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Abstract

The theoretical basis as well as the practical methods of empirical continuation of the differential cross section into the nonphysical region of the $\cos\vartheta$ variable are discussed. The equivalence of the different methods is proved. A physical applicability condition is given and the published applications are reviewed. In many cases the correctly applied procedure turns out to provide nonsignificant or even incorrect structure information which points to the necessity for careful and statistically complete analysis of the experimental data with a physical understanding of the analysed process.

Аннотация

В настоящей работе рассматриваются математическое обоснование и практические методы эмпирического продолжения дифференциального поперечного сечения в комплексной плоскости $\cos\vartheta$. Доказывается тождественность разных практических методов. Указываются физические условия, при которых метод применим, и дан обзор имеющихся в научной литературе применений. Если метод применён правильно, тогда в большинстве случаев извлечённая структурная информация статистически незначима или не согласуется с результатами других методов. Это подчёркивает важность статистически полного анализа данных и учитывания физических критерий применимости.

Kivonat

Jelen munkában a differenciális hatáskeresztmetszetnek a $\cos\vartheta$ változó nemfizikai tartományába történő empirikus folytatása elméleti alapjait és gyakorlati módszereit ismertetem. Bebizonyítom a különböző gyakorlati módszerek egyenértékűségét. Egy fizikai alkalmazhatósági feltétel birtokában ellenőrizten a szakirodalomban közölt alkalmazásokat. Nagyon sok esetben a következetesen alkalmazott folytatási eljárás statisztikusan nem szignifikáns vagy esetleg nem helyes szerkezeti információt ad. Ez arra mutat, hogy gondosabban kell az eljárást alkalmazni, melynek során az elemzett folyamat lefolyásának módját is meg kell vizsgálni.

1. INTRODUCTION

Recently, the new technique of empirical continuation of the differential cross section (Kisslinger 1972, Dubnicka et al. 1973, Borbély 1974) has penetrated nuclear physics with success. It is based on the fact that at certain nonphysical values of the variables the process can be described exactly by simple means. Actually, for transfer processes one has a pole singularity in the $\cos\vartheta$ plane, the residue in which contains information on the structure of the involved particles. In the most favourable cases this structure information can be extracted with such a small error which, up till now, was not possible. The analytic continuation being unstable, the empirical continuation - which is essentially an analytic continuation of empirically given (i.e. with errors) functions - is a very delicate problem. The difficulties of giving error estimates were only seemingly overcome by mathematical methods (Ciulli 1969, Cutkosky 1969), because in practice one lacks the additional information necessary for any error estimate (Miller and Viano 1973). Therefore the applications were mostly governed by the availability of experimental data in suitable numerical form and the results were checked only by comparing them with similar results of other methods. The connection between the two different empirical continuation methods, namely the extrapolation and the singularity subtraction methods, was also not studied. Of late, the results of the method have begun to be questioned (Nichitiu 1975, Locher and Mizutani 1978 a, b) too. In connection with this see the contributed paper of Borbély and Nichitiu (1975) to the Zurich Polarizetion Symposium and the author's comment to Plattner's (1975) lecture there.

The aim of this paper is to present in a coherent manner those ideas and methods, which are useful in practice to provide a means for understanding the underlying physical meaning in order to formulate applicability conditions which make it possible to judge the correctness of the result within the framework of the method itself. The key to it is provided by the

equivalence of the extrapolation and subtraction methods and by the easy physical interpretation of the latter. In possession of the applicability condition one can review the published applications of the method. Many of them turn out to be made under doubtful physical conditions and to provide incorrect results. The formal reasons are partly the omission of a statistically complete analysis and partly an incorrect comparison with the results of other methods following an incomplete treatment of spins (Blokhintzev et al. 1977). It is necessary to find out the physical reasons for the failure and work out methods of checking and improving the method to minimize the hazards of obtaining incorrect results. Though mostly simple or even trivial methods are used, the present paper can be used as an introduction only in combination with a study of the literature. In some cases the most trivial facts should be stressed since one may meet just the opposite statements in the literature. Since my endeavour has been to avoid polemics, I have tended not to give references in such cases.

In this paper only applications to low energy nuclear physics are dealt with. With the exception of one case only the continuation of the cross section in the $\cos\vartheta$ plane is discussed. By the forward dipersion relation approach a similar technique is applied to the continuation of the amplitude in the energy plane; nevertheless, it is not possible to discuss here the possibilities of that method because of the author's complete lack of experience in that field. Fortunately there is not much need for it as this task has already been undertaken by Locher and Mizutani (1978 a, b). I hope, however, that the conclusions with some modifications could also be applied in that field with a somewhat different physical background. Lately, the continuation of the polarizations in the $\cos\vartheta$ plane, rather than that of the cross section, has been proposed by Amado et al. (1978) and by Borbély (1978). Polarizations provide more detailed structure information. The physical background of the method is the same as for the continuation of the cross section. Though in this paper only the latter case is dealt with, the conclusions can be directly applied for polarizations.

For the empirical continuation method the starting point is the analycity of the function to be continued, as it is a sufficient and in some sense necessary condition of the convergence of polynomial extrapolations (Walsh 1960). The analycity of the cross section can be derived from that of the amplitude, the latter is a well studied problem (Shapiro 1965, Schnitzer 1965). The singularities of the two functions are, of course, connected with each other. The cross section of a transfer process has the following typical singularity structure in the $\cos\vartheta$ plane (Cutkosky and Deo 1968 a, b): i/ the physically accessible region is the $[-1,1]$ interval with no singularity inside it, ii/ the singularities are on the real axis, in many cases the transfer pole is nearer to the physical region than the left and right hand side cuts generated by other direct mechanisms. The cross section has a second order pole coming from the transfer pole of the amplitude, a first order pole from the interference of the amplitude pole with the regular background, and a background regular at the pole. The strength of the second order pole is defined entirely by the residues of the amplitudes, as in the case of non-zero spins one might have several amplitudes having a transfer pole. This connection is given by Borbély (1976) for the case when in one of the vertices only zero orbital momentum is present. For other more complicated cases this problem - as well as the connection with the residue of the forward elastic scattering amplitude in the energy variable and with similar quantities - is treated in detail by Blokhintzev et al. (1977). The reader is referred to their paper because such problems concern only the interpretation of the pole strength and the comparison with other results but they are not directly connected with the applicability of the continuation method itself. For the limited purposes of this paper it is sufficient to assume that the strength of the cross section pole is in all cases given by the product of two corresponding vertex constants only. For vertex constants I use the normalization of Borbély et al. (1968), Dolinsky et al. (1973), Borbély (1976) and Blokhintzev et al. (1977). The equivalent values of vertex constants given in other units as well as the values of other quantities are taken from the review paper of Blokhintzev et al. (1977).

Having information on the analycity structure of the cross section one can try to extract the strength of the pole by analysing experimental data. For this purpose one has two possibilities.

The first method is to remove the pole multiplying the cross section by a simple factor and then to use the mathematical theory of extrapolation to determine its strength. In section 2, I cite mathematical theorems giving the foundations of this extrapolation procedure. The impractical nature of the various error estimates given so far and a very important technique for improving the convergence of the extrapolation procedure, namely the optimal conformal mapping, are also discussed. To conclude this section the practical scheme of extrapolation is presented, and its convergence is proved on the basis of the previously cited theorems.

The second method is to determine the strength of the singularity from the asymptotical behaviour of expansion coefficients, this is discussed in section 3. After presenting the mathematical basis and discussing some difficulties in determining the asymptotical region by mathematical methods, a very simple and transparent physical condition is given for this purpose. Various generalizations of the method are discussed and their statistical equivalence to each other as well as to the extrapolation method is proved.

In section 4 various practical ways of improving the performance of the continuation method are discussed and, by using them in section 5, the interesting fact is demonstrated that in practical cases the systematic error of the extracted singularity strength given by the continuation method itself could be of the order of less than 1 %. The somewhat academic problem of the model independence of the method is discussed in section 6.

All the published applications to the continuation of the cross section in the $\cos\vartheta$ variable are carefully reviewed in section 7 using the above developed methods with a brief review of the theory of Coulomb effects. Finally, the conclusions reached are presented.

2. EXTRAPOLATION OF ANALYTIC FUNCTIONS

If it is possible to remove a singularity from the cross section by some means, then one can extract the strength of this singularity by extrapolating the regular function. To be more precise, later on the $z = \cos\vartheta$ plane is considered with the $[-1,1]$ interval being the physical region. As an example of removing the singularity, in the case of an amplitude pole at $z = z_p$, $(z_p - z)^2 \sigma(z)$ is regular at $z = z_p$. Polynomials fitted to the measured values of the regular function provide an obvious way of extrapolation (Chew 1958, Taylor et al. 1959). Polynomials are simple to handle in computation and, what is more, an elaborated theory of such approximations to analytic functions is available (Walsh 1960)*.

*References to this useful book do not mean that the actual theorem is Walsh's.

2.1. Mathematical aspects of extrapolation

Analytic continuation is a unique process, nevertheless at first sight it is not possible to extrapolate an analytic function beyond that region where its values are available only with some error, irrespective of the error's tending to zero. The point is that one can easily contruct a sequence of regular functions (for instance, powers of the variable) which are arbitrarily small inside the physical region while they are arbitrarily large outside it. But if a priori information (a priori here means that one cannot prove it using the measured values alone) on the size of the regularity region is available, then one has a convergence theorem: the best polynomial approximation in the physical region converges to the true value in its neighbourhood, too.

To give the latter statement a more precise meaning one needs to cite some theorems from Walsh's book (1960) on polynomial approximations. Let us

consider the ellipses with focii at $z=\pm 1$. The ellipse with the sum of semi-axes R is denoted by C_R while $C_1 \equiv C$ is the $[-1,1]$ interval. The minimum value of R so that $f(z)$ is not regular inside C_R is denoted by ϱ. Sometimes the ellipse C_ϱ is called convergence ellipse.

Theorem 1 (Walsh 1960, theorem 4.7 and the text following). A necessary and sufficient condition of $f(z)$ being regular inside C_ϱ and having a singularity on C_ϱ $(\varrho > 1)$ is the existence of such polynomials $p_n(z)$ of degree n, a constant M (which can depend on R but is independent of n and z), and arbitrary $R < \varrho$, so that

$$\max |f(z) - p_n(z)| \leq M/R^n, \quad z \text{ in } C, \; n = 0,1,2, \ldots \qquad (1)$$

and the nonexistence of such polynomials for $R > \varrho$. Such polynomials show a maximum convergence to $f(z)$ in C.

Theorem 2 (Walsh 1960, theorem 4.8). If (1) is valid for some R, then

$$\max |f(z) - p_n(z) \leq M'(\sigma/R)^n \quad z \text{ on } C_\sigma . \qquad 2$$

The phenomenon that the convergence in a region with a sufficient high speed implies it in a broader region too, is called superconvergence. Theorem 2 is extremely important for us; the above mentioned powers of the variable are excluded from consideration if they do not show the necessary speed of convergence to zero inside the physical region. But note that apart from the mere existance of the constant M' we have no information on its value, i.e. we have no error estimate (see later, too).

The theorems citèd so far establish the existence of an extrapolation procedure. An important nonexistence theorem is the following.

Theorem 3 (Walsh 1960, theorem 4.11). Polynomials showing maximum convergence cannot coverge uniformly in a region containing any part of the

ellipse C_ϱ inside its interior. In other words, apart from possible isolated points one necessarily has a divergence outside C_ϱ. It is the reason for calling it convergence ellipse. If theorems 1, 2 and 3 are compared one obtains that the very same condition, namely the maximum convergence of the polynomial sequence in the physical region, ensures the convergence of the extrapolation inside C_ϱ and the divergence outside it.

One can complete the extrapolation scheme by pointing out some practical ways of finding out polynomials showing maximum convergence.

Theorem 4. The following sets of polynomials show a maximum convergence:

i/ Chebyshev approximations (Walsh 1960, theorem 5.2).

ii/ Interpolating polynomials with suitably distributed interpolating points (Walsh 1960, theorem 7.6).

iii/ Polynomials $\pi_n(z)$ given by the minimum of the integral

$$\int_{-1}^{+1} |f(z) - \pi_n(z)|^2 \, w(z) \, dz \qquad (3)$$

with a positive and continuous weight function $w(z)$ (Walsh 1960, theorem 5.3, see also theorems 5.12 and 5.13 for weaker conditions on the weight function).

Case iii/ is the most important since functional (6) minimized in a least squares fit to measured points is an integral sum of integral (3) (see later). In this way one has a practical means of carrying through an extrapolation convergent outside the physical region.

2.2. Difficulties of an effective error estimate

The only difficulty we have to face is that the rate of convergence is slower outside the physical region than inside it and we have no estimate for the constant M' in formula (2). One cannot decide whether the extrapolated polynomial which fits the data within their statistical errors is

near enough, i.e. within its own statistical or a previously prescribed error, to the true value. To solve this problem one should take some additional information into consideration. One can suppose that on the convergence ellipse an upper bound for the function is known, or some Hölder type inequality is valid there, and so on (Ciulli 1969, 1973, Ciulli et al. 1975). Based on them one can estimate the possible error of the extrapolated value. Because of the very essence of this approach such estimates always contain at present inaccessible (because they concern the nonphysical region) parameters, therefore they are not useful. A statistical approach was proposed by Cutkosky (1969), who assumed that on the convergence ellipse the function takes on random values according to a Gaussian distribution with a zero mean value and an unknown but determined from the experimental data dispersion. Apart from some inner problems of the formulation (see Ciulli's (1973) review) one might rightly ask, together with Miller and Viano (1973): how can we check the validity of the basic statistical assumption? For reactions between particles of simple structure one has only a very limited number of singularities on the convergence ellipse (even after the optimal conformal mapping, see later), so one doubts its validity. Nevertheless, for composite particles, like the reaction $^{31}P(d,p)^{32}P$, one has an enormous number of densely located singularities, so in such cases it might describe the real situation. But in any case of a special consequence of this approach, like a contradiction between the so-called Cutkosky's (1969) convergence test function and a traditional χ^2 test, the validity of the basic assumption should be checked carefully. Here again inaccessible information concerning the nonphysical region should be used. One can conclude that within the framework of the extrapolation method no practically effective applicability condition has been formulated. One of the basic aims of this paper is to fill this gap.

2.3. Optimal conformal mapping

A very important technique for improving the performance of the method is the optimal conformal mapping proposed by Cutkosky and Deo (1968 a,b)

and by Ciulli (1969). According to theorems 1 and 2 the possible rate of convergence is defined by the location of the nearest singularity which determines the value of ϱ, i.e. the upper bound for R. If it is possible to map an additional region of regularity into the interior of the new convergence ellipse, then for the new plane the value of ϱ increases (Ciulli 1969). Therefore for the class of functions which have left and right hand side cuts on the real axis with the further singularities on the edges of these cuts, the optimal conformal mapping is that which maps the upper and lower edges of the cuts on the new convergence ellipse. As for the details, the reader is referred to the original literature. Such a mapping usually does not alter the theoretical basis of the method but is very useful from a practical point of view. In many cases only the faster convergence ensures the success of the extrapolation with the given errors of the experimental data. Later it will be demonstrated that in favourable cases one can use the original $\cos\vartheta$ variable, too.

Usually one extrapolates to the nearest singularity. Sometimes it happens that in the other direction one has a singularity nearer the physical region. In such cases it is unavoidable to apply some mapping which moves this singularity to such a distance that the new convergence ellipse contains in its interior the point one extrapolates to. The necessity for such mapping directly follows from theorem 3. Note also that if experimental data are available only in some part of the physical region - or what is equivalent, one deliberately uses only some part of the data - then in the above formulation one should understand this interval as the physical region. Before making the analysis the most simple thing is to use a linear transformation mapping it on the $[-1,1]$ interval (see the summary section of the paper by Cutkosky and Deo (1968 b)). Since experimental data at $\vartheta = 0^0$ and 180^0 are rarely available, stricly speaking one should always include such a transformation into the optimal conformal mapping. Obviously in most cases it makes no practical difference.

2.4. Practical scheme of the extrapolation

In practice, one has measured values of the function to be extrapolated only at some discrete points of the suitably chosen variable. By the usual least squares procedure one should fit the extrapolating polynomials to the experimental points (Chew 1958, Taylor 1959) rather than to find out the maximum convergence polynomials of various subcases of theorem 4 for a continuous function. Nevertheless, the convergence of such a procedure can be proved, too.

From a theoretical point of view it is not important which polynomial system (i.e. Chebyshev, Legendre, Laguerre, etc. polynomials) we choose to fit the data. As the condition we define their coefficients from is the same, the fitted coefficients represent the same information and one can easily calculate the coefficients of a given polynomial system from the coefficients of another one. But, naturally, a fixed number of terms in the usual expansions are not equivalent as different weight functions are used to determine the coefficients. Perhaps this is the reason for some confusion in choosing the polynomial system. Very often the polynomial system is specially chosen to simplify the least squares procedure (Kisslinger 1972, Borbély 1974), which gives practical advantages, particulary in handling the statistical errors.

If the polynomials $B_n(x)$ are chosen to satisfy the orthonormality relations

$$\sum_{i=1}^{N_i} B_n(x_i)\, w_i\, B_k(x_i) = \delta_{nk} \qquad (4)$$

and the recurrence relations

$$B_{n+1}(x) = (\alpha_n x + \beta_n) B_n(x) + \gamma_n B_{n-1}(x) \,, \qquad (5)$$

where the w_i weights are positive and the α_n, β_n, γ_n constants are defined by formula (4), then

$$\chi^2 = \frac{1}{N_i - N} \min_{\{A_n\}} \sum_{i=1}^{N_i} w_i \left\{ f_{exp}(x_i) - \sum_{n=1}^{N} A_n B_n(x_i) \right\}^2 \qquad (6)$$

is given with

$$A_n = \sum_{i=1}^{N_i} w_i \; f_{exp}(x_i) \; B_n(x_i) \; , \qquad (7)$$

where $f_{exp}(x)$ is the function to be fitted. If one works in the $z=\cos\vartheta$ plane, for the extrapolation method

$$f_{exp}(z) = (z_p - z)^2 \sigma(z)$$

while the w_i weights are given by its errors

$$w_i = (1 / \Delta f_{exp}(x_i))^2 .$$

The rms error of A_n is $\pm$ 1 and these coefficients are not correlated. The strength of the pole is given by the fitted and extrapolated polynomials

$$\lim_{z \to z_p} (z_p - z)^2 \sigma(z) \approx \sum_{n=1}^{N} A_n B_n(z_p) \; , \qquad (8)$$

where the sum contains all the significant coefficients and only those. The number of significant terms N of formula (8) can be determined from two equivalent conditions. i/ The actual values of the A_n coefficients with $n > N$ do not contradict the hypothesis that their "true" expectation values are smaller than their rms errors. ii/ With N terms the expectation value of $\chi^2 \approx 1.0$ is reached. It is clear that one should include all the significant terms as they are necessary even for the description of the data in the physical region, and one should reject all the nonsignificant terms because they make the error of the result larger, and they do not add any significant contribution to its value. If one deviates from this traditional rule of statistcs the reasons should be clarified.

Luckily enough functional (6), with the aid of which one defines the

extrapolating polynomial, is an approximating integral sum of functional (3), $N \ll N_i$ is a necessary condition of having a good approximation. Using it one can easily prove the convergence of this extrapolation scheme, too. For this purpose it is sufficient to construct a relatively smooth weight function $w(x)$ from the available weight set $\{w_i\}$ and to take more and more "measured" points into consideration always ensuring that sum (6) be a good approximation of integral (3). If the number of points is increased, the number of significant terms N slowly increases - with theorem 4 proving the maximum convergence. Note that the just described method allows the planning of experiments. With some preliminary information one can even minimize the error of the result by suitably choosing the new measured points.

It is clearly a hypothesis only that the exact values of the non-significant coefficients in formula (8) are zero, but one has no other practical condition at hand to define the truncation of the sum. It should be noted that the above discussed Cutkosky's convergence test (Cutkosky 1969) was applied by Dubnicka and Dumbrajs (1973, 1974 a, b) to define the number of terms in formula (8). In a large number of cases, however, when I checked their analysis (see section 7) I found no contradiction to the χ^2 test.

Including more and more terms in sum (8) one gets increasingly larger errors for the result. If, with the significant terms only, one already has a larger error than the value of the result, this fact cannot be considered as a failure of the method. One has a clear indication that the result is not significant and one has to use better data. Strictly speaking, the method fails in those cases when the deviation between the extracted strength of the pole and its true value is considerably larger than the statistical error provided by the analysis. Methods to control the systematic error of the method will be given later.

3. THE METHOD OF ASYMPTOTICAL COEFFICIENTS

In this section a different method, the singularity subtraction method, and its generalizations are discussed. This method was first applied by Cutkosky and Deo (1968 a) and was developed further by Borbély (1974). I give the most general formulation of it here and for obvious reasons I propose to call it the method of asymptotical coefficients. In the formulation of this method one use values of the function only in the physical region, so there is no need to solve the above discussed instability problem. On the other hand, as will be proved, the extrapolation method and the method of asymptotical coefficients are essentially equivalent. Because of this, the same problem presents itself in a different form. The formulation of the method makes it possible to give a physical applicability condition rather than a mathematical one. A further advantage is the greater felxibility for singularities other than a pole.

3.1. Mathematical basis of the method

To formulate the mathematical basis of the method let us consider orthonormal polynomials $p_k(z)$ with a positive continuous weight $n(z)$:

$$\int_C p_k(z)\, p_\ell(z)\, n(z)\, dz = \delta_{k\ell} \tag{9}$$

and let us consider the expansion coefficients a_k of an integrable function $f(z)$

$$a_k = \int_C f(z)\, p_k(z)\, n(z)\, dz . \tag{10}$$

Note that actually the a_k expansion coefficients provide the polynomial which minimizes functional (3) (Walsh 1960, § 6.5). Therefore expansion and approximation coefficients are the same provided that the weight is fixed, so later on they are not necessarily distinguished from each other.

Theorem 5 (Walsh 1960, theorem 6.7). If $f(z)$ has a singularity on C_ϱ and is regular inside it, then

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1/\varrho \tag{11}$$

It means that the asymptotical behaviour of a_n is roughly given by an exponential which depends on the position of the nearest singularity or singularities only with a "smoother" pre-exponential factor $F(n)$, viz.

$$a_n \sim F(n)\, e^{-n \ln \varrho} \tag{12}$$

Naturally, $F(n)$ depends on the singularity type. Popov (1964) gave the asymptotical behaviour of Legendre polynomial expansion coefficients for different singularities of the amplitude.

If there is only one singularity on C_ϱ, one can determine its strength by comparing its expansion coefficients with those of $f(z)$ in the asymptotical region where the contribution of other singularities fades out. Since, according to formula (12), with increasing n the contribution of other singularities located further decreases with higher speed, such a region necessarily exists. But it is not so easy to decide whether the statistical errors of the experimental data makes it possible to reach it. Borbély (1974) gave the condition that in the asymptotical region within their statistical errors the coefficients should show a behaviour corresponding to the location and type of the singularity. In particular the strength of the singularity extracted from different coefficients should be consistent statistically. Because of this I call it the self-consistency condition. In fact Borbély (1974) gave this condition in a different but equivalent form (see subsection 3.3). When this condition is applied I call the method the singularity subtraction method. It will be proved that the extrapolation and subtraction methods are statistically equivalent. The self-consistency condition is a necessary one only (Borbély 1975), the physical understanding of the process which is analysed can help to give a sufficient condition.

3.2. A physical applicability condition

It is well known that for transfer processes the pole contribution in the traditional formalism corresponds to PWBA matrix elements with no distortion in the initial and final channels and with the asymptotically valid (in the r space) Hankel type wave functions for the transferred particle (e.g. Borbély et al. 1968). For sufficiently high partial wave amplitudes (with obvious notations $\ell \geq kR$) it is a good approximation.

Let us suppose that with a given accuracy the partial wave amplitudes with $\ell > L$ are determined by the pole. In this case one has for the amplitude in the physical region

$$t(z) \approx P_L(z) + g/(z_p - z) , \qquad (13)$$

where the polynomial $P_L(z)$ of degree L describes the deviation from the pole contribution in the low partial wave amplitudes. For the cross section one has

$$(z_p - z)\,\sigma(z) \approx P_{2L+1}(z) + G/(z_p - z) \qquad (14)$$

This formula immediately implies that the asymptotical region in the expansion of $(z_p - z)\sigma(z)$ is that with $n > 2L+1$. However simple this condition might be, it gives a firm control over the method, since one usually has a good estimate for the interaction radii, or the value of L is known from other sources. Note that in the latter case the L=k.R connection can serve as a definition for the interaction radius if one wants to work entirely with amplitudes and avoid the position space. Apart from the quantitative side of the problem, qualitatively the above condition implies that one should have a well measured exchange peak as this is the most sensitive region to the highest partial wave amplitudes (see peripheral model calculations by Borbély and Dolinsky (1969)).

Formulae (13) and (14) also make it possible to discuss the disturbing effect of compound resonances (Cutkosky and Deo 1968 b, Nichitiu 1975) and of the strong contribution from low partial wave amplitudes in general. Let us suppose that we fix the pole contribution in partial wave amplitudes with $\ell > L$ and change the other amplitudes. No matter that low partial wave amplitudes are small or large, they have no effect on the expansion/approximation coefficients of $(z_p - z)\,\sigma(z)$ with $n > 2L+2$, once the weight (i.e. the errors of the data) is fixed. But in practice, naturally, the weight is not fixed, a larger polynomial background in formula (14) means larger statistical fluctuations as well, which sooner or later completely absorb the fixed pole defined terms. Therefore large low partial wave amplitudes, and resonances in particular, make it more difficult to reach the asymptotical region. In their presence one can expect a decrease in the number of significant terms for the extrapolation method, too (see the following proof of the equivalence of the two methods and the discussion of the $^{31}P(d,p)^{32}P$ reaction).

3.3. Equivalence of the extrapolation and subtraction methods

The equivalence of the extrapolation and subtraction methods can be proved as follows. Let us suppose that in the fit

$$(z_p - z)\,\sigma(z) \approx \sum_{n=1}^{N-1} A_n B_n(z) + \sum_{n=N}^{M} A_n B_n(z) \qquad (15)$$

the N-th and higher order significant coefficients show a pole behaviour in the sense of the above formulated self-consistency condition. Specifically, it means that one has N significant terms (Borbély 1974) in the fit

$$(z_p - z)^2\,\sigma(z) \approx \sum_{n=1}^{N} A'_n B'_n(z)\,. \qquad (16)$$

But one does not necessarily suppose that the true asymptotical region is reached, i.e. the pole behaviour can be fortuitous. Now, one can change the experimental values well within their error in such a way that they give the fitted values provided by the right hand side of formula (16). In this way

the result of the extrapolation method is not changed. The subtraction method applied to this pseudoexperimental data can, of course, give only the same result as the extrapolation method. This follows from the construction of these data since their values are given exactly by a second and first order pole and a polynomial background. But as the data are changed well within their errors, the subtraction method results for the experimental and pseudoexperimental data can differ from each other only within their own errors, which immediately implies the statistical equivalence of the subtraction and extrapolation methods.

The equivalence of the two methods shows that the problems in determining the asymptotical region with the aid of the self-consistency condition essentially coincide with the difficulties in determining the sufficient number of terms for extrapolation. The equivalence also helps one to understand that the restriction for the optimal conformal mapping, namely that the improvement given by it is only an asymptotical one, is irrevelant. In the subtraction method one has to reach the asymptotical region of the pole contribution, which is well within the asymptotical region of the background singularity contribution, where the mapping always gives some improvement. The real problem, however, is that in changing the position of the singularity one changes its strength as well, which in a given expansion coefficient might compensate for the faster exponential decrease - at least partially. Here, the strength of the singularity is understood in a broad sense as by the mapping one can also change its character.

3.4. Generalizations

The pole is a very special type of singularity. With simple factors it is very easy to remove from the cross section the pole itself or its interference term with the background amplitude. It is not possible in the case of more complicated singularities For elastic scattering with Coulomb interaction, for instance, at $\cos\vartheta = 1$ one has the Rutherford pole of second

order, an interference branch point of complicated structure and a regular nuclear background. A conformal mapping which removes the branch point introduces new singularity in the background. In such a situation one has a straightforward generalization of the subtraction method. Namely, one should describe the significant expansion coefficients in the asymptotical region, i.e. where the background is not important, simultaneously with as many singularities as needed. It is still possible because the pre-exponential factor in the asymptotical behaviour depends on the type of the singularity. For this purpose, however, one needs several coefficients and due to the larger number of fitted parameters, a more strict physical control. The method remains the same if the singularities are not at the same place on C_ϱ or even if they are on different ellipses. The self-consistency condition can be applied in all these cases to define the asymptotical region, but additional physical considerations are also desirable. Applying a method similar to the one previously used for proving the equivalence of the extrapolation and subtraction methods one can easily prove that this method of asymptotical coefficients is statistically equivalent to a direct fit of the unknown parameters of the singularities and a polynomial background to the experimental data themselves. From a computational point of view the method of asymptotical coefficients is simpler as one fits only several expansion coefficients rather than all the experimental data.

To make clear the connection of the various methods one can summarize them as follows. The extrapolation method removes the first and second order poles of the cross section and extrapolates $(z_p-z)^2\sigma(z)$ to $z=z_p$ by regular polynomials. In general, such an extrapolation is possible if one can handle the difficulties of a simultaneous removal of the pure and interference singularities. The singularity subtraction method removes the interference pole and determines the strength of the remaining pure pole of $(z_p-z)\,\sigma(z)$ by using the significant expansion coefficients in the asymptotical region. The asymptotical region can be determined by the self-consistency condition, in particular from the analysis of $(z_p-z)^2\sigma(z)$, or from physical considera-

tions. In general, such a method can be used if one can eliminate all the other singularities except the studied one from the cross section at the place of the studied singularity. The method of asymptotical coefficients is a generalization of the subtraction method, one describes the coefficients in the asymptotical region with as many singularities as needed. All these methods are statistically equivalent to each other, provided they can be applied to the given types of singularities. They are equivalent also to a direct fit to the experimental data, as has been discussed just above. Because all these methods continue into the nonphysical region an empirically given not necessarily regular analytic function, I propose to call them empirical continuation methods.

The method of asymptotical coefficients can be applied for determining the location of the singularity, too. The only modification needed is that one does not describe the asymptotical region of the coefficients with a fixed position singularity, but fits the position too. The asymptotical region could be defined by the self-consistency condition. Such a procedure is far more practical than the methods reviewed by Ciulli (1973) and Ciulli et al. (1975). I checked the applicability of this method in the case of a model energy dependence of the cross section given by the effective range formula for the n-n scattering. This is important because when measuring the relative kinetic energy spectrum in an n-n final state interaction peak with a sufficient but experimentally still feasible accuracy, one can extract the location of the virtual bound state providing a constraint on the n-n scattering length and effective range parameters. This case illustrates again a very attractive feature of the continuation method, its ability to forecast the error of the result. Having sufficiently accurate preliminary information one can always build up realistic methods for this purpose.

4. SOME WAYS OF IMPROVING THE PERFORMANCE OF THE CONTINUATION METHOD

There are ways of improving the performance of the continuation methods. Broadly speaking, one has two possibilities. Firstly, the optimal conformal

mapping, as one applies it following Cutkosky and Deo (1968 a,b) and Ciulli (1969), is optimal for a given class of functions. But not the entire region of regularity is mapped inside the convergence ellipse for a particular function, as not all points of the cuts are singularities for the given function. To be able to exploit a larger region of regularity one should perform an additional mapping either by studying the structure of nonphysical Riemann sheets - which is a difficult task, or by minimizing the number of parameters needed to represent the experimental data choosing some suitable parameters in a test function for the mapping. In more practical terms, one can fit the data instead of a polynomial $P_m(z)$, for instance, with a "double" polynomial $P_n(Q_k(z))$, where the argument of P_n is itself a polynomial Q_k (naturally, $n*k \leq m$). The argument of P_n could be a rational fraction or a similar function. The second possibility is to use such functions to fit the data which have singularities and these singularities can serve as a model for that of the fitted function. The point is that one can always describe a limited section in the asymptotical region of the expansion coefficients by a pole, for instance, since slightly changing its position it is possible even to imitate a different pre-exponential behaviour. In practice, one can apply rational fraction approximations (Borbély and Nichitiu 1975, 1976), where the positions of the poles are also fitted. Finally, one can combine these two methods, by using for instance, a rational fraction argument for a rational fraction. These possibilities are well known in the theory of approximations (see, for instance, the book of Bakhalov (1975) for an estimate of their speed of convergence in the physical region). Here, I have given only a "physical" interpretation. One should be aware that these approaches have no such an elaborated theory as have polynomial approximations. More complicated computational methods should also be used and one should explicitly limit the values of the fitted parameters, as in an empirical fit one can easily introduce singularities in the regularity region (Borbély and Nichitiu 1976).

A simple and important case in practice is when one does not fit the location of the model pole but chooses it applying physical considerations. If a singularity at $z=z_t$ prevents reaching the asymptotical region of

the expansion coefficients, i.e. one has a considerable contribution from it in the highest significant coefficients, then one can analyse $(z_t-z)\sigma(z)$ by the standard polynomial approximations. It is equivalent to a pole fixed at the singularity plus a polynomial background approximation to $\sigma(z)$, naturally in addition to the usual transfer pole. One can also analyse $(z_t-z)^2\sigma(z)$, because in this case the second and first order poles give more possibilities to describe the singularity. One can justify this method from another point of view, too. It is clear that such factors do not necessarily remove the singularity, therefore in the asymptotical behaviour one has the same exponential. One can easily prove, using the explicit expressions for the pre-exponential factors in formula (12) given by Popov (1964), that one has a faster decrease for the new singularity - in complete agreement with one's "physical intuition". The shortcoming of this simple procedure is the increase in the number of parameters needed to describe the contribution of other singularities. One should therefore apply such factors with restraint.

5. CHECKING OF THE RESULT FOR A SMALL ERROR CASE

As the very small error for the pole strength given by the empirical continuation method in the most favourable cases is usually doubted, I think it is not without benefit to study in some detail the smallest error case met by me so far. It is the analysis of the d(d,p)t reaction at E_d=25.3 MeV (Borbély 1974), where the statistical error of the pole strength is 1.5 %. One has 41 points with a typical error of 3.5 % (van Oers and Broskman 1963), so if one knew the shape of the angular distribution, a fit of the normalization would give an error of 3.5 %/ $\sqrt{41}$=0.5 % approximately. Therefore the "effectiveness" of the method is 0.5 %/1.5 %=0.33, which is quite reasonable. The main problem here, however, is not the magnitude of the statistical error which can easily be calculated, but the magnitude of the systematic error connected with the contribution of the background singularities in the asymptotical region of the expansion coefficients. To check it one has two possibilities. Firstly, the application of the just discussed methods of improvement at least changes even if does not decrease its value, therefore

comparing the results given by them one has information on the magnitude of the systematic error. Secondly, the above formulated physical applicability condition helps to define the asymptotical region. A peripheral model fit to the data (Borbély 1971, see the details there) gives that one has some distortion in the $\ell=2$ partial wave amplitudes. Applying the consideration leading to formula (14) and including the symmetry of the cross section due to the identity of the deuterons (see the general formulae of Borbély and Dolinsky (1967)) one obtains that for $(z_p^2-z^2)\sigma(z^2)$ the first 4 terms contain distortion effects, the higher index terms are free of them. It is obviously valid for the $\cos\vartheta$ plane. One can apply a comformal mapping and/or introduce a factor of $(z_t^2-z^2)^2$ to suppress the contribution of the branch point which generates the left and right hand cuts at $z_t=\pm 3.37$. Borbély (1974) performed the analysis with the optimal conformal mapping and without the suppression factor. The results of the singularity subtraction method for the four possible cases are given in table 1. As the selfconsistency condition gives that the n=4 and 5 terms belong to the asymptotical region, I give the pole strength $G_d^2\,G_t^2$ following from them in all cases, though the above consideration shows that for the no mapping no suppression case only the n=5 term can be used. Therefore in the latter case the result following from the n=4 term is given in parentheses. The comparison of the other three cases with n=4 shows that in the best case with the mapping and suppression factor, one has a remarkable small systematic error of the order of the statistical error or less, i.e. about 1 %. Note that the n=5 results show negligible changes, one has an even smaller systematic error for them. But, naturally, the statistical error for them is much larger. The above consideration concerns only the error of the continuation method itself, and the estimates do not contain the normalization error and other systematic errors of the experimental data. The correlation of the experimental errors as well as Coulomb effects (see later) are also not included. I emphasize that such a small systematic error is the consequence of the fact that, as a peripheral model fit (Borbély 1971) shows it, the cross section is dominated by the pole defined large partial wave amplitudes and the background singularities are unusually far from the pole

($z_p=\pm 1.33$ while $z_t=\pm 3.37$). Therefore the background contribution fades out very rapidly from the expansion coefficients. In other less favourable cases one can even get incorrect results, as can be seen from the following.

6. THE PROBLEM OF MODEL INDEPENDENCE

No doubt the reader is now sufficiently confused with all this trickery about improvement of approximations, therefore it is time to discuss the model dependence or independence of the method. First of all I should like to point out that the formulae (Borbély 1976, Blokhintzev et al. 1977) used for the strength of the transfer pole are exact in the sense that no "form-factor effects" (Kisslinger 1973) nor "nuclear radius effects" (Dubnicka and Dumbrajs 1974 a) are neglected, simply because one has no such effects in the pole itself. Again, that one uses or can use a suitably chosen variable, i. e. one applies conformal mapping, cannot be considered as a model (Choudhary 1976). The analytic continuation being unique, its result is independent of the choice of the variable. On the other hand, the result of an empirical continuation can depend on the variable one uses, as it directly affects the error of the various approximations with a sometimes severely limited number of terms. Therefore the fact of the unavoidable approximation is the point one should discuss. Here one cannot avoid making some allowance for a "model element" in the method (Borbély 1974, but see also Borbély 1975). It should be clear, however, that the mere fact of approximation in not equivalent to a model as the term is used in nuclear physics. In this respect the shell-model, for instance, with its average field and residual interaction, or the DWBA with its optical model wave functions used in the nuclear interior, cannot be compared to the empirical continuation method. The method is asymptotically exact in the sense that the error of the result tends to zero as one has better and better experimental data. A simple condition, namely that the number of significant terms should be higher than roughly speaking $2kR$, can be given determining when the result is near to the asymptotical exactness. The only assumption about the asymptotical region one makes based on this condition is not a model as defined above. But sometimes, if the

exact analycity properties are not known (see later the effects of the long range Coulomb interaction), one can use approximate expressions for the singularities which might be so rough that they can serve as a model only.

7. APPLICATIONS

In nuclear physics taken in the strict sense, Amado (1959) was the first to propose the application of the empirical continuation method. He, however, proposed to extrapolate $\sigma_{exp}(z)/\sigma_{theor}(z)$, where σ_{theor} is given by the Butler theory. In this way, additional poles at the zeros of $\sigma_{theor}(z)$ are introduced strongly hindering, if not paralyzing, the continuation procedure. Dullemond and Schnitzer (1963) checked the extrapolation method on data calculated by DWBA. For a possible explanation for their failure see the paper of Borbély (1976). Following the introduction of optimal conformal mapping, a long series of papers was published with successful applications.

In this section these applications of the continuation method are discussed in the light of the above ideas. They are grouped according to the extracted structure information with increasing mass number. Many of them claimed to have been successful turn out to give doubtful results. It can be seen not only by performing the analysis consistently and comparing the results correctly with other methods, but by applying the above given physical applicability condition as well. It is possible to estimate the reliability of the results by very simple means, which I shall try to illustrate in the following. To gain experience in avoiding doubtful applications it is necessary to point out the error made; the continuation method can only profit from a clear-cut determination of its possibilities. Naturally, one can speak only of the present day possibilities; sufficient improvements in the continuation technique can change the situation in the future.

A quick erosion of the reputation of a method can easily result from a situation in which many of the applications gave virtually incorrect results;

this process has already begun (Nichitiu 1975, Locher and Mizutani 1978 a,b). One should be aware that every method has its own limitations; applications beyond the possibilities should not devaluate the method itself provided that well defined applicability conditions are available and it is possible to check the result within the framework of the method. In favourable cases the empirical continuation method did produce such a small error which was not feasible until now. One should not forget that the method uses directly measurable cross section data and the evaluation is simple and transparent. Within its field of applicability the continuation method is in some sense superior to other methods in extracting structure information (advantages and disadvantages are illustrated by Borbély (1976)), therefore there is no doubt that it will have more applications in the future.

7.1. The deuteron vertex constant

From a theoretical point of view the best known vertex constant is the $d \to p+n$ one. We know it from the residue in the bound state pole which appears in the energy plane for n-p scattering - as the energy dependence is well described by the effective range formula. Moreover, the various realistic nucleon-nucleon potentials also give the same value through the asymptotical normalization of the deuteron wave function. This is natural since they are adjusted to describe the parameters of the effective range formula. The value of G_d^2 is 0.43 ± 0.01 fm (Blokhintzev et al. 1977), the error is an estimate of the maximum possible systematic errors, i.e. it is not statistical in character. Later, one needs its squared value $G_d^4 = 0.18 \pm 0.009$ fm^2. Because it is known so well, the deuteron vertex constant could serve as a test of the continuation method.

7.1.1. Analysis of the n-d scattering data

In the $\cos\vartheta$ plane the simplest process giving information on the structure of the deuteron is the d(n,d)n proton exchange, which dominates the backward peak in n-d scattering (Borbély 1971). The analycity structure

of the cross section was studied by Kisslinger (1972), while the continuation method was applied by Kisslinger (1972), Dubnicka and Dumbrajs (1974 a), Borbély and Nichitiu (1975) for the analysis of the data. To be specific, I choose the E_d=7.01 MeV data of Bonner et al. (1969) and I consider the extrapolation method. Kisslinger reported an agreement with the standard value with N=4 terms but without any statistical analysis in the sense that neither the value of χ^2 for the fit nor the error of the result were given. He also found that three terms are not sufficient for a correct result. Dubnicka and Dumbrajs reported an agreement with the standard value with N=3 terms and χ^2=0.56. Our results, which were briefly reported at the Zurich Polarization Symposium, give that with N=3(4) and χ^2=0.56(0.47) the deuteron vertex constant is G_d^4=0.24±0.01(0.18±0.04) fm^2. One can see that there is an agreement with Kisslinger's normalization as four terms gave the standard value, but he ambiguously included the last term, which is not significant in the physical region. The inclusion of an ambiguous number of nonsignificant terms make the method quite undefined. By definition they only make the error larger while changing the result to a lesser degree: in this case the pole strength was changed from 0.24±0.01 fm^2 to 0.18±0.04 fm^2, which is statistically the same value with a larger error. On the other hand, our analysis agrees with that of Dubnicka and Dumbrajs, except for their normalization. It partly comes from their incomplete treatment of spins (for more details, see the review of Blokhintzev et al. (1977)). In all cases when they reported an agreement with other results, their results were virtually incorrect. One concludes that the small number of available significant terms does not allow extrapolation to the pole. The physical reason is clear. One can safely suppose that there is some distortion in the ℓ=1 partial wave amplitudes which immediately implies that $(2\ell+1)+2=5$ significant terms in the $\cos\vartheta$ plane are necessary to get a correct result. As after the conformal mapping one is very near to applicability, one can try to achieve an improvement by suppressing the forward cut contribution. The extrapolation of $(x_t-x)^2\sigma(x)$ with N=3 terms gives the result G_d^4=0.17±0.01 fm^2 where in the mapped plane the forward cut is at $x=x_t$. That this result is correct we know not from the analysis itself, but from outside sources of information. Locher

and Mizutani (1978 a) checked the reliability of the continuation method results for n-d scattering using a realistic model. Their minimum error estimate really proves to be conservative if one considers the above uncertainties. At the same time their criticism was generalized to all cases of cross section continuation without any distinction. A generalization of error estimates received by "phenomenological" models is justified only if the "microscopic" reason for the inaccuracy is understood and one has some information on it in the new case, too.

One can conclude that the n-d data because of their large errors are not sufficiently accurate to allow the method to be safely applied. More accurate data are avaliable for p-d scattering with a similar transfer process dominating at backward angles. To apply the method one should be able to handle the modifications caused by the presence of the Coulomb interaction.

7.1.2. Coulomb effects

There are essentially three kinds of Coulomb effects. First of all, for elastic scattering there is the Rutherford pole and interference cut at $\cos\vartheta = 1$. According to theorem 3 it is not possible to extrapolate outside the physical region in the $\cos\vartheta$ plane. In principle, it is easy to handle this difficulty: one should redefine the physical region considering a point at a small finite distance from the singularity as the new border of it. For the new physical region it is trivial to apply the machinery of the optimal conformal mapping and to move the Coulomb singularity to a considerable distance from the physical region (see the summary section of the paper by Cutkosky and Deo (1968 b)). In this case on the edge of the physical region the derivative of the mapping is very large (a typical value is 20), therefore the angle resolution of the experiments might be not sufficient to perform the analysis without taking it into account. Sometimes it might be suitable to exclude the most forward points from the analysis. Moreover, during the mapping the "strength" of the Rutherford singularity (in a general sense, as

its character is also changed) considerably increases, therefore it is often necessary to suppress its contribution.

The second type of Coulomb effects is that for the transfer pole which can still be included in a DWBA type description. It was spoken of earlier that the pole comes from undisturbed plane waves and Hankel function asymptotics of the bound state wave functions. One can still obtain closed formulae in the case of Coulomb distortion of the plane waves and/or Coulomb modified bound state wave function asymptotics (Coulomb vertex). Coulomb distortion was studied by Ter-Martirosyan (1955)(see also Morinigo 1964); Coulomb vertex effects were discussed, apart from other authors, by Dzhamalov and Dolinsky (1971). In both cases the pole is changed into a branch point. On the level of amplitudes they were included in the angular extrapolation method by Kisslinger and Nichols (1975). The restriction to the extrapolation of amplitudes rather than the more easily available cross section, which in addition contains less systematical error, is essential. The inclusion of Coulomb vertex effects for the cross section was attempted by Borbély (1975). In the same spirit, one can handle Coulomb distortion too. But one should be aware that only the leading term of the singularity is included and it is necessary to rely on the faster decrease of the pre-exponential factor in formula (12) for the interference singularity or a possible singularity introduced by the procedure itself for the background term as compared with the pre-exponential factor of the studied singularity. The neglected effects are the order of 1/N, where N is the number of terms for the fit (Popov 1964). In the given case (Borbély 1975), N≈8-10. These procedures are therefore incomplete in some sense.*

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*There is also an error in formula (4) of the above mentioned paper (Borbély 1975), the powers η should be replaced by 2η. This formula was derived from formula (17) of Dzhamalov and Dolinsky (1971), which correctly describes the singularity but does not contain the constraint that the formfactor depends on q^2 rather than on q. The inclusion of it gives the power 2η without changing the other factors.

In a straightforward manner the complete inclusion of Coulomb effects can be achieved within the framework of the method of asymptotical coefficients taking into account several singularities simultaneously.

There is a third kind of Coulomb effect which is usually neglected because its existence is not well known. Dolinsky and Mukhamedzhanov (1975) showed that because of the long range of the Coulomb interaction, other mechanisms different from the transfer might have singularities coinciding with the transfer singularity. No closed formulae, however, were obtained by them. Essentially because of it one cannot make a thorough analysis of the available data. It is worth doing only if all Coulomb effects can be handled correctly or at least one has reliable estimates on their neglibility.

Before proceeding with the discussion of applications one should study the problem of approximating the branch point singularity by a pole when some Coulomb effects are neglected. In such cases the extrapolation method, for instance, lacks any meaning at first sight, as one extrapolates by regular polynomials to a not entirely removed singularity. For small Coulomb effects, however, the physical region is only negligibly influenced by definition, therefore the significant approximation coefficients are not changed considerably and the extrapolated value is near that true value which would be the result in the absence of Coulomb modifications (Borbély 1975). The same consideration can be applied to the subtraction method, too. In the case of small effects, usually only the very high index approximation coefficients are changed considerably, but in most cases they are inaccessible since they are not significant. If the pole is changed into another singularity, then the value of Coulomb corrections in the empirical continuation methods depends on the number of terms used for the continuation, too.

7.1.3. Analysis of p-d scattering data

There is very little to add to the already published material on p-d scattering (Borbély and Nichitiu 1975, 1976). As a first approximation one

might neglect the transformation of the pole into a branch point by Coulomb effects and take into account only the Rutherford elastic scattering singularity on the forward edge of the physical region. For the very wide range of bombarding energy of E_p=3.0–46.3 MeV this pole approximation for the transfer singularity with the necessary suppression of the Rutherford singularity in the $\cos\vartheta$ plane (but the analysis itself is performed in the mapped plane) gives G_d^4=0.159±0.004 fm^2. The result is stable as a function of the bombarding energy, against the changing of the suppression factor and of the extrapolating function. A rational fraction extrapolation also gives the same value. One has a significant difference with the standard value, it probably comes from neglecting the Coulomb effects in the pole. For the cross section the Coulomb distorted transfer amplitude of Ter-Martirosyan (1953) gives a second order pole, which is renormalized in comparison with the undisturbed pole, and a branch point from the interference with the background. The inclusion of the easily calculable renormalization factor gives the standard value of G_d^4=0.189±0.003 fm^2. A more complete analysis with the method of asymptotical coefficients is not possible because of the lack of theoretical formulae for the complete inclusion of Coulomb effects. Note that Coulomb effects in the pole can be surprisingly large, at E_p=10.0 MeV the renormalization factor for the cross section is 1.084.

The adverse effect of the Rutherford singularity can be demonstrated on the example of p–d scattering at E_p=6.78 MeV (Grotzschel et al. 1974). As was discussed earlier, in the case of n–d scattering after suppressing the right hand cut, N=3 terms are sufficient to give the correct strength of the pole whereas for p–d scattering N=11 terms are needed (Borbély and Nichitiu 1976). The analysing procedure is the same in both cases, the only difference is that for p–d scattering it is not the triangle singularity at z_t=3.85 that gives the right hand cut but the Rutherford singularity at z_t=1.0. The dractic difference in the number of terms sufficiently illustrates the negative effect of the Rutherford singularity. Therefore one should be careful in the case of elastic scattering with Coulomb interaction. A well measured backward exchange peak itself does not guarantee the success of the

method. It is because one extrapoletes beyond the nearest singularity defined convergence ellipse in the $\cos\vartheta$ plane.

Chowdhury et al. (1976) applied the extrapolation method to the very old p–d scattering data of Sherr et al. (1947) at E_p=1.5, 2.1 and 2.5 MeV. The analysis is statistically incomplete as even the errors of the experimental data are not known. Because of the Rutherford singularity, four terms at most cannot be sufficient for extrapolating to the pole. In addition, the Rutherford singularity is completely neglected, therefore extrapolation outside the convergence ellipse was attempted, which makes the whole procedure most questionable. For more detailed discussion, see the h–p scattering case later.

7.2. Three nucleon bound state vertex constants

For three nucleon bound state vertex constants the empirical continuation method can play a more important role. Actually, for the t→d+p vertex constant it was demonstrated earlier that the method can result in a very small error. One should add here that the experimental data used are not at the same level as those of present day experimental possibilites (the experiments of Hutson et al. (1971) can serve as an example), therefore a sufficient improvement is still feasible. The importance of an accurate value for the vertex constants consists in the fact that they have the same status as binding energies, various electric and magnetic moments, and so on. They should be reproduced in three nucleon calculations as well. At present the calculation are not accurate enough for matching their error with the experimental one (Kim and Tubis 1974), but it is only a matter of time.

7.2.1. Analysis of single nucleon transfer processes

Until now the empirical continuation method was applied to obtain the t→d+n and the h→d+p (h ≡ ^{3}He) vertex constants. The single nucleon transfer

reactions d(d,p)t, d(d,n)h provide an excel ent possibility; the results are given by Borbély (1974, 1975). There is no need to discuss it in more detail, since is was partly done when the accuracy of the E_d=25.6 MeV results was checked. I should repeat that Coulomb vertex effects are taken into account incompletely by Borbély (1976) - as was mentioned earlier. It does not alter the conclusion of the significant difference between the triton and helion vertex constants, as specific Coulomb vertex effects are small compared with the difference (see there).

7.2.2. Analysis of two nucleon transfer

Elastic scattering with two nucleon transfer was also analysed. Dubnicka et al. (1973) and Dubnicka and Dumbrajs (1974 a) considered n-t elastic scattering; Kisslinger (1973) and Kisslinger and Nichols (1975) studied p-h scattering. The deuteron exchange pole is, however, so near to the uncorrelated n+p exchange cut that one should be very cautious when applying the method (Locher and Mizutani 1978 a). For n-t scattering at E_n=3.5 MeV, where the method was actually applied (Dubnicka et al. 1973), the cut is at z_u=-4.17, the pole is at z_p=-3.52. The situation is very similar for p-h scattering, where at E_p=13.6 MeV, z_u=-2.67 and z_p=-2.38. In the case of n-t scattering three or four significant terms cannot be sufficient to reach the asymptotical region of the expansion coefficients where one has only pole contribution without any contamination from the cut. This is clear by taking into account the difficulties met in n-d scattering, where the backward peak is dominated by the pole without any nearby cut. A similar conclusion was reached by Locher and Mizutani (1978 a) based on an analysis using model data. Though Dubnicka and Dumbrajs (1974 a) reported complete agreement with other results, I checked their analysis and found a 2.2 times larger pole strength for the cross section than the standard value; this was because of their already mentioned nonstandard normalization. Apart from this, I successfully reproduced their analysis, i.e. the number of significant terms, the value of χ^2, relative error of the results, and so on.

As opposed to this, I was completely unsuccessful in reproducing Kisslinger's (1973) analysis of the p-h scattering, the results of which were confirmed by Kisslinger and Nichols (1975). Locher and Mizutani (1978 b) pointed out that in this case Kisslinger used a nonstandard normalization. My analysis, however, shows essential differences from his . In the above cited papers very few of the details were given, such a crucial quantity as the number of terms used for extrapolation is missing. As such details have not been accessible, I can give only the results of my analysis for E_p=13.6 MeV (Hutson et al. 1971) as an example in table 2. I use the conformal mapping as described by Kisslinger (1973), i.e. no attention was paid to the Rutherford singularity on the edge of the physical region, the fitting polynomials are the same orthonormal polynomials of formula (14) as used there. The polynomials are normalized to one and the numerically found overlap of the tenth polynomial with the first one is 1.4×10^{-13} instead of an exact zero, which shows the extreme stability of the recurrence relation of formula (5) used for the calculation of these polynomials. In some sense this figure characterizes the magnitude of possible numerical errors as it is difficult to point out any other possible source of numerical instability. One can see that a large number of terms is needed to represent the data in the physical region, which is a direct consequence of the neglected Rutherford singularity. If all the significant terms are included (N=9), the extracted strength is statistically insignificant. As the transfer pole is outside the convergence ellipse degenerated to the physical region itself*, theorem 3 can be applied

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*Strictly speaking, the convergence ellipse has its focii at the two extreme measured points used in the analysis (which in the mapped plane are at x_{max}=0.961 and at x_{min}=-0.989 rather than at $x=\pm 1.0$) and it goes through the Rutherford singularity at x_t=1.0 , so it is slightly larger than the physical region. Nevertheless, it is far from containing the transfer pole at x_p=-3.36 inside its interior.

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thereby justifying the empirically found divergence of the extrapolation. Naturally I have tried to handle the Rutherford singularity with the methods

described for the analysis of p-d scattering, too, but I was unsuccessful in extracting the correct pole strength in the entire energy range of E_p=4.5-31.0 MeV. It is not possible to cope simultaneously with the difficulties caused by the Rutherford singularity and the n+p exchange cut so near the deuteron exchange pole.

7.3. Four nucleon bound state vertex constants

The empirical continuation method was applied to extract four nucleon bound state vertex constants in two cases. Kisslinger (1972) reported an approximately correct result (Blokhintzev et al. 1977) for the extrapolation of the n-α elastic cross section to the h exchange pole. The reliability of his result was doubted by Locher and Mizutani (1978 a) who emphasized the difficulties in separating the continuum exchange cut from the pole at a large distance from the physical region. No well measured exchange peak is present in the analysed data therefore the possibility of reaching the asymptotical region is uncertain. Moreover, I checked the analysis at E_n=10.0 MeV and found that with χ^2=2.0 four terms are needed to describe the data of Hoop and Barshall (1966) instead of either the 5 or 6 terms used by Kisslinger. The strength of the extracted pole given by four terms is already not significant, i.e. it has an error compatible with its value. One has to restore the numerical form of the data using a figure, therefore some distortion is quite feasible. Even so, my check shows that Kisslinger must have used nonsignificant terms to get a seemingly correct result.

The α-d scattering was analysed by Dubnicka and Dumbrajs (1974 b) to extract the $\alpha \to d+d$ vertex constant. They reported a very low average value of G^2=0.43 fm. After correcting their nonstandard normalization, this value is equivalent to G^2=23 fm (Blokhintzev et al. 1977). It is in agreement with other available empirical (Baryshnikov and Blokhintzev 1971) and theoretical (Baryshnikov et al. 1974) results not referred to by them. Despite the agreement, one cannot speak of a successful application. Not only does the histogram of the values for the pole strength (i.e. the values of r^2 in their

notation) show a completely flat distribution, but there is also a strikingly close correlation with the reaction cross section given by Gruebler et al. (1975), as pointed out by Nichitiu (1975). In fact, the properly normalized reaction cross section describes the sometimes very large fluctuations of the r values completely within their errors. It shows that the asymptotical region of the coefficients is not reached, as is suggested by the previously demonstrated adverse effect of the Rutherford singularity and by the absence of well measured and large backward peaks in the data. In other words, not only are the deuteron and the n+p exchange (at E_d=10.0 MeV the pole is at z_p=-3.93, the cut is at $_u$=-4.18 in the cos ϑ plane, while x_p=-1.73 and x_u=-1.95 in the mapped plane) not separated from each other, but they are not separated from other singularities and resonances. Note that with the n-t and p-h case, this is the third case when the attempt to extract the strength of the deuteron exchange pole has failed.

7.4. Vertex constants for light nuclei

Neutron separation vertex constants for light nuclei ($A \geq 6$) were determined by Dubnicka and Dumbrajs (1974 a), Borbély (1976), and Borbély et al. (1976). It is highly improbable that one can successfully apply the method to neutron scattering data in order to extract the heavy particle exchange pole strength, as was attempted by Dubnicka and Dumbrajs. The usual bad statistics of such experimental data does not make it possible to cope with the difficulties caused by the necessity to interchange the order of the near-located forward singularities with that of the distantly located backward exchange pole and simultaneously separate the pole from the branch points near it (see the paper of Dubnicka and Dumbrajs (1974 a)). The situation is completely analogous to the previous case of d-α scattering. In addition, one has no information on the existence of a heavy particle exchange peak. Note that forward mechanisms often give a small backward peak, therefore this peak cannot be automatically attributed to backward mechanisms. There is a strong correlation of the extracted pole strength with the total cross section (Nichitiu 1975) which - apart from the inner statistical contradic-

tion of the results - shows the failure.

Borbély (1976) applied the method to (d,p) reactions on light nuclei checking the results by comparing them with peripheral model calculations. The number of significant terms in formula (14), however, was not analysed there. Predicted as well as empirical (provided by the peripheral model fit) values of kR are available; one can easily prove that the number of obtained significant terms is sufficient. The only problem that presents itself is in the case of the $^{31}P(d,p)^{32}P(3.447\ MeV)$ reaction. If one has a reaction leading to several excited states of the final nucleus, then in general one might expect that the number of necessary terms does not change for them, as it basically depends on slowly changing wave numbers and nuclear radii. A sudden decrease in the actual number of significant terms in the extrapolation method (in the given case from 9-10 to 7) most probably means that the pole defined terms disappeared in the statistical fluctuation of the low partial wave amplitude contribution. It is therefore quite improbable that the result for this case is correct.

The continuation method was also applied to (d,p) reactions on some 1p shell nuclei and the results are briefly presented by Borbély et al. (1976). The uncertainty of the magnitude and of the reliability of the experimental errors used in the analysis makes it difficult to judge the reliability of these results within the framework of the continuation method. It can be done only by comparing them with the results of other methods or by repeating the analysis with other data.

CONCLUSIONS

Looking over the mass of applications discussed in the previous section, one can briefly draw the conclusions concerning reaction mechanisms as follows: i/ For single nucleon transfer the method can be safely applied provided that the data are accurate enough to meet the physical applicability condition following from formula (14). The Rutherford singularity in the

elastic scattering is a great obstacle to the method. In possession of very accurate data it is still possible to extract the correct strength in the sense that the deviation from the standard value in the p-d case is smaller than the changes resulting from Coulomb modifications of the transfer pole. ii/ All attempts to extract the strength of a deuteron exchange pole have failed. One possibility for improving the method might be the explicit removal of the nearby square root type n+p exchange cut by a mapping. It is not difficult to modify the optimal conformal mapping in this manner. iii/ No correct application to three or more nucleon transfer is known.

From a methodological point of view it is necessary to perform a statistically complete analysis, with the traditional χ^2 test providing the number of included terms. Using additional physical information and/or applying methods described in sections 4 and 5 one can decide whether the given number of terms is sufficient.

Table 1.

Results for different analyses of the d(d,p)t E_d=25.3 MeV reaction by the singularity subtraction method.

Optimal mapping	Suppression factor	Pole strength $G_d^2\ G_t^2$ in fm^2	
		n=4	n=5
no	no	(0.466±0.007)	0.496±0.034
no	yes	0.526±0.008	0.501±0.038
yes	no	0.493±0.007	0.501±0.039
yes	yes	0.509±0.007	0.501±0.039

Table 2.

Divergence of the extrapolation method for h-p scattering at E_p=13.6 MeV, completely neglecting all Coulomb effects.

n,N	A_n a/	G_h^4 fm^2 b/		χ_N^2
3	516.1	1.813	±0.004	390.4
4	-91.8	-0.179	±0.022	25.1
5	0.8	-0.073	±0.14	26.2
6	15.2	15.1	±1.0	16.0
7	-14.5	-78.2	±6.5	5.8
8	8.3	290.	±45.	2.2
9	-3.3	-684.	±296.	1.7
10	0.4	184.	±2000.	1.8

a/ The A_n coefficients are calculated from formula (7); note that they have an rms error of ±1.

b/ G_h^4 is the extracted strength of the cross section pole given by formula (8) with N terms. The correct result is about 1.1 fm^2.

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