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ON GRAVITATIONAL WAVE ENERGY IN EINSTEIN GRAVITATION THEORY

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<u>Аннотация</u>

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На примерах точных волновых решений уравнений Эйнштейна показано, что стандартная общепринятая формулировка проблемы энергии-импульса с использованием псевдотензоров дает либо нулевое либо знакопеременное значение для энергии гравитационных волн. Показано, что если в теории гравитации Эйнштейна строго переходить к пределу слабого поля, то теория дает однозначный нулевой результат для слабых гравитационных волн. Известный ненулевой результат возникает за счет некорректного перехода к приближению слабого поля в теории гравитации Эйнштейна.

<u>Abstract</u>

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By the example of precise wave solutions for the Einstein equations it is shown that a standard commonly adopted formulation of energy-momentum problem with pseudotensors provides us either with a zero or sign-variable values for the energy of gravitational waves. It is shown that if in the Einstein gravitation theory a strict transition to the limits of weak fields is realised then the theory gives us an unambiguous zero result for weak gravitational waves. The well-known non-zero result arises due to uncorrect transition to weak field approximation in the Einstein gravitation theory.

I N T R O D U C T I O N

Examples of precise wave solutions for the Einstein equations, i.e., Taub waves[1] and Bondi-Pirani-Robinson ones[2] have been considered in the present work. A standard commonly-adopted Einstein formulation of the energy-momentum problem with pseudotensors is shown to provide us either with a zero or sign-variable values for the gravitational wave energy.

In connection with these results there arises quite a natural question what the sense and meaning of the usually derived positive value of gravitational waves in weak field approximation are. We will show that in the Einstein gravitation theory an accurate transition to weak field limit gives us an unambiguous zero result for any gravitational waves. The well known non-zero result occurs because of incorrect transition to the weak field approximation, when the Einstein gravitation theory is replaced with a nongeometrised theory of the tensor field in flat space-time so that the precise relations of the Einstein gravitation theory are violated.

2. GENERAL FORMULAE

In the Einstein gravitation theory for the free gravitational field one may take as a Lagrangian density either scalar density $\sqrt{-g}R$ or the quantity $\sqrt{-g}G = \sqrt{-g}g^{ik}(\Gamma^m_{in}\Gamma^n_{km} - \Gamma^n_{ik}\Gamma^m_{nm})$ different from $\sqrt{-g}R$ by divergence.

Then using a standard procedure of infinitesimal displacements one may obtain an expression either for the Lorentz pseudotensor

$$(\tau_i^k)_L = \sqrt{-g}R\delta_i^k + [\frac{\partial\sqrt{-g}R}{\partial(\partial_k g_{mn})} - \partial_\ell \frac{\partial\sqrt{-g}R}{\partial(\partial_k\partial_\ell g_{mn})}]\partial_i g_m + \frac{\partial\sqrt{-g}R}{\partial(\partial_k\partial_\ell g_{mn})}\partial_i\partial_\ell g_{mn},$$

or for the Einstein pseudotensor

$$(\tau_i^k)_E = -\sqrt{-g}G\delta_i^k + \frac{\partial\sqrt{-g}G}{\partial(\partial_k g_{mn})}\partial_i g_{mn}.$$

For free gravitational field that satisfies the Einstein equations the pseudotensor τ_i^k may be expressed as a divergence from the pseudotensor of spin σ_i^{nk} antisymmetric in upper indices

$$\tau_i^k = \partial_n \sigma_i^{nk}. \qquad (1)$$

For the Lorentz pseudotensor the density σ_i^{nk} has the form

$$(\sigma_i^{nk})_L = \sqrt{-g}\,g^{km}g^{n\ell}(\partial_\ell g_{im} - \partial_m g_{i\ell}),$$

and for the Einstein pseudotensor it is

$$(\sigma_i^{nk}) = \frac{g_{im}}{\sqrt{-g}}\partial_\ell[-g(g^{mn}g^{k\ell} - g^{km}g^{n\ell})],$$

According to (1) for periodical solutions (wave packets) after averaging over the period (packet length) there takes place an equality

$$< r_o^o > = \int r_o^o \, dt = 0$$

i.e., in the Einstein gravitation theory an average value of energy for any periodical waves is equal to zero.

3. EXAMPLES OF PRECISE SOLUTIONS FOR EINSTEIN EQUATION

3.1. Taub waves /1/

For flat Taub waves

$$ds^2 = A(dt^2 - dx^2) - B(dy^2 + dz^2).$$

Wave solutions $A = A(u)$, $B = B(u)$ where $u = t - x$ satisfy the field equation

$$B''/B \quad \frac{1}{2}(B'/B)^2 + A'B'/(AB) = 0.$$

For the energy density r_o^o we obtain the following expression

$$(r_o^o)_L = -\partial_o (BA'/A); \quad (r_o^o)_E = 0.$$

i.e., *the Lorentz energy density is in average equal to zero for* any periodical solution, and the Einstein energy density is strictly equal to zero.

3.2. Bondi-Pirani-Robinson waves /2/

$$ds^2 = dt^2 - dx^2 - L^2(e^{2\beta}dy^2 + e^{-2\beta}dz^2),$$

where L, β are the functions of $u = t - x$, satisfying the field equation

$$L'' + (\beta')^2 L = 0.$$

Fixing a specific function $B(u)$ one defines the function L .For the quantities $(r^{o}_{o})_{E}$ and $(r^{o}_{o})_{E}$ we derive the expression

$$(r^{o}_{o})_{L} = 0; \qquad (r^{o}_{o})_{E} = 4[(L')^2 - L^2 (B')^2] = 4(LL'),$$

that being averaged provides us with a zero periodical solutions.

Let us treat some specific examples of possible precise solutions.

a) We will choose the function β in the form $\beta = \omega u$ where $\omega = const.$

Here the function L admits the solution of the form

$$L = D \sin(\omega u + \delta); \quad D, \ \delta = const.$$

The Einstein energy density is a sign-variable function

$$(r^{o}_{o})_{E} = 4\omega^2 D^2 \cos 2(\omega u + \delta)$$

The average value for $(r^{o}_{o})_{E}$ is equal to zero.

b) Let us choose the function β in the form $\beta = c \cos \omega u$.

where c is a constant. The Einstein equations take the form of the equation $L'' + L c^2 \omega^2 \sin^2 \omega u = 0$ which is known to possess periodical solutions. For these solutions the mean value for the Einstein energy density turns out to be equal to zero.

Thus for the treated examples of the Bondi-Pirani-Robinson waves the Lorentz energy density is strictly equal to zero, and the Einstein energy density equals zero in average for periodical solutions. The list of such examples may be extended. A number of such examples of precise wave solutions (Bondi waves[4], Brinkmann-Peres waves[5,6], Kaigorodov-Pestov waves[7], Dangvu waves[8]) was treated in work[3].

4. WEAK FIELD APPROXIMATION IN EINSTEIN GRAVITATION THEORY

Practical calculations of energy emitted by some system in the form of gravitational waves are always made in the frame of perturbation theory in weak approximation (with respect to the emitted waves) for asymptotically flat space-time. In this the variables are replaced

$$g_{ik} = \gamma_{ik} + \phi_{ik} \tag{2}$$

and the quantity ϕ_{ik} is treated as a new field variable against flat space-time background with a metric tansor γ_{ik}. Indeed substitution of variables (2) is always possible and does not change the physical sense and the results of theory.

In all the physical theories in the flat space-time a symmetric energy momentum tensor is defined in a standard way $t^{ik} = -2\delta L/\delta\gamma_{ik}$. This tensor is known to differ from the canonical energy-momentum tensor by divergence of antisymmetric spin tensor, and this divergence does not influence any physical results. A symmetric energy-momentum tensor may also be obtained through symmetrization of a canonical energy-momentum tensor (Resenfeld-Belinfante procedure[9,10]).

After replacing variables (2) one may formally consider the Einstein gravitation theory and it is usually treated as a nonlinear theory of tensor field φ_{ik} against the space time background (see for instance[11]). Then the field theory equation is $\delta(\sqrt{-g}R)/\delta\phi_{ik} = 0$. As for the symmetric energy-momentum tensor density of gravitational field in the flat space-time with a metric tensor γ_{ik} it is $t^{ik} = -2\delta(\sqrt{-g}R)/\delta\gamma_{ik}$. . Note that substitution of variables (2) is strictly symmetric with respect to the quantities γ_{ik} and ϕ_{ik} . Therefore in any geometrised gravitation theory, and in the Einstein one in particular, where the initial precise Lagrangian density depends on the sum $g_{ik} = \gamma_{ik} + \phi_{ik}$ we have precise relations

$$\delta(\sqrt{-g}R)/\delta\phi_{ik} = \delta(\sqrt{-g}R)/\delta\gamma_{ik} = 0 \qquad (3)$$

for free gravitatinal field, i.e., the equations for the field ϕ_{ik} are fulfilled and at the same time symmetric energy-momentum tensor of free gravitational field is strictly equal to zero.

Equality (3) holds in any approximation of the Einstein gravitation theory and in the usual weak field approximation as well. To be convinced one is to decompose quite accurately the two parts of equality (3) into a perturbation theory series, not rejecting arbitrarily various terms in the parts of precise equality (3).

In the weak field approximation when it is assumed that in equality (2) $|\phi_{ik}| << |\gamma_{ik}|$ variations (3) are usually expanded into a series over the fields ϕ_{ik} :

$$-2\delta(\sqrt{-g}\,R)/\delta\phi_{ik} = -2\delta(\sqrt{g}R)/\delta\gamma_{ik} = \underset{(1)}{t}^{ik} + \underset{(2)}{t}^{ik} + \dots \qquad (4)$$

Both the *field equations* and energy-momentum density tensor have strictly the same decompsition terms ϕ_{ik}, the sum of these terms being equal to zero . (Explicit forms for $\underset{(1)}{t}^{ik}$ and $\underset{(2)}{t}^{ik}$ are given in Appendix).

As a rule when calculating the energy of gravitational waves in weak field approximation, one, following Einstein deals only with the terms of the first order in ϕ_{ik} in field equations, i.e., one assumes that

$$-2\delta(\sqrt{-g}R)/\delta\phi_{ik} = \underset{(1)}{t}^{ik} = 0 \qquad (5)$$

leaving at the same time in the expressions for the energy-momentum density tensor only the terms of the second order in ϕ_{ik} neglecting those of the first order, i.e., one assumes that

$$-2\delta(\sqrt{-g}R)/\delta\gamma_{ik} = \underset{(2)}{t}^{ik} . \qquad (6)$$

In the Einstein gravitation theory where the precise Lagrangian density $\sqrt{-g}R$ depends only on the sum (2), approximations (5) and (6) are senseless. They are far from being correct.

Following precise relation (4) with an accuracy up to the terms of the second order over ϕ_{ik} there takes place an equality

$$\underset{(2)}{t}^{ik} = -\underset{(1)}{t}^{ik}$$

i.e., the second order terms in the expression for energy-momentum density tensor are quite precisely compensated by the first order terms, as a result energy-momentum density tensor turns out to be equal to zero.

Thus "satisfactory" results usually obtained with the Einstein method in weak gravitational wave energy calculations and present almost in all works devoted to gravitational theory are a consequence of an incorrect transition from precise relations of the Einstein gravitation theory to "weak field approximation". These results have nothing to do with the Einstein gravitation theory and cannot be treated as an argument in favour of this theory. In reality they may be related to some nongeometrised theory of tensor field, where relation (3) of the Einstein gravitation theory does not take place.

In the Einstein gravitation theory where the Lagrangian density of the gravitational field depends only on $g_{ik} = \gamma_{ik} + \phi_{ik}$ gravitational waves carry no energy.

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Appendix

EXPLICIT EXPRESSIONS FOR $\underset{(2)}{t}^{ik}$ AND $\underset{(1)}{t}^{ik}$

$$L_g = \sqrt{-g}R = \mathfrak{L}^{(1)} + \mathfrak{L}^{(2)} + \dots ,$$

where

$$\mathfrak{L}^{(1)} = -\partial_k (\gamma^{mn}\Gamma^k_{mn} - \gamma^{k\ell}\Gamma^m_{\ell m});$$

$$\mathfrak{L}^{(2)} = -\partial_k (-\phi^{mn}\Gamma^k_{mn} + \phi^{k\ell}\Gamma^m_{\ell m} + \frac{\phi}{2}\gamma^{mn}\Gamma^k_{mn} - \frac{\phi}{2}\gamma^{mn}\Gamma^k_{mn}) - \gamma_{ik}(\Gamma^n_{im}\Gamma^m_{kn} - \Gamma^m_{ik}\Gamma^n_{mn});$$

A symmetric tensor of energy-momentum in first order approximation is

$$\underset{(1)}{t}^{ik} = -(\partial^i \partial^k \phi - \quad \phi\gamma^{ik} - \partial^i \partial_p \phi^{pi} + \quad \phi^{ik} + \partial_e \partial_p \phi^{ep} \gamma^{ik}).$$

This expression coincides with the field equation $(-2)\delta L/\delta\phi_{ik} = 0$ in the first order. The relevant pseudotensor τ_i^k is a divergence from the antisymmetric spin tensor

$$\tau^k_{i(1)} = \partial_n(\partial_\ell \phi_{im} - \partial_m\phi_{i\ell})\,\gamma^{km}\gamma^{n\ell}.$$

In the second order the expression for the energy-momentum symmetric tensor is

$$\begin{aligned} t_{ik(2)} = {} & (\phi^{\ell s} - \frac{\phi}{2}\gamma^{\ell s})(\partial_i\partial_k\phi_{s\ell} + \partial_\ell\partial_s\phi_{ik} - \partial_\ell\partial_k\phi_{si}) - \\ & - (\gamma_{ik}\phi^{\ell s} - \gamma_{ik}\phi\gamma^{\ell s}\frac{1}{4} - \frac{1}{2}\phi_{ik}\gamma^{\ell s})(\phi_{\ell s} + \partial_\ell\partial_s\phi - 2\partial_\ell\partial_p\phi^p_s) - \\ & - \partial_\ell\phi^{\ell s}(\partial_i\phi_{sk} + \partial_k\phi_{si} - \partial_s\phi_{ik}) + \frac{1}{2}\partial_\ell\phi(\partial_i\phi^\ell_k + \partial_k\phi^\ell_i - \partial^\ell\phi_{ik}) + \\ & + \frac{1}{2}\partial_k\phi^{\ell s}\partial_i\phi_{\ell s} + \partial_\ell\phi_{mi}\partial^\ell\phi^m_k - \partial_m\phi_{i\ell}\partial^\ell\phi^m_k - \\ & - \gamma_{ik}(\frac{3}{4}\partial_p\phi^{\ell s}\partial^\ell\phi_{\ell s} - \partial_\ell\phi^{\ell s}\partial_p\phi^p_s + \partial_\ell\phi^{\ell s}\partial_s\phi - \frac{1}{2}\partial_p\phi_{s\ell}\partial^\ell\phi^{ps} - \frac{1}{4}\partial_\ell\phi\partial^\ell\phi). \end{aligned}$$

Here the canonical energy-momentum tensor is equal to

$$\tau_i^{(2)k} = \partial_n[(\partial_\ell\phi_{im} - \partial_m\phi_{i\ell})(-\gamma^{km}\phi^{n\ell} - \gamma^{n\ell}\phi^{mk} + \frac{\phi}{2}\gamma^{km}\gamma^{n\ell})].$$

The sum of $t^{(1)}_{ik}$ and $t^{(2)}_{ik}$ provides us with the field equation with the accuracy up to the second order:

$$t^{(1)}_{ik} + t^{(2)}_{ik} = 0.$$

Due to the field equations the second and first order terms in an expression for t^{ik} are precisely compensated. Therefore one cannot reject, as it is usualyy done, $t^{ik}_{(1)}$ as a diverging term and keep only $t^{ik}_{(2)}$.

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