CONF-781088--1

The Center for Particle Theory

THE UNIVERSITY OF TEXAS AT AUSTIN

UNSTABLE QUANTUM STATES AND RIGGED HILBERT SPACES*

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Abstract

We apply rigged Hilbert space techniques to the quantum mechanical treatment of unstable states in nonrelativistic scattering theory. We discuss a method which is based on representations of decay amplitudes in terms of expansions over complete sets of generalized eigenvectors of the interacting Hamiltonian, corresponding to complex eigenvalues. These expansions contain both a "discrete" and a "continuum" contribution. The former corresponds to eigenvalues located at the second sheet poles of the S matrix, and yields the exponential terms in the survival amplitude. The latter arises from generalized eigenvectors associated to complex eigenvalues on background contours in the complex plane, and gives the corrections to the exponential law.

1. Introduction

When considering unstable states in quantum theory, one is confronted with the following problem (see e.g. [1] and references therein). From the analogy with the classical case, one might believe that the quantum survival probability of the unstable state h, $P(t) = |A(t)|^2$, where $A(t) = (h,\exp(-iHt)h)$ is the survival amplitude, decays exponentially. However, by using some of the Paley-Wiener theorems and the physical hypothesis that the Hamiltonian is bounded below, it is seen that as t approaches infinity, P(t) cannot decrease faster than $\exp(-\alpha t^q)$, $\alpha > 0$, $q < 1$, so that P(t) vanishes slower than $\exp(-it/\Gamma)$ for any positive Γ. Moreover, if one assumes that h is a state with finite expectation value for the energy, it is easily seen that $dP(t)/dt$ vanishes at $t = 0$. These two facts imply that a quantum unstable state cannot have an exact exponential decay, and the need arises for methods allowing for the separation of the exponential term from a background relevant at short and long times only.

It has been many times suggested, and in many cases proven, that resonances in scattering theory should be connected either to poles of the analytic continuation of the resolvent of the Hamiltonian, suitably restricted to some manifold, or to poles of the resolvent of certain nonself-adjoint operators associated with the

*Invited paper at the VII International Colloquium on Group Theoretical Methods in Physics, Austin, Texas, 1978. Supported in part by NATO Research Grant No. 1380 and by CNR Contratto di Ricerca No. 77.01543.63.

**A Fulbright-Hays Act grant is gratefully acknowledged.

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Hamiltonian (see e.g. [2-5], [6, vol. IV]).

Another recurring argument in the literature is that resonances, i.e. peaks in cross sections, are to be associated with unstable states, as stated for example in any book on quantum scattering theory; for clear formulations of this see [1,7].

Also, from an intuitive point of view, physicists like to think of resonances as associated to complex eigenvalues of the Hamiltonian, the real and negative imaginary part of one such "eigenvalue" being respectively the energy and half-width of the resonance. For a rigorization of this idea along lines different from those we develop here, see [3,8,9].

The aim of this lecture is to show how to give a rigorous ground to the last point above, by the use of analytic eigenfunctionals in the framework of rigged Hilbert spaces (or Gel'fand triplets) [10-12].* This technique is further shown to provide a method, which is intuitively easy, for the analytic continuation of a reduced resolvent of the Hamiltonian across the absolutely continuous (a.c.) spectrum in such a way that its poles are the same as those of the continued S-matrix. As a by-product, the connection between unstable states and resonances is stressed once again. Our method stems from [15], where analytic continuation techniques were used to separate the exponential decay law from the background in the framework of the Friedrichs model [16], and the present lecture and a forthcoming paper [17] are intended to provide a rigorization and generalization of [15]. Our approach is related to that of ref. [8], the main difference being that we allow, and desire, the existence of generalized eigenvectors with complex eigenvalues not restricted to the positions of resonances.

2. Rigged Hilbert spaces and analytic functionals

In this section, we recall some essential features of rigged Hilbert spaces, in connection with the diagonalization of self-adjoint (s.a.) operators. We also review a few facts about analyticity of families of functionals in rigged Hilbert spaces, that depend on a complex parameter z. Finally, we examine the case when such families are families of eigenfunctionals of a s.a. operator, and connect their isolated singularities with eigenfunctionals and associated functionals.

Rigged Hilbert spaces were introduced and studied [10, vol IV; 11,12] in order to give a rigorous and direct meaning to the eigenket and eigenbra formalism developed by Dirac. We shall sketch the formalism by means of an example, the operator $T = -id/dx$ with domain in the Hilbert space $\mathcal{H} = L^2(-\infty,+\infty)$ such that T is

*The appearance of "complex eigenvalues" for self-adjoint operators was first noticed and treated by physicists, in connection with the reduction of unitary representations of Lie groups with respect to noncompact subgroups (see e.g., [13]). Rigged Hilbert space techniques in this connection were first introduced, to our knowledge, in [14].

s.a. The eigenvalue equation $-idf_p(x)/dx = pf_p(x)$ has as only solutions plane waves $f_p(x) = \exp(ipx)$ that do not belong to H; eigenkets make no sense within H. However, we know that they are associated to Fourier transform, i.e. $\langle\exp(-ipx)|h\rangle \equiv \int_{-\infty}^{+\infty}\exp(-ipx)h(x)dx = \tilde{h}(p)$ can be defined for any h in H to yield a unitary transform from H onto $L^2(-\infty,+\infty)$ w.r.t. the variable p; then Th is transformed into the function $\langle\exp(-ipx)|Th\rangle = p\langle\exp(-ipx)|Th\rangle = p\tilde{h}(p)$. In general, given any s.a. operator A, there is a unitary operator that diagonalizes A, that is to say that transforms the Hilbert space onto a space of L^2 functions of a real variable λ upon which A acts as the operator of multiplication by λ; such a unitary operator is called the generalized Fourier transform associated to A (see e.g. [6, vol. I]).

Rigged Hilbert space technique consists essentially in finding I) a dense submanifold ϕ II) included in the domain of T, III) invariant with respect to T, endowed with a topology such that IV) the identity embedding of ϕ into H is continuous and V) the action of T from ϕ into ϕ is continuous; finally, things must be arranged in such a way that if $\varphi \in \phi$, $\langle\exp(-ipx)|\varphi\rangle$ defines a linear continuous functional upon ϕ, i.e. it belongs to the dual space ϕ'. It has to be noted that because of I) and IV), H is continuously and densely embedded into ϕ'. Hence, vectors in ϕ' can be approximated by vectors in ϕ, a physically important fact. Then the relationship:

$$\langle\exp(-ipx)|T\varphi\rangle = p\langle\exp(-ipx)|\varphi\rangle \tag{1}$$

valid for almost every p, is to be understood not only as defining the function $p\tilde{\varphi}(p)$ in $L^2(-\infty,\infty)$, Fourier transformed of $-id\varphi(x)/dx$, but also in the following way. For any p, $\langle\exp(-ipx)|\varphi\rangle$ defines a linear continuous functional upon ϕ, and so does $\langle\exp(-ipx)|T\varphi\rangle$ because T is continuous upon ϕ; therefore T can be seen as acting upon $\langle\exp(-ipx)|$ and transforming it into the new functional $\langle\exp(-ipx)|T$ that acts upon φ according to $\langle\exp(-ipx)|T\varphi\rangle$. Then (1) simply means that the functional $\langle\exp(-ipx)|$ is eigenvector of the extension of T in the space ϕ', with eigenvalue p. In the example at hand, one can choose ϕ to be S, the space of test functions for tempered distributions (whose space is denoted by S') endowed with the usual topology [10 vol. I; 6 vol. I]. We know that the functionals $\langle\exp(-ipx)|$ are in S', so that they are eigenvectors of T in S', even though they do not belong to H. Moreover they form a complete set, in the sense that the scalar product between any two vectors φ and ψ in S is given by:

$$(\varphi,\psi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{\langle\exp(-ipx)|\varphi\rangle}\langle\exp(-ipx)|\psi\rangle dp \ , \tag{2}$$

as follows from unitarity of the Fourier transform. Such eigenvectors can be approximated by vectors in H, as is well known. Therefore, if we are looking for a "complete set of eigenvectors" for any given s.a. operator A acting in a Hilbert space H, we may start looking for a submanifold ϕ such that conditions I)-V) are satisfied w.r.t. A. Because of technical reasons, we have also to require that the topology on the submanifold ϕ be such that ϕ is VI) nuclear, VII) barrelled and

VIII) complete. For the definitions we do not give here, see f.i. [10, vol. II; 11]. A triple of spaces $\phi \subset H \subset \phi'$ satisfying conditions I), IV), VI), VII) and VIII) is a rigged Hilbert space (notice that in ref. [10, vol. IV] this term denotes a smaller class of spaces). The triple $S \subset H \subset S'$ is an example of such space. A theorem of Maurin states that a "complete family of eigenvectors" always exists, that is to say the following. Given any s.a. operator A in a Hilbert space H, there exists a rigged Hilbert space $\phi \subset H \subset \phi'$ that satisfies II), III), and V) w.r.t.A. In the dual ϕ' of ϕ there exists a family of eigenvectors $\langle\lambda|$ of A,

$$\langle\lambda|A = \lambda\langle\lambda| \quad \text{or} \quad \langle\lambda|A\varphi\rangle = \lambda\langle\lambda|\varphi\rangle \tag{3}$$

for any $\varphi \in \phi$ and any λ in the spectrum Σ of A. This set is complete, i.e. for any φ and ψ in ϕ it holds:

$$(\varphi,\psi) = \int_\Sigma \overline{\langle\lambda|\varphi\rangle}\langle\lambda|\psi\rangle d\mu(\lambda) \tag{4}$$

μ being some measure on Σ. Otherwise stated, this means that the transform from ϕ into $L^2(\Sigma,\mu)$ given by $|\varphi\rangle \to \langle\lambda|\varphi\rangle$, where $\langle\lambda|\varphi\rangle$ is to be considered for fixed φ a function of the parameter λ in Σ, can be extended to the whole of H and then coincides with one of the possible generalized Fourier transforms that diagonalize A [10, vol. IV; 11].

Returning to the example of the derivative operator, we see that here, as well as in the general case, the choice of the rigged Hilbert space is not unique: for example, instead of S we could have chosen for ϕ the space $\mathcal{D}$ of test functions with compact support [10 vol. I; 6 vol. I]. Of course, also this choice is such that Maurin theorem is satisfied, and indeed we know that the plane waves $\langle\exp(-ipx)|$ are distributions in $\mathcal{D}'$ as well. However, here another interesting feature arises: for any complex z, the functions exp(-ipz) are distributions in $\mathcal{D}'$, that is to say the integrals

$$\langle z|\varphi\rangle \equiv \int_{-\infty}^{+\infty} \exp(-izx)\varphi(x)dx \tag{5}$$

converge for any test function with compact support and are continuous w.r.t. the topology of $\mathcal{D}$. Moreover, these distributions are such that, for any complex z and for any φ in $\mathcal{D}$, $\langle z|T\varphi\rangle = z\langle z|\varphi\rangle$, i.e. the eigenvalue equation has solutions also for eigenvalues that are not in the spectrum of T; however, the associated eigenvectors in $\mathcal{D}'$ do not play any role in Maurin theorem. This is a general feature: solutions of the eigenvalue problem may arise for eigenvalues in the spectrum as well as for eigenvalues outside the spectrum, that do not play any role in Maurin theorem [13,14].

Another important feature arises in this example: for any fixed φ in $\mathcal{D}$, $\langle z|\varphi\rangle$ as given in (5) is an analytic function of z, because one can differentiate under the integral. We will say that a family of vectors $\langle G(z)|$ in ϕ' depending on the complex parameter z is analytic in a region Ω if for any φ in ϕ the ordinary function of z given by $\langle G(z)|\varphi\rangle$ is analytic in Ω [10, vol. I]. An isolated singularity for an analytic family $\langle G(z)|$ is a point z_0 in the complex plane such that, for at least some φ in ϕ, $\langle G(z)|\varphi\rangle$ is singular at z_0 and the family $\langle G(z)|$ is analytic in some punctured disc about z_0 [10, vol. I]. The integral $\int_\gamma \langle G(z)|dz$ has to be understood as that particular vector in ϕ', call it $\langle c|$, such that for any φ in ϕ, it holds $\langle c|\varphi\rangle = \int_\gamma \langle G(z)|\varphi\rangle dz$ where, for fixed φ, the r.h.s. is an ordinary Riemann integral [10, vol. I]; it can be shown that, under assumptions milder than I)-VIII), if γ is a rectifiable path, $\int_\gamma \langle G(z)|dz$ always converges to a unique $\langle c|$ in ϕ' whenever $\langle G(z)|$ is analytic on γ. Also, with the same assumptions, Cauchy theorems for integrals of ordinary analytic functions hold true for integrals of analytic families in ϕ'. The same is true for Taylor expansions or Laurent expansions about an isolated singularity z_0, i.e.:

$$\langle G(z)| = \sum_{n=-\infty}^{+\infty} \langle c_n(z_0)|(z-z_0)^n \tag{6}$$

$$\langle c_n(z_0)| = \frac{1}{2\pi i}\int (z-z_0)^{-n-1}\langle G(z)| \ , \tag{6'}$$

the series converging in a suitable sense in ϕ' [10, vol. I]. Moreover, the principle of identity of analytic functions, or principle of analytic continuation, holds true in our setting. We are interested in the case when a family of eigenvectors $\langle G(\lambda)|$ of a s.a. operator A, $\langle G(\lambda)|A = \lambda\langle G(\lambda)|$, is the restriction to the spectrum Σ of A of a family $\langle G(z)|$ analytic in some region $\Omega \supset \Sigma$. Then, provided there is in Σ a proper accumulation point which is also in Ω, the family $\langle G(z)|$ satisfies $\langle G(z)|A = z\langle G(z)|$ everywhere in Ω by the principle of identity of analytic functions. Next, consider the coefficients $\langle c_n(z_0)|$ of the Laurent expansion about an isolated singularity z_0 for such an analytic family of eigenvectors. From (6') it is readily seen that [17]:

$$\langle c_n(z_0)|A = z_0\langle c_n(z_0)| + \langle c_{n-1}(z_0)| \ . \tag{7}$$

Therefore, if $\langle c_{n-1}(z_0)|$ vanishes, $\langle c_n(z_0)|$ is an eigenvector with eigenvalue z_0, otherwise it is an associated vector. If the singularity is a pole of order N (or a regular point) then $\langle c(z_0)|$ (or the first nonvanishing coefficient) is an eigenvector with eigenvalue z_0.

3. Singularities of the continued reduced resolvent and analytic families of eigenvectors of the Hamiltonian

Here we envisage a s.a. operator H with suitably good properties and show how the singularities of the analytic continuation of its resolvent, properly reduced, across the spectrum, are associated with corresponding singularities of analytic families of eigenvectors of H. In the case where H is interpreted as the Hamiltonian, these singularities give rise to exponentially decaying discrete contributions to survival amplitudes of nonstationary states.

Suppose that there exists a rigged Hilbert space $\phi \subset \mathcal{H} \subset \phi'$ that satisfies conditions I)-VIII) w.r.t. H and that the complete family of eigenvectors $\langle G(\lambda)|$ of H in ϕ' which appear in Maurin theorem is the restriction to the spectrum Σ of H of a family $\langle G(z)|$ analytic in some region $\Omega \supset \Sigma$. For the sake of clarity, assume Σ to be simple and an interval, possibly infinite, of the real axis, and that formula (4) can be written with μ the Lebesgue measure. Throughout this section, we consider valid these assumptions. From sec. 2, we know that $\langle G(z)|H = z\langle G(z)|$ holds everywhere in Ω. Consider also the space ϕ^{x} of antilinear continuous functionals on ϕ. If $\langle G(\bar{z})|$ is in ϕ' then the functional $|F(z)\rangle$ defined by:

$$\langle \varphi|F(z)\rangle = \overline{\langle G(\bar{z})|\varphi\rangle} \tag{8}$$

is in ϕ^{x}. If $\langle G(\bar{z})|A = \bar{z}\langle G(\bar{z})|$ we have $H|F(z)\rangle = z|F(z)\rangle$. The family $|F(z)\rangle$ is of course analytic in $\bar{\Omega}$, the complex conjugate of the region Ω; for such families of antilinear eigenvectors everything can be stated as it was in sec. 2 for $\langle G(z)|$. The introduction of such antilinear functionals could be avoided, but we will not dwell upon this here [15,17].

Let f(z) be an entire analytic function whose restriction to Σ is bounded, and consider the following identity for any φ and ψ in ϕ:

$$(\varphi,f(H)\psi) = \int_{\Sigma}\overline{\langle G(\lambda)|\varphi\rangle}f(\lambda)\langle G(\lambda)|\varphi\rangle d\lambda \ . \tag{9}$$

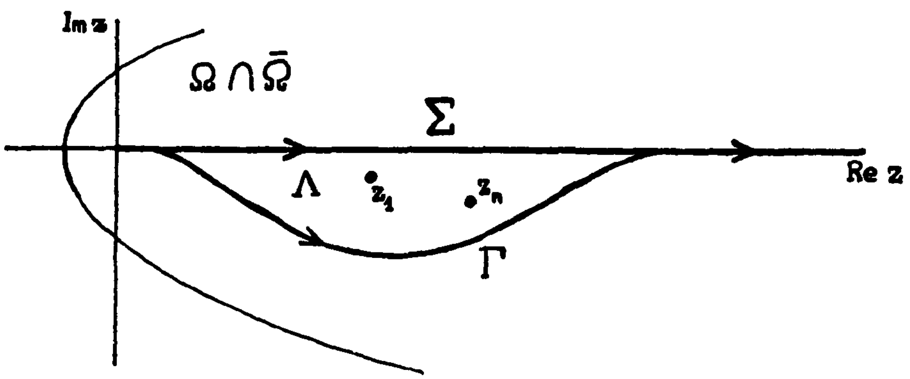
If Γ is a path inside $\Omega \cap \bar{\Omega}$, obtained by deforming Σ, with both end points in Σ, and $z_1,z_2,\ldots,z_n$ are the singular points of $\langle G(z)|$ and/or $|F(z)\rangle$ in the region Λ enclosed by Σ and Γ, Cauchy theorem yields (see figure)

$$(\varphi,f(H)\psi) = \int_{\Gamma}\langle\varphi|F(z)\rangle f(z)\langle G(z)|\psi\rangle dz - 2\pi i\Sigma_i \mathrm{Res}_{z_i}[\langle\varphi|F(z)\rangle f(z)\langle G(z)|\psi\rangle] \ , \tag{10}$$

the last term at the r.h.s. being the sum over the residues. If, say, $|F(z)\rangle$ is regular in Λ, whereas $\langle G(z)|$ is singular only at z_0, a simple pole, (10) becomes:

$$(\varphi,f(H)\psi) = \int_{\Gamma}\langle\varphi|F(z)\rangle f(z)\langle G(z)|\psi\rangle - 2\pi i\langle\varphi|F(z_0)\rangle f(z_0)\langle c_{-1}(z_0)|\psi\rangle \ , \tag{11}$$

$\langle c_{-1}(z_0)|$ being an eigenvector of H in ϕ' with eigenvalue z_0, defined by (6) and (6'). In the general case of a higher order singularity, there would appear various coefficients of the Laurent expansions of $\langle G(z)|$ and $|F(z)\rangle$, coefficients which we know to be either eigenvectors or associated vectors with eigenvalues given by the location of the singularities. Therefore, we have expanded scalar products in terms of a "complete" family of eigenvectors and associated vectors of H with complex eigenvalues; singularities of the analytic family of eigenvectors give a discrete



contribution to such expansions. Let us stress here that such eigenvectors can always be approximated by means of vectors in H. Now let $f(z) = \exp(-izt)$ and interpret H as the Hamiltonian of some quantum system. In the case that leads to (11), we have for the survival amplitude A(t) of a state φ in Φ:

$$A(t) = \int_\Gamma \langle\varphi|F(z)\rangle\langle G(z)|\varphi\rangle \exp(-izt)dz - 2\pi i \exp(-iz_0 t)\langle\varphi|F(z_0)\rangle\langle c_{-1}(z_0)|\varphi\rangle \quad (11')$$

Assuming Γ lies in the lower halfplane of the complex energy, so that $\mathrm{Im}\, z_0 < 0$, we have associated with the pole at z_0 two eigenvectors, $\langle c_{-1}(z_0)|$ and $|F(z_0)\rangle$, and a time behaviour $\exp(-iz_0 t)$, i.e. exponential decay, explicitly separated from a background given by the integral over Γ. In the case of a higher order pole at z_0, the discrete contribution arising from the singularity would be of the form $P(t)\exp(-iz_0 t)$, with P(t) a polynomial.

Next, let us consider the resolvent $R(w) = (H-w)^{-1}$. Whenever w is not in Σ, R(w) is in the space $L(\phi,\phi^x)$ of continuous operators from ϕ into ϕ^x, where ϕ^x is endowed with the so-called weak topology [10, vol. II; 11]; as such, it has a continuation across Σ, $R^{II}(w)$, which is also in $L(\phi,\phi^x)$ (but not in $L(H,H)$!), its singularities being the same as those of the families $|F(z)\rangle$ and $\langle G(z)|$. Let us remark that only matrix elements between vectors in ϕ make sense for an operator in $L(\phi,\phi^x)$. To see that R(w) is continuable, consider, for any φ and ψ in ϕ, the identity: $(\varphi,R(w)\psi) = \int_\Sigma \overline{\langle G(\lambda)|\varphi\rangle}\langle G(\lambda)|\psi\rangle(\lambda-w)^{-1}d\lambda$. Application of Plemelj formulas, together with a suitable definition of convergence for sequences of operators in $L(\phi,\phi^x)$, and the fact that $|F(z)\rangle\langle G(z)|$ is in $L(\phi,\phi^x)$, yield that:

$$R^{II}(z) = R(z) + 2\pi i|F(z)\rangle\langle G(z)| \quad (12)$$

is the analytic continuation of R(w) across Σ from the upper halfplane into the second Riemann sheet. Clearly its singularities are those of $|F(z)\rangle$ and $\langle G(z)|$.

4. A critical comment

Even though we have been able to connect to each other singularities of the continued resolvent, discrete contributions to the survival amplitude with exponential decay law and eigenvectors (or associated vectors) of the Hamiltonian with complex eigenvalues, we are not in general justified in considering these objects associated with true physical resonances. Indeed, the analytic structure of the eigenvectors critically depends on the choice of ϕ. As a matter of fact, given any z_0 with $\mathrm{Im}\, z_0 < 0$, we can find a ϕ such that the corresponding family of eigenvectors has a simple pole at z_0. Correspondingly, poles for the continued resolvent can be located arbitrarily and therefore, for a Hamiltonian with at least part of the spectrum as in sec. 3,unstable states can be found with arbitrary energy and lifetime. In order to be able to select the true resonances in any particular case, we need additional input from the physics of the problem. We do this in the next section by using a second Hamiltonian and scattering theory.

5. Connection with the second sheet poles of the S matrix

A suitable way, though not the only possible one [1,4,5,6,18], to select physical resonances and to associate them to singularities of analytic families of eigenvectors of the Hamiltonian H, may consist in the introduction of a suitable "unperturbed" Hamiltonian H^0 and in assuming that resonances are connected to the singularities of the continued S matrix. Accordingly, assume we have two Hamiltonians, an "unperturbed" one H^0 and a "perturbed" one H in the Hilbert space $\mathcal{H}$, and let P^0 and P be the orthogonal projections on the absolutely continuous (a.c.) subspaces of H^0 and H respectively (for the definition, see f.i. [6, vol. I]). Assume the existence of the Møller wave operators W_+ and W_- for the pair (H^0,H), defined by $\lim_{t\to\mp\infty}\|\exp(-iH^0t)P^0h - \exp(-iHt)W_\pm P^0h\| = 0$, where h is any vector in $\mathcal{H}$. The S operator is then defined as $S = W_-^*W_+$ and it is unitary in $P^0\mathcal{H}$ (the space of scattering asymptotes). We recall, because it will be important later, the meaning in time independent scattering theory of the wave operators. There is a generalized Fourier transform F^0 that diagonalizes H^0, which transforms the vector h in $\mathcal{H}$ into the function $[F^0(h)]_i(E)$, where the subscript i takes into account degeneracy, say angular momentum, and the vector H^0h into $E[F^0(h)]_i(E)$; moreover there are two generalized Fourier transforms F^+ and F^- that diagonalize H, i.e. $h \to [F^+(h)]_i(E)$, or $h \to [F^-(h)]_i(E)$, such that Hh is transformed by them into $E[F^+(h)]_i(E)$ and $E[F^-(h)]_i(E)$ respectively and such that

$$[F^+(W_+P^0h)]_i(E) = [F^0(P^0h)]_i(E) \tag{13}$$

and

$$[F^-(W_-P^0h)]_i(E) = [F^0(P^0h)]_i(E) \ . \tag{13'}$$

Let $S_{ij}(E)$ be the diagonalization of the S operator w.r.t. H^0, i.e. $[F^0(SP^0h)]_i(E) = \Sigma_j S_{ij}(E)[F^0(P^0h)]_j(E)$ (in the formal theory of scattering, one has: $\langle E,i|S|E'j\rangle = \delta(E-E')S_{ij}(E)$). Then it also holds

$$[F^-(Ph)]_i(E) = \Sigma_j S_{ij}(E)[F^+(Ph)]_j(E) \ . \tag{14}$$

Now we assume that the S matrix elements $S_{ij}(E)$ can be continued from above into the unphysical sheet and we stick to the usual assumption that resonances are to be associated with poles of such continuation. Next, we assume that a single rigged Hilbert space $\phi \subset \mathcal{H} \subset \phi'$ exists such that assumptions II), III), V) of sec. 2 are satisfied w.r.t. both the pairs H^0, P^0 and H, P. Let us remark that we do not assume that the same be true for $W_\pm$ and S, and indeed in general it is not true. Moreover, assume that the generalized Fourier transforms F^0 and F^+ above are implemented by complete families of eigenvectors of H^0 and H, $\langle G_i^0(E)|$ and $\langle G_i^+(E)|$ respectively. Then it is easily seen that

$$\langle G_i^-(E)| \equiv \Sigma_j S_{ij}(E)\langle G_j^+(E)| \tag{15}$$

is a complete family of eigenvectors of H in ϕ' that implements F^-. The families $\langle G_i^+(E)|$ and $\langle G_i^-(E)|$ are the "in" and "out" states of the formal theory of scattering, here in a rigorous setting. Finally, let Σ be the a.c. spectrum of both Hamiltonians and consider the restrictions of the family $\langle G_i^0(E)|$ to $P^0\phi$ and of the families $\langle G_i^\pm(E)|$ to $P\phi$. From now on the symbols $\langle|$ and $|\rangle$ will denote such restrictions for the respective families. Assume that for such restrictions the following hold: a) the family $\langle G_i^0(E)|$ is the restriction to Σ of a family $\langle G_i^0(z)|$ analytic in a simply connected region $\Omega_0 \supset \Sigma$ in the plane of the complex energy z; b) the families $\langle G_i^+(E)|$ and $|F_i^-(E)\rangle$ are continuous boundary values on Σ of families $\langle G_i^+(z)|$ and $|F_i^-(z)\rangle$ analytic in some simply connected region Ω_1 in the lower halfplane, $\Omega_1 \subset \Omega_0$. It follows that $\langle G_i^-(E)|$ is analytic in $\bar{\Omega}_1$ and can be continued into Ω, the intersection of Ω_1 with the region of analyticity of the continued S matrix, because of (15) (provided the series converges suitably). The singularities in Ω of the continued family $\langle G_i^-(z)|$ are then exactly the same as those of the continued S matrix (apart from pathological cases in which zeros of $\langle G_i^+(z)|$ cancel poles of $S_{ij}(z)$). Now apply formula (12) to the resolvent of the unperturbed Hamiltonian by means of the eigenvector families $\langle G^0(z)|$ and $|F^0(z)\rangle$; no singularities arise for this operator in Ω_1, because Ω_0 is simply connected and includes Ω_1. Remark that (12) is valid here and in the following for the resolvents restricted to ϕ, only when suitable conditions on the not a.c. parts of the spectra hold; otherwise, they are valid for the respective restrictions to $P\phi$ and $P^0\phi$. Then, apply to the perturbed Hamiltonian formulas (10), (12) and (11') (the last one in case S has only one simple pole in Ω), by inserting the family $\langle G_i^-(z)|$ and letting Γ to be a path in Ω. These formulas show that resonances, i.e. singularities z_i in the analytic S matrix continued from above into the unphysical sheet are associated with: 1) eigenvectors and associated vectors of H with eigenvalues z_i, that can be approximated by vectors in $\mathcal{H}$ just as plane waves can be; 2) singularities, of the same kind, of the continued resolvent of H, whereas the continued resolvent of H_0 is regular; 3) discrete contributions, that decay exponentially, to the survival amplitude of vectors in $P\phi$. Moreover, the exponential contributions of resonances to the survival amplitude are straightforwardly separated from the background.

6. The example of the Friedrichs model

The Friedrichs model [16], equivalent to the Lee model in the first sector with an unstable V particle, provides a simple example in which the theory sketched above can be implemented. Let $\mathcal{H} = \mathbb{C} \oplus L^2(0,+\infty)$, so that a vector in $\mathcal{H}$ is a pair $(h^0;h(E))$ where h^0 is a complex number and h(E) is in $L^2(0,+\infty)$. H^0 acts according to $H^0h = (Mh^0;Eh(E))$, with $M > 0$. H acts according to $Hh = (Mh^0 + g\int_0^\infty \bar{f}(E)h(E)dE;Eh(F) + h^0gf(E))$, where f(E) is a suitable cut-off function and g is a coupling constant. If we take f(E) to be the restriction to $[0,+\infty)$ of a function in Z, the space of Fourier transforms of test functions with compact support [10, vol. I], and if we assume that $f(0) = 0$, $f(E) \neq 0$ for $E > 0$, and $g^2 < M/\int_0^\infty |f(E)|^2E^{-1}dE$, we have the following. Choose $\phi = C + Z$: then all the assumptions of sec. 5 are satisfied and

P is the identity. In particular, as $P^0\phi = Z$, it holds $\langle G^0(E)|\varphi\rangle = \varphi(E)$ and $\langle G^0(z)|\varphi\rangle = \varphi(z)$, so $\langle G^0(z)|$ is analytic entire. If we define: $\alpha(z) = z - M - g^2\int_0^\infty |f(E)|^2(z-E)^{-1}dE$, we have:

$$\langle G^\pm(E)|\varphi\rangle = gf(E)[\alpha(E\mp i0)]^{-1}\{\varphi^0 + g\int_0^\infty \overline{f(E')}\varphi(E')(E\mp i0-E')^{-1}dE'\} + \varphi(E) .$$

The S operator on $P^0\mathcal{H}$, diagonalized w.r.t. H^0, is $S(E) = \alpha(E-i0)[\alpha(E+i0)]^{-1}$, with continuation into the unphysical sheet: $S(z) = \alpha(z)[\alpha(z) + 2\pi i g^2\overline{f(\bar{z})}f(z)]^{-1}$. For g small enough and nonzero there is a resonance pole at a point $z_0 = E_0 - i(\Gamma_0/2)$, $\Gamma_0 > 0$, close to M, solution of the equation $\alpha(z) + 2\pi i g^2\overline{f(\bar{z})}f(z) = 0$ [17].

The theory sketched in sec. 5 holds also for degenerate perturbations; we refer to [19] for other interesting models where this theory applies.

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