(\underset{\sim}{\Sigma}.\hat{\underset{\sim}{\ell}} + \hbar)$ corresponding to the eigenvalues E, $j(j+1)\hbar^2$, $\mu\hbar$ and $-\kappa\hbar$ respectively and have the form (Rose 1961, p.159)

$$\psi_{\kappa,\mu,E}(\underset{\sim}{x},\zeta) = \frac{1}{r} \begin{pmatrix} g_{\kappa,E}(r) & \chi_{\kappa}^{\mu}(\Omega,\zeta) \\ if_{\kappa,E}(r) & \chi_{-\kappa}^{\mu}(\Omega,\zeta) \end{pmatrix} . \qquad (1.9)$$

The spinor spherical harmonics are

$$\chi_{\kappa}^{\mu}(\Omega,\zeta) = \sum_m C(\ell,\tfrac{1}{2},j;\ \mu-m,m)\, Y_{\ell}^{\mu-m}(\Omega)\, \chi_m(\zeta), \qquad (1.10)$$

where the notation and phase conventions for the Clebsch-Gordon coefficient $C(j_1,j_2,j;\ m_1,m_2)$ and spherical harmonic $Y_{\ell}^{m}(\Omega)$ is that of Rose (1957), the $\chi_m(\zeta)$ are the two component Pauli spinors, $\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & \underset{\sim}{\sigma} \end{pmatrix}$ where the $\underset{\sim}{\sigma}$ are the Pauli 2 x 2 spin matrices, and the values of the total and orbital angular momentum quantum numbers j and ℓ respectively are obtained from κ via

$$j = |\kappa| - \frac{1}{2},\ \ell = j \pm \frac{1}{2} = \begin{cases} \kappa & \kappa > 0 \\ -\kappa - 1 & \kappa < 0 \end{cases}$$

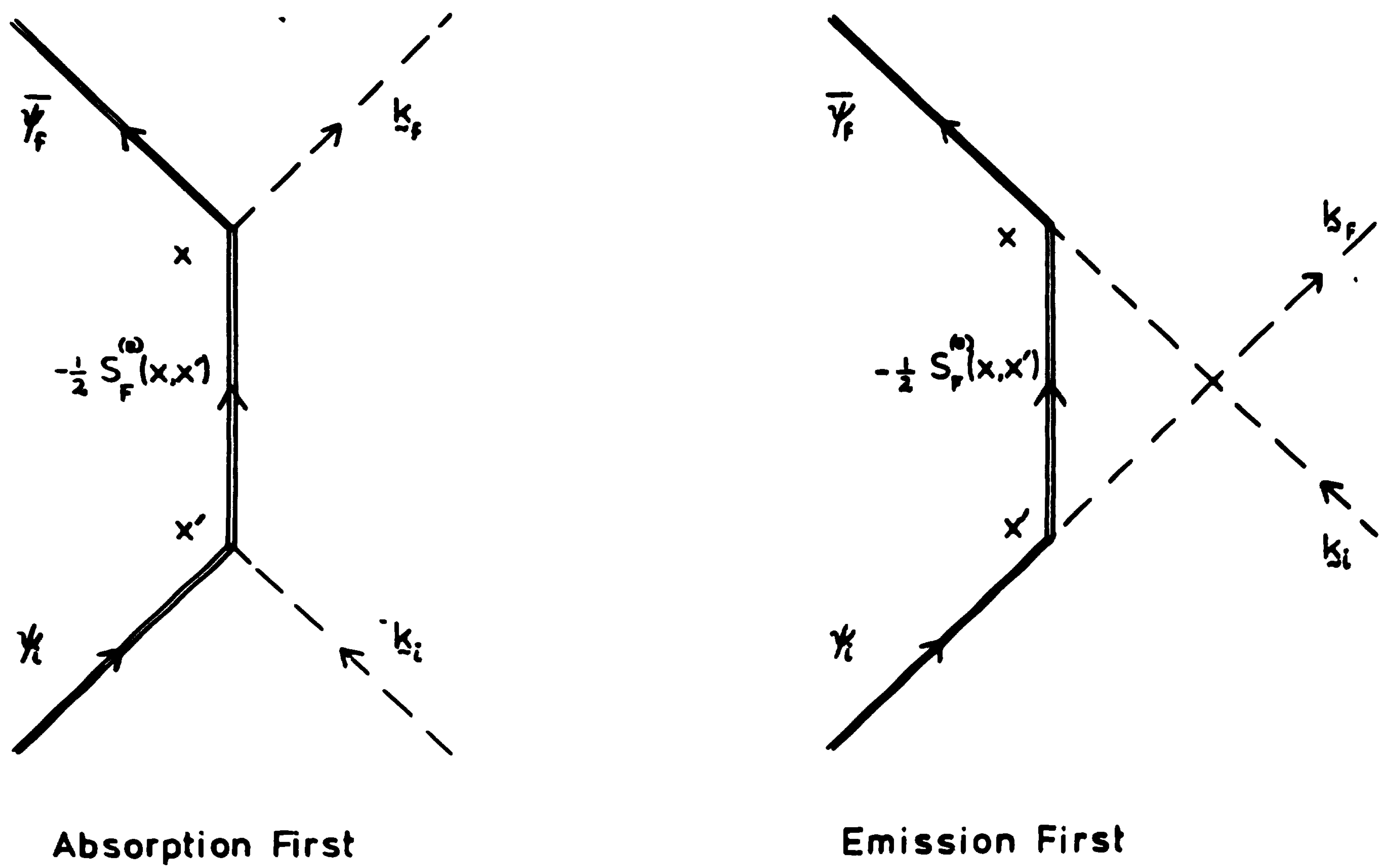

Figure 1. Compton Scattering by a Bound Electron.

The parameter κ takes all non-zero integral values.

The radial functions are solutions of the coupled first order equations

$$\hat{A}_{\kappa,E}(r) \begin{pmatrix} g_{\kappa,E}(r) \\ f_{\kappa,E}(r) \end{pmatrix} = 0 \tag{1.11}$$

where

$$\hat{A}_{\kappa,E}(r) = \begin{pmatrix} \frac{d}{dr} + \frac{\kappa}{r} & -\{E + mc^2 - V(r)\}/\hbar c \\ \{E-mc^2-V(r)\}/\hbar c & \frac{d}{dr} - \frac{\kappa}{r} \end{pmatrix} . \tag{1.12}$$

In this angular momentum representation the bound electron propagator (1.7) is then (Brown and Schaefer 1956)

$$S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',t-t') = - \sum_{\mu,\kappa,E} [\epsilon(t-t')+\epsilon(E)]\psi_{\kappa,\mu,E}(\underset{\sim}{x})\, \psi^{\dagger}_{\kappa,\mu,E}(\underset{\sim}{x}')\, e^{-iE(t-t')/\hbar} \tag{1.13}$$

where

$$\epsilon(y) = \begin{cases} 1 & y > 0 \\ 0 & y = 0 \\ -1 & y < 0 \end{cases} ,$$

and equation (1.2) becomes for our static central field

$$[\hat{H}_D^{CF}(\underset{\sim}{x}) - i\hbar \frac{\partial}{\partial t}]\, S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}', t-t') = 2i\hbar\beta\, \delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t'). \tag{1.14}$$

If we now introduce the mixed representation $S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E)$ (Brown and Schaefer 1956) of $S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',t-t')$ through the Fourier transform

$$S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',t-t') = \frac{i}{\pi} \int_{-\infty}^{\infty} S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E)\beta\, e^{-iE(t-t')/\hbar} dE \tag{1.15}$$

then $S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E)$ satisfies

$$(\hat{H}_D^{CF} - E)\, S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E) = \delta(\underset{\sim}{x}-\underset{\sim}{x}') , \tag{1.16}$$

that is $S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E)$ is a Green function for the Dirac equation.

$S_F^{(e)}(\underset{\sim}{x}, \underset{\sim}{x}',E)$ arises naturally in the evaluation of S-matrix elements involving the bound electron propagator. As an illustration we consider bound Compton scattering for which the S-matrix element describing the "absorption first" process (figure 1) is proportional to (Whittingham 1971)

$$S_{fi}^{(a)} = \iint d^4x' d^4x\, \bar{\psi}_f(x)\, \gamma.\epsilon^{(f)*} e^{-ik_f.x}\; S_F^{(e)}(x,x')\; \gamma.\epsilon^{(i)} e^{ik_i.x'} \psi_i(x'). \tag{1.17}$$

Here $k = (\underline{k}, i\omega/c)$ and $\epsilon = (\underline{\epsilon}(\underline{k}), 0)$ are the photon 4-momentum and 4-polarization respectively. With the aid of (1.15) we can integrate over t,t' and E in (1.17) to obtain

$$S_{fi}^{(a)} = -4\pi i \hbar^2 \delta(E_f + \hbar\omega_f - E_i - \hbar\omega_i) \iint d^3x' \, d^3x$$
$$\times \bar{\psi}_f(\underline{x})\gamma.\epsilon^{(f)*} e^{-i\underline{k}_f.\underline{x}} S_F^{(e)}(\underline{x},\underline{x}', E_i + \hbar\omega_i)\gamma.\epsilon^{(i)} e^{i\underline{k}_i.\underline{x}'} \psi_i(\underline{x}'). \quad (1.18)$$

The "emission first" process involves $S_F^{(e)}(\underline{x},\underline{x}',E_i-\hbar\omega_i)$.

2. The Green function $S_F^{(e)}(\underline{x},\underline{x}',E)$

In § 1 it was shown that the S-matrix elements for quantum electrodynamical processes involving bound virtual electron-positron states can be expressed in terms of the Green function $S_F^{(e)}(\underline{x},\underline{x}',E)$. If we introduce the two linearly independent solutions of (1.11), the solution ${}^{o}y_{\kappa E}(r) \equiv \begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix}$ regular at the origin and the solution ${}^{\infty}y_{\kappa,E}(r)$ regular at infinity, then $S_F^{(e)}(\underline{x},\underline{x}',E)$ can be expressed as (Brown and Schaefer 1956)

$$S_F^{(e)}(\underline{x},\underline{x}',E) = \sum_{\mu,\kappa} \left\{ \frac{\theta(r-r')-1}{\Delta_\kappa(E)} {}^{o}\psi_{\kappa,\mu,E}(\underline{x}) {}^{\infty}\psi^*_{\kappa,\mu,E}(\underline{x}') - \frac{\theta(r-r')}{\Delta_\kappa(E)} {}^{\infty}\psi_{\kappa,\mu,E}(\underline{x}) \; {}^{o}\psi^*_{\kappa,\mu,E}(\underline{x}') \right\} \quad (2.1)$$

where, for example

$${}^{o}\psi_{\kappa,\mu,E}(\underline{x}) = \frac{1}{r} \begin{pmatrix} {}^{o}g_{\kappa,E}(r)\, \chi_\kappa^\mu \; (\Omega,\zeta) \\ i\,{}^{o}f_{\kappa,E}(r)\chi_{-\kappa}^\mu \; (\Omega,\zeta) \end{pmatrix} \quad (2.2)$$

and

$${}^{o}\psi^*_{\kappa,\mu,E}(\underline{x}) = \frac{1}{r}\left({}^{o}g_{\kappa,E}(r)\chi_\kappa^{\mu\dagger}(\Omega,\zeta)^\dagger \; , \; -i\,{}^{o}f_{\kappa,E}(r)\, \chi_{-\kappa}^{\mu\dagger} \; (\Omega,\zeta)\right). \quad (2.3)$$

$\Delta_\kappa(E)$ is the Wronskian of the two solutions of (1.11), that is

$$\Delta_\kappa(E) = {}^{o}g_{\kappa,E}(r)\; {}^{\infty}f_{\kappa,E}(r) - {}^{\infty}g_{\kappa,E}(r)\; {}^{o}f_{\kappa,E}(r). \quad (2.4)$$

As both ${}^{o}y_{\kappa,E}(r)$ and ${}^{\infty}y_{\kappa,E}(r)$ satisfy (1.11) we have immediately

$$\left({}^{o}f_{\kappa,E}(r) \; , \; {}^{o}g_{\kappa,E}(r)\right) \hat{A}_{\kappa,E}(r) \begin{pmatrix} {}^{\infty}g_{\kappa,E}(r) \\ {}^{\infty}f_{\kappa,E}(r) \end{pmatrix} + \left({}^{\infty}f_{\kappa,E}(r) \; , \; {}^{\infty}g_{\kappa,E}(r)\right) \hat{A}_{\kappa,E}(r) \begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix}$$
$$= 0,$$

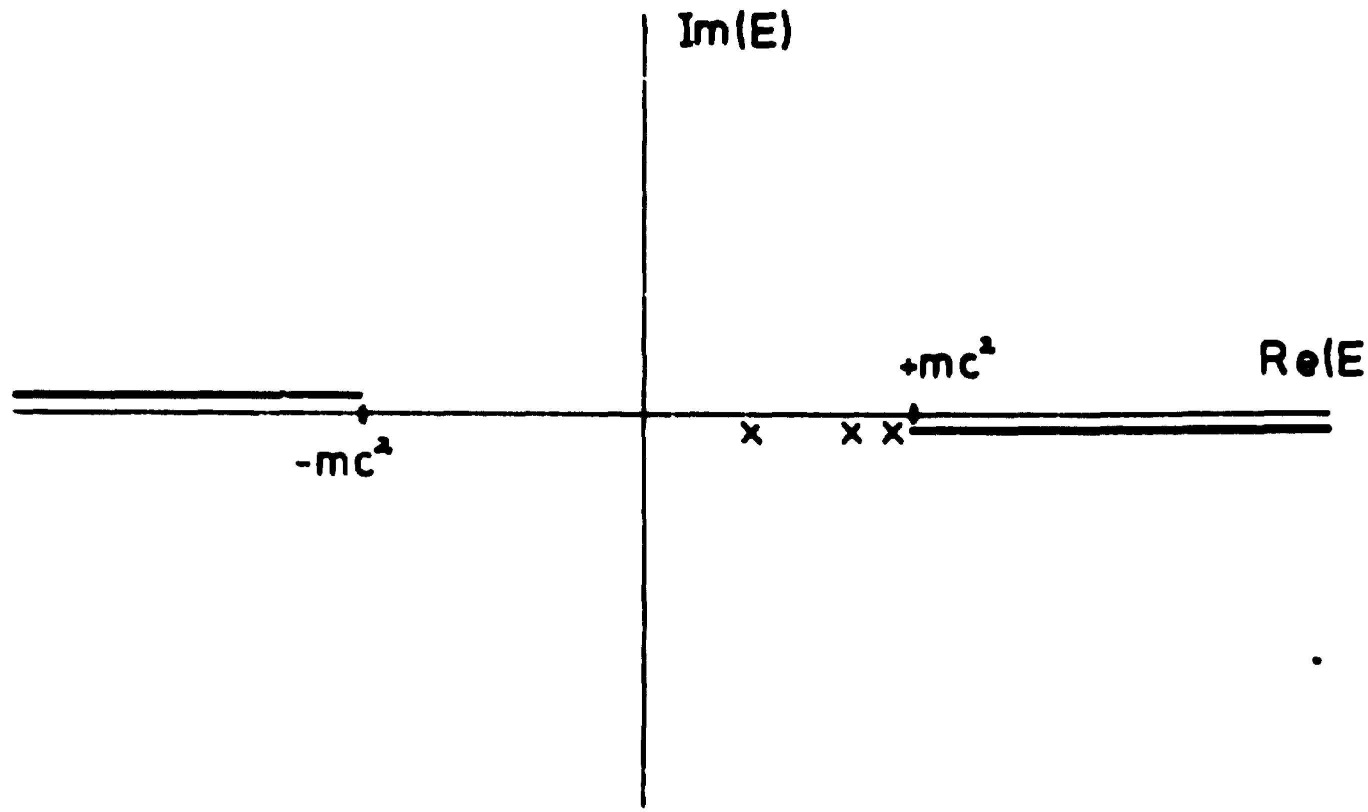

Figure 2. Poles and Branch Lines of $S_F^{(a)}(\underset{\sim}{x}, \underset{\sim}{x}', E)$

that is

$$\frac{d}{dr}\{{}^{0}g_{\kappa,E}(r)\,{}^{\infty}f_{\kappa,E}(r) - {}^{\infty}g_{\kappa,E}(r)\,{}^{0}f_{\kappa,E}(r)\} = \frac{d}{dr}\Delta_{\kappa}(E) = 0\,. \quad (2.5)$$

Thus, as indicated by the notation, the Wronskian for solutions of the radial Dirac equations is independent of r.

We shall see later that the solutions ${}^{0}y_{\kappa,E}(r)$ and ${}^{\infty}y_{\kappa,E}(r)$ become proportional when E is equal to a discrete energy eigenvalue E_n. Thus $\Delta_{\kappa}(E=E_n) = 0$ for all r.

Brown and Schaefer (1956) show that $S_F^{(e)}(\underline{x},\underline{x}',E)$ given by (2.1) is a solution of (1.16) and has all the desired properties of a Green function, that is, considered as a function of E, $S_F^{(e)}(\underline{x}, \underline{x}',E)$ has simple poles in the range $0 < E < mc^2$, two branch lines $E \geq mc^2$ and $E \leq -mc^2$, and is regular elsewhere (figure 2). Also, as a function of r(or r') with r'(or r) fixed, $S_F^{(e)}(\underline{x}, \underline{x}',E)$ is finite at infinity and at the origin for E not equal to a discrete energy eigenvalue.

To ensure that the Green function satisfies the correct boundary conditions in the limit of large r (or r'), we note that the radial Dirac equations (1.11) have the asymptotic solutions

$$\begin{aligned} g_{\kappa,E}(r) &\xrightarrow[r\to\infty]{} (E+mc^2)^{\frac{1}{2}}\,[a\,e^{\lambda r} + b\,e^{-\lambda r}] \\ f_{\kappa,E}(r) &\xrightarrow[r\to\infty]{} (mc^2-E)^{\frac{1}{2}}\,[a\,e^{\lambda r} - b\,e^{-\lambda r}] \end{aligned} \quad (2.6)$$

where

$$\lambda = (m^2c^4 - E^2)^{\frac{1}{2}}/\hbar c.$$

For $|E| < mc^2, \lambda$ is real and we must choose

$$\begin{pmatrix} {}^{\infty}g_{\kappa,E}(r) \\ {}^{\infty}f_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} \pm\, b\,(mc^2 \pm E)^{\frac{1}{2}}\, e^{-\lambda r}\,. \quad (2.7)$$

For $|E| \geq mc^2$ we have

$$\begin{aligned} &\lambda \to ip\,, \quad p = (E^2 - m^2c^4)^{\frac{1}{2}}/\hbar c \\ &(mc^2-E)^{\frac{1}{2}} \to i(E-mc^2)^{\frac{1}{2}} \end{aligned} \quad (2.8)$$

and therefore we choose

$$\begin{pmatrix} {}^{\infty}g_{\kappa,E}(r) \\ {}^{\infty}f_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} a \begin{pmatrix} (E + mc^2)^{\frac{1}{2}} \\ i(E-mc^2)^{\frac{1}{2}} \end{pmatrix} e^{ipr} \quad (2.9)$$

so that, on the half line $E \geq mc^2$, ${}^{\infty}y_{\kappa,E}(r)$ represents outgoing electrons while on the half line $E \leq -mc^2$ it represents outgoing positrons.

If we substitute the angular momentum representation (2.1) for $S_F^{(e)}(\underline{x},\underline{x}',E)$ into S-matrix elements such as (1.18) and use wave functions of the form (1.9) for the initial and final electron (or positron) states, then all angular integrations can be effected using the orthonormality properties of the spinor spherical harmonics $\chi_\kappa^\mu(\theta,\phi)$. Thus the S-matrix elements can be expressed in terms of radial integrals over the various solutions of the radial Dirac equations (1.11) (see, for example, Whittingham 1971, §4). In the remaining sections of this report we will obtain analytical results for the radial functions ${}^{(o,\infty)}g_{\kappa,E}(r)$ and ${}^{(o,\infty)}f_{\kappa,E}(r)$ for $|E| < mc^2$ $(E \neq E_n)$ and $|E| \geq mc^2$ for the case of a point Coulomb potential. The corresponding Wronskians will be evaluated.

3. Radial functions for $|E| < mc^2$.

If we introduce the functions $\phi_1(r)$ and $\phi_2(r)$ via

$$\begin{pmatrix} g_{\kappa,E}(r) \\ f_{\kappa,E}(r) \end{pmatrix} = (mc^2 \pm E)^{\frac{1}{2}} e^{-\lambda r}(\phi_1 \pm \phi_2) \qquad (3.1)$$

and assume that ϕ_1 and ϕ_2 can be written as power series

$$\begin{pmatrix} \phi_1(\rho) \\ \phi_2(\rho) \end{pmatrix} = \rho^\gamma \sum_{n=0}^{\infty} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \rho^n , \qquad (3.2)$$

where $\rho \equiv 2\lambda r$, then for the point-Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{\alpha Z\hbar c}{r} \qquad (3.3)$$

the recurrence relations for α_n and β_n yield the relation

$$\gamma^2 = \kappa^2 - (\alpha Z)^2 .$$

α is the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137} .$$

For solutions regular at the origin we must choose

$$\gamma = + [\kappa^2 - (\alpha Z)^2]^{\frac{1}{2}} \qquad (3.4)$$

and we obtain (Rose 1961, p.175; Akheizer and Berestetskii 1965, p.123)

$$^{o}\phi_1(\rho) = \frac{\gamma - \sigma}{\zeta - \kappa} \rho^{-\frac{1}{2}} M_{k'-1,\gamma}(\rho)$$

$$^{o}\phi_2(\rho) = \rho^{-\frac{1}{2}} M_{k',\gamma}(\rho). \qquad (3.5)$$

We have introduced

$$\sigma = \frac{\alpha Z E}{\hbar c \lambda} \quad , \quad k' = \sigma + \frac{1}{2} \quad , \quad \xi = \frac{\alpha Z m c}{\hbar \lambda} \ . \qquad (3.6)$$

The Whittaker function $M_{k',\gamma}$ is related to the confluent hypergeometric function ${}_1F_1$ $(a,b;\rho)$ via (Abramowitz and Stegun 1965), equation 13.1.32)

$$M_{k',\gamma}(\rho) = \rho^{\gamma+\frac{1}{2}} e^{-\rho/2} {}_1F_1 \left(\frac{1}{2} + \gamma - k', \ 2\gamma + 1, \rho \right) . \qquad (3.7)$$

Thus the solutions regular at the origin are

$$\begin{pmatrix} {}^0g_{\kappa,E}(r) \\ {}^0f_{\kappa,E}(r) \end{pmatrix} = (mc^2 \pm E)^{\frac{1}{2}} (2\lambda r)^{\gamma} \ e^{-\lambda r} \left\{ \frac{\gamma - \sigma}{\xi - \kappa} \ {}_1F_1 \ (\gamma - \sigma + 1, 2\gamma + 1, 2\lambda r) \right.$$
$$\left. \pm {}_1F_1 \ (\gamma - \sigma, \ 2\gamma + 1, 2\lambda r) \right\} . \qquad (3.8)$$

The solutions regular at infinity are (Rose 161, p.176)

$$^{\infty}\phi_1(\rho) = (\kappa + \xi)\rho^{-\frac{1}{2}} W_{k'-1,\gamma}(\rho)$$

$$^{\infty}\phi_2(\rho) = \rho^{-\frac{1}{2}} \ W_{k',\gamma}(\rho) \qquad (3.9)$$

where the Whittaker function $W_{k',\gamma}$ is (Abramowitz and Stegun 1965, equation 13.1.34)

$$W_{k',\gamma}(\rho) = \frac{\Gamma(-2\gamma)}{\Gamma(\frac{1}{2}-\gamma-k')} \ M_{k',\gamma}(\rho) + \frac{\Gamma(2\gamma)}{\Gamma(\frac{1}{2}+\gamma-k')} M_{k',-\gamma}(\rho) . \qquad (3.10)$$

This solution behaves as $\rho^{-\gamma}$ near the origin due to the term $M_{k',-\gamma}$.

For the point-Coulomb potential(3.3) the discrete energy eigenvalues are

$$E_{n,\kappa} = \left[1 + \left(\frac{\alpha Z}{\gamma + n - |\kappa|}\right)^2\right]^{-\frac{1}{2}} mc^2 \qquad n=1,2,3... \qquad (3.11)$$

and therefore

$$\frac{1}{2} + \gamma - k' = - n \ . \qquad (3.12)$$

Thus $\Gamma(\frac{1}{2} + \gamma - k') = \infty$ and $W_{k',\gamma}$ is proportional to $M_{k',\gamma}$, that is for $E = E_n$ the solution regular at the origin and the solution regular at infinity coalesce and become the (unnormalized) eigenfunction belonging to the eigenvalue E_n.

As the Wronskian is independent of r we can choose to evaluate it at $r = \infty$ where we can use the asympototic relation (Abramowitz and Stegun 1965, equation 13.5.1)

$${}_1F_1(a,b,z) \to \frac{\Gamma(b)}{\Gamma(a)} \ e^z z^{a-b} \left\{ \sum_{n=0}^{S-1} \frac{(b-a)_n (1-a)_n}{n!} \ z^{-n} + O(|z|^{-S}) \right\}$$
$$+ \frac{\Gamma(b-a)}{\Gamma(a)} \ z^{-a} \left\{ \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} \ (-z)^{-n} + O(|z|^{-R}) \right\} (3.13)$$

$(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+n-1)$$

$$(a)_o = 1.$$

The asymptotic form of the solutions regular at the origin is therefore

$$\begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} (mc^2 \pm E)^{\frac{1}{2}}\, e^{\lambda r} \left\{ \frac{\gamma-\sigma}{\xi-\kappa} \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma-\sigma+1)} (2\lambda r)^{-\sigma} \pm \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma-\sigma)} (2\lambda r)^{-1-\sigma} \right\}$$

$$= (mc^2 \pm E)^{\frac{1}{2}}\, \frac{\gamma-\sigma}{\xi-\kappa}\, \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma-\sigma+1)}\, e^{\lambda r} (2\lambda r)^{-\sigma}. \tag{3.14}$$

Noting that

$$W_{k',\gamma}(z) \xrightarrow[z\to\infty]{} e^{-z/2} z^{k'} \{1 + \frac{\gamma^2 - (k'-\frac{1}{2})^2}{z} + O(|z|^{-2})\} \tag{3.15}$$

then the asymptotic form of the solutions regular at infinity is

$$\begin{pmatrix} {}^{\infty}g_{\kappa,E}(r) \\ {}^{\infty}f_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} (mc^2 \pm E)^{\frac{1}{2}} (2\lambda r)^{-\frac{1}{2}} \Big\{ (\kappa+\xi)\, e^{-\lambda r} (2\lambda r)^{k'-1}$$

$$\pm e^{-\lambda r} (2\lambda r)^{k'} \left(1 + \frac{(\kappa-\xi)(\kappa+\xi)}{2\lambda r}\right) \Big\}$$

$$= \pm (mc^2 \pm E)^{\frac{1}{2}}\, e^{-\lambda r} (2\lambda r) . \tag{3.16}$$

In obtaining this result we have used the relation

$$\gamma^2 - (k'-\tfrac{1}{2})^2 = (\kappa-\xi)(\kappa+\xi) .$$

With the asymptotic forms (3.14) and (3.16) the Wronskian becomes

$$\Delta_\kappa(E) = -2\,(m^2c^4 - E^2)^{\frac{1}{2}}\, \frac{\gamma-\sigma}{\xi-\kappa}\, \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma-\sigma+1)}$$

$$= \frac{2\hbar c\lambda}{\kappa-\xi}\, \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma-\sigma)} \tag{3.17}$$

since $\Gamma(z) = (z-1)\,\Gamma(z-1)$.

4. Radial functions for $|E| > mc^2$

4.1 Solutions regular at the origin

The solutions regular at the origin can be obtained by the analytical continuation (2.8) of the solutions (3.8) for $|E| < mc^2$, that is

$$\begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix} = \begin{pmatrix} (E+mc^2)^{\frac{1}{2}} \\ i(E-mc^2)^{\frac{1}{2}} \end{pmatrix} (2pr)^{\gamma} e^{-ipr} \{- \frac{\gamma+i\nu}{\kappa+i\xi} {}_1F_1(\gamma+i\nu+1,2\gamma+1,2ipr)$$

$$\pm {}_1F_1(\gamma+i\nu,2\gamma+1,2ipr)\} \qquad (4.1)$$

where

$$\nu = i\sigma = \frac{\alpha ZE}{\hbar cp} . \qquad (4.2)$$

Since

$$|\gamma+i\nu|^2 = \gamma^2 + (\frac{\alpha ZE}{\hbar cp})^2 = \kappa^2 + (\frac{\alpha Zmc}{\hbar p})^2$$

$$= |\kappa + i\xi|^2 , \qquad (4.3)$$

we can rewrite (4.1) as

$$\begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix} = \begin{pmatrix} (E+mc^2)^{\frac{1}{2}} \\ i(E-mc^2)^{\frac{1}{2}} \end{pmatrix} |\frac{\gamma+i\nu}{\kappa+i\xi}| (2pr)^{\gamma} e^{-ipr}$$

$$\times \{(\gamma+i\nu) \left[- \frac{\gamma-i\nu}{\kappa+i\xi}\right] {}_1F_1(\gamma+i\nu+1,2\gamma+1,2ipr)$$

$$\pm (\gamma-i\nu) {}_1F_1(\gamma+i\nu,2\gamma+1,2ipr)\} . \qquad (4.4)$$

Thus with the definition

$$e^{2i\eta(\gamma)} = - \frac{\kappa-i\xi}{\gamma+i\nu} = - \frac{\kappa-imc^2\nu/E}{\gamma+i\nu} \qquad (4.5)$$

and use of the Kummer transformation (Abramowitz and Stegun 1965, equation 13.1.27)

$${}_1F_1(\gamma+i\nu,2\gamma+1,2ipr) = e^{2ipr} {}_1F_1 (\gamma-i\nu+1,2\gamma+1,-2ipr)$$

$$= e^{2ipr} {}_1F_1{}^{*}(\gamma+i\nu+1,2\gamma+1,2ipr) \qquad (4.6)$$

we finally obtain for the solution regular at the origin

$$\begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix} = N \begin{pmatrix} (E+mc^2)^{\frac{1}{2}} \\ i(E-mc^2)^{\frac{1}{2}} \end{pmatrix} (2pr)^{\gamma} \{(\gamma+i\nu) e^{-ipr+i\eta(\gamma)} {}_1F_1(\gamma+i\nu+1,2\gamma+1,2ipr)$$

$$\pm \text{ complex conjugate}\} . \qquad (4.7)$$

The constant N is unimportant in constructing $S_F^{(e)}(\underset{\sim}{x},\underset{\sim}{x}',E)$ since the Wronskian will also contain N as a factor. However for convenience later we will normalize the solutions (4.7) according to

$$\int d^3x \, \psi^{\dagger}_{\kappa,\mu,E}(\underset{\sim}{x}) \, \psi_{\kappa',\mu',E'}(\underset{\sim}{x}) = \delta_{\mu\mu'}\delta_{\kappa\kappa'} \, \delta(E-E'), \qquad (4.8)$$

that is

$$\int_0^\infty \{g_{\kappa,E'}(r)\, g_{\kappa,E}(r) + f_{\kappa,E'}(r)\, f_{\kappa,E}(r)\}\, dr = \delta(E-E') \qquad (4.9)$$

The normalization constant is (Rose 161,p.193)

$$N(\gamma) = \frac{e^{\pi\nu/2}\,|\Gamma(\gamma+i\nu)|}{2(\pi p\hbar c)^{\frac{1}{2}}\Gamma(2\gamma+1)} . \qquad (4.10)$$

The asymptotic form of (4.7) is then (Rose 1961,p.194)

$$\begin{pmatrix} {}^{0}g_{\kappa,E}(r) \\ {}^{0}f_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} \pm \left(\frac{E\pm mc^2}{\pi p\hbar c}\right)^{\frac{1}{2}} \begin{matrix}\cos \\ \sin\end{matrix} [pr + \delta_\kappa(\gamma)] \qquad (4.11)$$

where the phase is

$$\delta_\kappa(\gamma) = \nu \,\ell n\, 2pr - \arg\Gamma(\gamma+i\nu) + \eta(\gamma) - \pi\gamma/2. \qquad (4.12)$$

From (4.5) the phase contribution $\eta(\gamma)$ is given by

$$\eta(\gamma) = \frac{1}{2}\,[\arg(-\kappa+i\xi) - \arg(\gamma+i\nu)] .$$

Since $-\kappa+i\xi$ is in the second or first quadrant depending on κ being positive or negative resp ctively, and $\gamma+i\nu$ is the first quadrant, we have

$$\eta(\gamma) = \begin{cases} \frac{1}{2}\,[\pm\pi - \tan^{-1}\left|\frac{mc^2\nu}{\kappa E}\right| - \tan^{-1}\left|\frac{\nu}{\gamma}\right|\,] & \kappa > 0 \\ \frac{1}{2}\,[\tan^{-1}\left|\frac{mc^2\nu}{\kappa E}\right| - \tan^{-1}\left|\frac{\nu}{\nu}\right|\,] & \kappa < 0 \end{cases} \qquad (4.13)$$

To remove this ambiguity in $\eta(\gamma)$ for $\kappa > 0$ we insist that, in the limit $Z \to 0$ (4.11) must reduce to the free field solutions for which (Rose 1961,p.206)

$$\delta_\kappa(Z=0) = -\frac{\ell+1}{2}\pi = \begin{cases} -\frac{\kappa+1}{2}\pi & \kappa > 0 \\ +\frac{\kappa}{2}\pi & \kappa < 0 \end{cases} . \qquad (4.14)$$

Since in the limit $Z \to 0$

$$\nu \to 0, \gamma \to |\kappa| \quad , \arg\Gamma(\gamma+i\nu) \to \arg\Gamma(|\kappa|) = 0$$

we have

$$\delta_\kappa(\gamma) \to \eta(Z=0) - \frac{\pi|\kappa|}{2} ,$$

that is

$$\eta(Z=0) = \begin{cases} -\frac{\pi}{2} & \kappa > 0 \\ 0 & \kappa < 0 \end{cases} . \qquad (4.15)$$

Thus we have finally

$$\eta(\gamma) = \begin{cases} -\frac{1}{2}\,[\pi + \tan^{-1}\left|\frac{mc^2\nu}{\kappa E}\right| + \tan^{-1}\left|\frac{\nu}{\gamma}\right|\,] & \kappa > 0 \\ \frac{1}{2}\,[\tan^{-1}\left|\frac{mc^2\nu}{\kappa E}\right| - \tan^{-1}\left|\frac{\nu}{\gamma}\right|\,] & \kappa < 0 \end{cases} \qquad (4.16)$$

4.2 Solution irregular at the origin

In order to construct a second linearly independent solution of the Dirac equation for $|E| > mc^2$ it is convenient to introduce the solutions obtained from (4.7) by changing the sign of γ (Yennie et al 1954, Rawitscher 1958), that is

$$\begin{pmatrix} \bar{g}_{\kappa,E}(r) \\ \bar{f}_{\kappa,E}(r) \end{pmatrix} = N(-\gamma) \begin{pmatrix} (E+mc^2)^{\frac{1}{2}} \\ i(E-mc^2)^{\frac{1}{2}} \end{pmatrix} (2pr)^{-\gamma}\{(-\gamma+i\nu)e^{-ipr+i\eta(-\gamma)}$$

$$\times\, {}_1F_1(-\gamma+i\nu+1,-2\gamma+1,2ipr) \pm \text{compl.conj.}\} \qquad (4.17)$$

where

$$e^{2i\eta(-\gamma)} = \frac{-\kappa+i\, mc^2\nu/E}{-\gamma+i\,\nu} \quad . \qquad (4.18)$$

As indicated in §3 these solutions will behave as $r^{-\gamma}$ at the origin. Their behaviour for large r is

$$\begin{pmatrix} \bar{g}_{\kappa,E}(r) \\ \bar{f}_{\kappa,E}(r) \end{pmatrix} \to \pm \left(\frac{E\pm mc^2}{\pi p\hbar c}\right)^{\frac{1}{2}} \begin{matrix}\cos \\ \sin\end{matrix}\ [pr+\delta_\kappa(-\gamma)] \qquad (4.19)$$

where

$$\delta_\kappa(-\gamma) = \nu\, \ln 2pr - \arg\Gamma(-\gamma+i\nu) + \eta(-\gamma) + \pi\gamma/2. \qquad (4.20)$$

From (4.18) we have

$$\eta(-\gamma) = \begin{cases} \frac{1}{2}\ [\tan^{-1}|\frac{\nu}{\gamma}| - \tan^{-1}|\frac{mc^2\nu}{\kappa E}|] & \kappa > 0 \\ \frac{1}{2}\ [\pm\pi + \tan^{-1}|\frac{\nu}{\gamma}| + \tan^{-1}|\frac{mc^2\nu}{\kappa E}|\ ] & \kappa < 0 \end{cases} \qquad (4.21)$$

so that, if we choose the positive sign in (4.21),

$$\eta(-\gamma) = \eta(\gamma) + \tan^{-1}|\frac{\nu}{\gamma}| + \frac{\pi}{2} \qquad (4.22)$$

and

$$\delta_\kappa(-\gamma) = \delta_\kappa(\gamma) + \pi\gamma + \tan^{-1}|\frac{\nu}{\gamma}| + \frac{\pi}{2} + \arg\Gamma(\gamma+i\nu) - \arg\Gamma(-\gamma+i\nu)\ . \qquad (4.23)$$

Recalling the identity (Abramowitz and Stegun 1965, equation 6.1.17)

$$\Gamma(-z) = \frac{-\pi}{z\,\sin\,\pi z\Gamma(z)}\ , \qquad (4.24)$$

that is

$$\arg\Gamma(-z) = -\arg z - \arg\sin\pi z - \arg\Gamma(z) \pm\pi \quad ,$$

then

$$\arg\Gamma(-\gamma+i\nu) = -\arg\Gamma(-\gamma-i\nu)$$
$$= \arg(\gamma+i\nu) + \arg[\sin\pi(\gamma+i\nu)] - \arg\Gamma(\gamma+i\nu) \pm\pi. \qquad (4.25)$$

From (4.23) we therefore obtain

$$\delta_\kappa(-\gamma) = \delta_\kappa(\gamma) + \delta \tag{4.26}$$

where

$$\delta = \pi\gamma_0 + \frac{\pi}{2} - \arg[\sin\pi(\gamma+i\nu)] \pm \pi \quad . \tag{4.27}$$

Unlike the non-relativistic result, the phases of the regular and irregular point-Coulomb solutions do not differ by $\pi/2$.

4.3 Solutions asymptotic to outgoing waves

For the bound electron propagator to be asymptotic to the free electron propagator in the limit of large r (or r') - where the external field is assumed to be zero - we require solutions of the radial Dirac equation which, for $|E| > mc^2$, have the form of outgoing waves at large r (see equation (2.9). We can construct these outgoing wave solutions from a linear combination of the regular and irregular solutions discussed in §4.1 and §4.2 respectively. We require

$$a \begin{pmatrix} {}^{o}g_{\kappa,E}(r) \\ {}^{o}f_{\kappa,E}(r) \end{pmatrix} + b \begin{pmatrix} \bar{g}_{\kappa,E}(r) \\ \bar{f}_{\kappa,E}(r) \end{pmatrix} \xrightarrow[r\to\infty]{} \pm \left[\frac{E\pm mc^2}{\pi p\hbar c}\right]^{\frac{1}{2}} e^{i[pr+\delta_\kappa(\gamma)]} \quad . \tag{4.28}$$

Using the asymptotic forms (4.11) and (4.19) together with the relation (4.26) between $\delta_\kappa(\gamma)$ and $\delta_\kappa(-\gamma)$, (4.28) yields the requirements

$$a + b\,e^{i\delta} = 2 \quad , \quad a + be^{-i\delta} = 0. \tag{4.29}$$

The solutions of these equations are

$$a = 1 + i \cot\delta \quad , \quad b = - i \operatorname{cosec} \delta \tag{4.30}$$

or, defining δ' via

$$\delta = \delta' + \frac{\pi}{2} \pm \pi \quad , \tag{4.31}$$

that is

$$\delta' = \pi\gamma - \arg[\sin\pi(\gamma+i\nu)] \quad , \tag{4.32}$$

then

$$a = 1 - i \tan\delta' \quad , \quad b = i \sec\delta' \quad . \tag{4.33}$$

The phase difference δ' may be written in the form

$$\begin{aligned} \delta' &= \pi\gamma - \tan^{-1}\left\{\frac{\cos\pi\gamma \ \sinh\pi\nu}{\sin\pi\gamma \ \cosh\pi\nu}\right\} \\ &= \pi\gamma - \tan^{-1}\{\cot\pi\gamma \ \tanh\pi\nu\}. \end{aligned} \tag{4.34}$$

In the ultrarelativistic limit $E >> mc^2$, $\nu \to \alpha Z$ and (4.34) agrees with the result of Yennie et al (1954).

Thus we have for our outgoing waves solution

$$\begin{pmatrix} {}^{-}g_{\kappa,E}(r) \\ {}^{-}f_{\kappa,E}(r) \end{pmatrix} = (1 - i \tan \delta') \begin{pmatrix} {}^{0}g_{\kappa,E}(r) \\ {}^{0}f_{\kappa,E}(r) \end{pmatrix} + i \sec \delta' \begin{pmatrix} \bar{g}_{\kappa,E}(r) \\ \bar{f}_{\kappa,E}(r) \end{pmatrix} . \qquad (4.35)$$

Having obtained our desired two sets of linearly independent solutions (4.7) and (4.35), we can now use their asymptotic forms for large r to evaluate their Wronskian. We have

$$\begin{aligned} \Delta_\kappa (E) &= {}^{0}g_{\kappa,E}(r)\, {}^{-}f_{\kappa,E}(r) - {}^{-}g_{\kappa,E}(r)\, {}^{0}f_{\kappa,E}(r) \\ &= i \sec \delta' \{ {}^{0}g_{\kappa,E}(r)\, \bar{f}_{\kappa,E}(r) - \bar{g}_{\kappa,E}(r)\, {}^{0}f_{\kappa,E}(r) \} \qquad (4.36) \end{aligned}$$

or, using (4.11) and (4.19)

$$\begin{aligned} \Delta_\kappa (E) &= - \sec \delta' \left(\frac{E+mc^2}{\pi p\hbar c}\right)^{\frac{1}{2}} \left(\frac{E-mc^2}{\pi p\hbar c}\right)^{\frac{1}{2}} \\ &\quad \times \{\cos [pr + \delta_\kappa(\gamma)] \sin[pr+\delta_\kappa(-\gamma)] \\ &\quad - \sin [pr+\delta_\kappa(\gamma)] \cos [pr+\delta_\kappa(-\gamma)]\} \\ &= -i \frac{\sec \delta'}{\pi} \sin [\delta_\kappa(-\gamma) - \delta_\kappa(\gamma)] \\ &= i/\pi. \qquad (4.37) \end{aligned}$$

5. Remarks

The expressions developed in this report for the various point-Coulomb solutions of the Dirac equation have been essential in assessing the accuracy and stability of the numerical methods used by the author in a series of calculations of the spectral distribution of γ rays incoherently scattered by the K electrons of medium to heavy elements (Whittingham, 1978 a,b). For this purpose computer routines were written to generate the Whittaker functions $M_{k',\gamma}$ and $W_{k',\gamma}$ for real and complex arguments. Aspects of the computation of these functions are discussed in the appendix of this report.

APPENDIX: COMPUTATION OF WHITTAKER FUNCTIONS

The Whittaker functions $M_{k',\gamma}$ and $W_{k',\gamma}$ are related to the regular and irregular Kummer functions $M(a,b,z)$ and $U(a,b,z)$ respectively via (Abramowitz and Stegun 1965, equations 13.1.32, 13.1.33)

$$M_{k',\gamma}(z) = e^{-z/2}\, z^{\gamma+\frac{1}{2}}\; M(\tfrac{1}{2}+\gamma-k',\ 2\gamma+1, z) \tag{A.1}$$

$$W_{k',\gamma}(z) = e^{-z/2}\, z^{\gamma+\frac{1}{2}}\; U(\tfrac{1}{2}+\gamma-k',\ 2\gamma+1, z). \tag{A.2}$$

The regular Kummer function $M(a,b,z)$ is just the confluent hypergeometric function ${}_1F_1(a,b,z)$, whereas the irregular Kummer function is given by (Abramowitz and Stegun 1965, equation 13.1.3)

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\,\frac{M(1+a-b,2-b,z)}{\Gamma(a)\,\Gamma(2-b)}\right\}. \tag{A.3}$$

For $|z| \leq 20$ the expansion (Abranowitz and Stegun 1965, equation 13.1.2)

$$M(a,b,z) = 1 + \frac{az}{c} + \frac{az}{c}\,\frac{(a+1)z}{2(c+1)} + \frac{az}{c}\,\frac{(a+1)z}{2(c+1)}\,\frac{(a+2)z}{3(c+2)} + \ldots \tag{A.4}$$

was used. $U(a,b,z)$ was then computed directly from (A.3).

For $|z| > 20$ the following expansions were used (Abramowitz and Stegun 1965, equations 13.5.1, 13.5.2):

$$\begin{aligned} M(a,b,z) = \Gamma(b)\, \Big\{ &\frac{z^{-a}}{\Gamma(b-a)}\Big[1 + \frac{a(1+a-b)}{z} + \frac{a(1+a-b)}{z}\,\frac{(a+1)(1+a-b+1)}{2z} \\ &+ \frac{a(1+a-b)}{z}\,\frac{(a+1)(1+a-b+1)}{2z}\,\frac{(a+2)(1+a-b+2)}{3z} + \ldots\Big] \\ &+ \frac{e^{z} z^{a-b}}{\Gamma(a)}\Big[1 + \frac{(b-a)(1-a)}{z} + \frac{(b-a)(1-a)}{z}\,\frac{(b-a+1)(1-a+1)}{2z} \\ &+ \frac{(b-a)(1-a)}{z}\,\frac{(b-a+1)(1-a+1)}{2z}\,\frac{(b-a+2)(1-a+2)}{3z} + \ldots\Big]\Big\}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} U(a,b,z) = z^{-a}\Big[1 &+ \frac{a(1+a-b)}{z} + \frac{a(1+a-b)}{z}\,\frac{(a+1)(1+a-b+1)}{2z} \\ &+ \frac{a(1+a-b)}{z}\,\frac{(a+1)(1+a-b+1)}{2z}\,\frac{(a+2)(1+a-b+2)}{3z} + \ldots\Big]. \end{aligned} \tag{A.6}$$

The Γ function of complex argument was computed by firstly using the relationship $\Gamma(z+1) = z\Gamma(z)$ to ensure that $|z| \geq 10$ and then using the expansion (Abramowitz and Stegun 1965, equation 6.1.41)

$$\ln \Gamma(z) - (z-\tfrac{1}{2})\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3}$$

$$+ \frac{1}{1260z^5} - \frac{1}{1680z^7} + \frac{1}{1188z^9} - \ldots \quad (A.7)$$

The routines for $M(a,b,z)$, $U(a,b,z)$ and $\ln\Gamma(z)$ were written in double precision arithmetic. The series (A.4),(A.5),(A.6) were terminated when the ratio of the last computed term to the sum of computed terms was less than 10^{-9}.

REFERENCES

Abramowitz, M., and Stegun, I.A., 1965, Handbook of Mathematical Functions (New York: Dover Publications)

Akhiezer, A.I., and Berestetskii, V.B., 1965, Quantum Electrodynamics (New York: Interscience).

Brown, G.E., and Schaefer, G.W., 1956, Proc. R.Soc.A, 233, 527-36.

Jauch, J.M., and Rohrlich, F., 1955, The Theory of Photons and Electrons (Reading, USA: Addison Wesley)

Rawitscher, G.H., 1958, Phys. Rev. 112, 1274-81

Rose, M.E., 1957, Elementary Theory of Angular Momentum (New York: Wiley)

Rose, M.E., 1961, Relativistic Electron Theory (New York: Wiley)

Whittingham, I.B., 1971, J. Phys. A, 4, 21-37

Whittingham, I.B., 1978a, To be published

Whittingham, I.B., 1978b, To be published

Yennie, D.R., Ravenhall, D.G., and Wilson, R.N., 1954, Phys. Rev. 95, 500-12.

JAMES COOK UNIVERSITY OF NORTH QUEENSLAND

POST OFFICE, JAMES COOK UNIVERSITY, QLD 4811 TELEPHONES: Douglas 793711; Pimlico 792193; TELEX: AA77069

NATURAL PHILOSOPHY RESEARCH REPORTS ISSUED BY THE PHYSICS DEPARTMENT 1967 - 1978

No.	Author	Title	Issued
1.	Ward, J.F.	The Measurement of Meteorological and Oceanographic Parameters of the Southern Ocean by Coherent Radar Backscatter Sounding.	May, 1967
2.	Gibson-Wilde, B.C.	An Investigation at Townsville into Anomalous Long-Range Transequatorial VHF propagation, (1961-1966).	June, 1967
3.	Ward, J.F.	The Measurement of Oceanographic and Meteorological Parameters of the Ocean by Coherent Radar Backscatter Sounding-Initial Results and their Interpretation.	April, 1968
4.	Coleman, M.	On the Feasibility of Using Balloon Sondes for the Direct Measurement of Stratospheric Turbulence.	March, 1973
5.	Ward, J.F.	Power Spectra from Ocean Movements Measured Remotely by Ionospheric Radio Backscatter.	April, 1969
6.	Withdrawn.		
7.	Ward, J.F.	The Tropical Meteorology Research Unit within the University in North Queensland.	July, 1969
8.	Dexter, P.E.	Correlation Experiment: Townsville H.F. Radar - "Eltanin" Oceanwave; Visual Observation	January, 1970
9.	Ward, J.F.	The Measurement of Oceanographic and Meteorological Parameters of the Ocean by Coherent Radar Backscatter - Experimental Procedure, Equipment, Data Analysis, Extension to Stage III of Project.	November, 1970
10.	Taylor, R.B. and Bode, L.	Levels of ^{64}Cu Energy	January, 1970
11.	Dexter, P.E.	Observations on the Backscatter of Ultrasonic Waves from a Randomly Rough, Moving Water Surface.	January, 1971
12.	Nicol, J.L.	Flying Doctor Aerials - The Effect of an Earth of Finite Conductivity on the properties of an H.F. Aerial.	April, 1972
13.	Gibson-Wilde, B.C. and Martin J.S.	A Ray Tracing Method for VHF Trans-equatorial Propagation.	July, 1972
14.	Taylor, R.B. & Bacon, B.,	Low Energy Nuclear Capture.	Not issued
15.	Taylor, R.B. & Hille, F.	Low Energy Nuclear Capture.	Not issued

No.	Author	Title	Issued
16.	Blyth, W.P.	Lie Algebras In Mathematical Physics.	Not issued.
17.	Nicol, J.L.	Field Tests and Performance of a New Type of Base Station Array for Operation near 5MHz	May, 1971
18.	Ward, J.F.	Phase Fluctuations of Radio Frequency Reference Emissions.	May, 1971
19.	Ward, J.F.	Effects of Phase Path Fluctuations over Ionospheric Modes in H.F. Radar Backscatter Observations.	May, 1971
20.	Leigh, R.M.	Analysis of Nimbus III, Medium Resolution, Infra-Red Data for the Australian Tropical Region.	August, 1971
21.	Ward, J.F.	Topics from Physics Dept., and T.R.U.M.A.P., James Cook University submitted for W.M.O. Report No. 11 on Tropical Meteorology	Sept., 1971
22.	Thiel, D.V.	Model Experiments at UHF in the Quasi-Far Field for Array Design.	February, 1972
23.	Ward, J.F.	Standardization of National Frequencies by H.F.Radio Propagation.	April, 1972
24.	Heron, M.L.	Ionospheric Influences in Standardization of National Frequencies by H.F.Radio.	Incorporated in No. 28.
25.	Taylor, R.B. & Bode, L.	A Study of the Gamma-Ray Spectrum from ^{63}Cu (nγ) ^{64}Cu.	July, 1972
26.	Ward, J.F.	The Measurement of Meteorological and Oceanographic Parameters of the Ocean by Coherent, Radar Backscatter - Performance of Large Azimuth Rotating Array (Kissing Point) and Systems Parameters.	May, 1973
27.	Heron, M.L.	150 MHz Polarisation Recorder for U.S.Navy Space Surveillance Satellites.	July, 1973
28.	Heron, M.L., Christy, D., and Ward, J.F.	National Frequency Standardization by High Frequency Radio Propagation. Propagation Anomalies, Doppler Measurement and Equipment, Radar Backscatter Methods.	February, 1974
29.	Ward, J.F.	Topics from Physics Dept., and T.R.U.M.A.P., James Cook University submitted for W.M.O. Report No. 13 on Tropical Meteorology.	December, 1973
30.	Thiel, D.V.	Precision VLF Phase Measurement for Delineation of Subsurface Conductivity Perturbations.	October, 1974
31.	Ward, J.F. Christy, D. & Dexter, P.E.	The Measurement of the Oceanographic and Meteorological Parameters of the Ocean - Synchronous Detection for Amplitude - Time and Spectral Data.	December, 1974
32.	Henderson, C.L. and Ward, J.F.	High Resolution Studies of Local and Ionospherically Propagated H.F.Radio Noise.	February, 1975

No.	Author	Title	Issued
33.	Robson, R.E.	Non-Linear Diffusion of Ions in a Gas.	October, 1974
34.	Ward, J.F.	Topics from Physics Department and T.R.U.M.A.P., James Cook University submitted for W.M.O. Report No. 14.	April, 1975
35.	Robson, R.E.	Diffusion Cooling of electrons in a Finite Gas.	August, 1975
36.	Robson, R.E.	Non-Linear Diffusion of Ions in a Gas: II Diffusion in finite enclosures.	August, 1975.
37.	Thiel, D.V.	The Physical Mechanism Behind VLF Phase Measurement for Delineation of Subsurface Conductivity Perturbations.	October, 1975
38.	Thiel, D.V. and Ward, J.F.	The Use of Direct Phase Difference Measurements in Electromagnetic Prospecting.	November, 1975
39.	Ward, J.F. and Dexter, P.E.	An Australian Ocean Radar - A Long-Range Doppler Radar for Ocean Surface Studies using Backscatter via the Ionosphere	November, 1975
40.	Ward, J.F.	Topics from Physics Department and T.R.U.M.A.P., James Cook University submitted for W.M.O. Report No. 15 on Tropical Meteorology.	March, 1976
41.	Robson, R.E.	On Plume Trapping by an Elevated Inversion	November, 1976
42.	Ward, J.F.	Phase Gradient Determination to High Precision for VLF Field Investigation above Variable Sub-surface strata.	December, 1976
43.	Thiel, D.V.	Precision VLF Phase Measurement for Delineation of Subsurface Conductivity Perturbations.	December, 1976
44.	Nicol, J.L.	To be issued.	
45.	Dexter, P.E. & Ward, J.F.	Analysis and Interpretation of Data Derived by Radio and Ultrasonic Backscatter from Rough Water Surfaces.	January, 1977
46.	Ward, J.F.	Limits and Improvements to the Reliability of Data in Long Range, H.F., Radar Oceanography	February, 1977
47.	Ward, J.F.	Topics from Physics Dept and TRUMAP James Cook University submitted for W.M.O. Report No.16 on Tropical Meterorology	June, 1977
48.	Whittingham, I.B.	Point Coulomb solutions of the Dirac equation: Analytical results required for the evaluation of the bound electron propagator in quantum electrodynamics.	December, 1977
49.	Ward, J.F.	Some coherency problems in digital phase communications.	October, 1977
50.	Ward, J.F.	The First Eight years of operations of the Tropical Research Unit for Meteorological and Atmospheric Physics - T.R.U.M.A.P.	September, 1977
51.	Potts, R. & Robson, R.E.	Preliminary Report on Operation of Acoustic Sounder at James Cook University	December, 1977
52.	Whittingham, I.B.	WKB solutions of the Dirac equation for a point-Coulomb potential and their application in the relativistic theory of bound Compton scattering	December, 1977

No.	Author	Title	Issued
53.	Whittingham, I.B.	Elastic scattering of low energy γ rays: I. Theoretical background.	February, 1978
54.	Whittingham, I.B.	Elastic scattering of low energy γ rays: II. Theoretical cross sections for scattering of ^{152}Eu γ rays by Pb.	February, 1978
55.	Ward, J.F.	Electrical and Magnetic Effects in Tropical Cyclones.	February, 1978