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PURE CLASSICAL SU(2) YANG-MILLS THEORY WITH POTENTIALS INVARIANT UNDER A U(1) GAUGE SUBGROUP

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ABSTRACT : The present article is devoted to pure SU(2) classical Yang-Mills theories whose potentials are invariant under a U(1) gauge subgroup. Such potentials are shown to be associated with classical Maxwell-like fields with magnetic sources as 't Hooft's monopole is associated with the Dirac magnetic monopole. Conversely we give Yang-Mills potentials corresponding to some Maxwell-like fields, in particular

i) static magnetic fields with emphasis on those with cylindrical symmetry (including the dipole and other multipoles)

ii) the "ephemerons" corresponding to an instantaneous magnetic multipole .

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I. Introduction

The concept of symmetry has always been a useful tool in physical problems and has already been used in gauge theories where the Lorentz group L or the four dimensional Euclidean group is involved besides the gauge groupe G. As famous examples, let us mention the spherical symmetry of the 't Hooft monopole where a SO(3) diagonal subgroup of L and G plays a particular role, and the use of self-duality for the 't Hooft Polyakov instanton.

We have exploited symmetry properties[1] by looking for SU(2) potentials A_μ which are invariant under a U(1) subgroup of the SU(2) gauge group. It happens that the problem is easy to solve in principle since, given a U(1) subgroup generated by a traceless matrix G(x), we are able to find the corresponding Yang-Mills potentials by solving an associated Maxwell-type problem (section II).

Conversely, given some types of Maxwell fields with magnetic sources, we are able to associate many classical Yang-Mills fields with different kinds of singularities including many kinds of magnetic multipole solutions generalizing 't Hooft monopole one.

In section III we investigate static magnetic sources with a special emphasis on those with cylindrical symmetry.

Section IV contains a generalisation of our results of sections II and III where the sources are magnetic but not static. Peculiar examples called "ephemerons" are investigated. They correspond to magnetic multipoles appearing suddenly and disappearing immediately. They are examples of truly four-dimensional problems which are not self-dual but instead satisfy

$$\varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 0 \tag{I.1}$$

Finally, in the conclusion, a few remarks are made about the possible extensions of the present work.

II. Yang-Mills potentials invariant under a U(1) gauge subgroup.

The main idea has been explained in a recent letter[1]. Let us recall it.

Let $A_\mu(x)$ be the SU(2) potentials of a gauge theory on space-time (with Euclidean metric) i.e. a set of four 2 x 2 Hermitian traceless matrices. Under a SU(2) gauge transformation U(x), A_μ transforms as

$$A_\mu' = U A_\mu U^{-1} - i U \partial_\mu U^{-1} \qquad (2.1)$$

Let $U_a(x)$ be a one dimensional subgroup of the gauge group generated by G(x) (a traceless Hermitian 2 x 2 matrix)

$$U_a(x) = \exp (i a G(x)) \qquad (2.2)$$

where a is a real parameter. For $A_\mu(x)$ to be invariant under this subgroup it is necessary and sufficient that

$$[G, A_\mu] + i \partial_\mu G \equiv i D_\mu G = 0 \qquad (2.3)$$

It must be noted that G(x) cannot be chosen arbitrarily since multiplying (2.3) by G(x), taking the trace and remembering that G^2 is a multiple of the unit matrix, we get the condition

$$\partial_\mu (G^2) = 0 \qquad (2.4)$$

Without loss of generality we will impose the normalisation

$$G^2 = \frac{1}{4} \qquad (2.5)$$

Our aim is to look for potentials $A_\mu(x)$ which, using the definitions of the covariant derivative

$$D_\mu = \partial_\mu + i [A_\mu , \ ] \qquad (2.6)$$

and of the fields

$$F_{\mu\nu} = -i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \qquad (2.7)$$

satisfy (2.3) and the Yang-Mills equations

$$D_\mu F_{\mu\nu} = o \quad (2.8.a)$$

$$\epsilon_{\mu\nu\rho\sigma} \; D_\nu F_{\rho\sigma} = o \quad (2.8.b)$$

where (2.8.b) is identically satisfied.

Let us first examine condition (2.3). Its general solution is easily obtained from standard vector techniques since the bracket is nothing else than the vector product. We get

$$A_\mu = -i[G, \partial_\mu G] + \alpha_\mu G \quad (2.9)$$

where $\alpha_\mu(x)$ is an arbitrary four vector.

Let us now require that A_μ, as given by (2.9), satisfy the Yang-Mills equations. The fields, as defined by (2.7), are

$$F_{\mu\nu} = -i[\partial_\mu G, \partial_\nu G] + (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu) G \quad (2.10)$$

This equation shows that $F_{\mu\nu}$ is parallel to G in internal space since, by (2.4), $\partial_\mu G$ and $\partial_\nu G$ are orthogonal to G. Therefore

$$F_{\mu\nu} = f_{\mu\nu} \; G \quad (2.11)$$

The Maxwell-like field $f_{\mu\nu}$ is easily obtained by multiplying (2.10) by G and taking the trace :
$f_{\mu\nu}$ is the sum of two terms

$$f_{\mu\nu} = h_{\mu\nu} + a_{\mu\nu} \quad (2.12)$$

where

$$h_{\mu\nu} = -2i \; Tr(G[\partial_\mu G, \partial_\nu G]) \quad (2.13)$$

$$a_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu \quad (2.14)$$

Putting (2.11) into (2.8) and using (2.3), one obtains

$$\partial_\mu \; f_{\mu\nu} = o \quad (2.15.a)$$

$$\epsilon_{\mu\nu\rho\sigma}\, \partial_\nu \, f_{\rho\sigma} = 0 \qquad (2.15.b)$$

Hence, we arrived at a Maxwell-like problem without sources (neither electric nor magnetic) in the domain D of space time where G is defined and twice derivable. If G were defined and derivable everywhere, we would be left with a free Maxwell-like field.

In the general case, the field $f_{\mu\nu}$ can be considered as created by two currents j^e_μ and j^m_μ (e and m for electric and magnetic-like currents) vanishing in D and defined by

$$\partial_\mu \, f_{\mu\nu} = j^e_\nu \qquad (2.16.a)$$

$$\frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, \partial_\nu \, f_{\rho\sigma} = j^m_\mu \qquad (2.16.b)$$

By analytic continuation of $f_{\mu\nu}$, the currents j^e_ν and j^m_μ vanish in a domain $D' \supset D$. In certain cases they are uniquely defined, for example, whenever D' is the whole space time minus one point or one line, etc... as our examples will show.

Under gauge transformations U, the transformation properties of G, α_μ, $h_{\mu\nu}$ and $f_{\mu\nu}$ are

$$\begin{aligned} G' &= U\, G\, U^{-1} \\ \alpha'_\mu &= \alpha_\mu - 2i\ \mathrm{Tr}(G(\partial_\mu\, U^{-1})U) \\ h'_{\mu\nu} &= h_{\mu\nu} - \partial_\mu(\alpha'_\nu - \alpha_\nu) + \partial_\nu(\alpha'_\mu - \alpha_\mu) \\ f'_{\mu\nu} &= f_{\mu\nu} \end{aligned} \qquad (2.17)$$

The important point being that $f_{\mu\nu}$ is a gauge independent quantity.

Finally let us remark that (2.16) may, at first sight, look quite symmetrical between electricity and magnetism. This is not so since, using (2.12) and (2.13), one finds that the magnetic current

$$j^m_\mu = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, \partial_\nu h_{\rho\sigma} \qquad (2.18)$$

is directly related to our fundamental quantity G while j^e_μ is connected to the less essential arbitrary α_μ (2.9). In what follows, we will restrict ourselves to examples where

$$j^e_\mu = 0 \tag{2.19}$$

To sum up, given G(x) and $\alpha_\mu(x)$, formulae (2.12), (2.13) and (2.14) provide us with a solution (A_μ, $F_{\mu\nu}$) of the Yang-Mills equations in a domain D and associated by (2.11), (2.16) with a Maxwell-like field $f_{\mu\nu}$.

III. U(1)-symmetric potentials associated with given static magnetic fields

In this section we will discuss the problem of finding suitable G's such that the fields $F_{\mu\nu}$ (2.11) obtained in a U(1) invariant way correspond to static and magnetic-like Maxwell $f_{\mu\nu}$ field. It is therefore convenient to set the following

inverse problem

let $f_{\mu\nu}$ be a static magnetic Maxwell-like field $f_{\mu\nu}(f_{ij} = \varepsilon_{ijk}f_k,\ f_{oi} = o)$ satisfying (2.15), (2.19)

$$\varepsilon_{ijk}\,\partial_j f_k = o \qquad (3.1.a)$$

$$\partial_i f_i = j_o^m \equiv \rho^m \qquad (3.1.b)$$

with no electric current and a static magnetic charge density $\rho^m(\vec{x})$. The problem is then to find a static G such that

$$f_{ij} = -2i\ \mathrm{Tr}(G\,[\partial_i G,\ \partial_j G]) \qquad (3.2)$$

It is clear that, if we have solved the inverse problem, we get a solution of the Yang-Mills equations valid at best outside the support of ρ^m and that the corresponding Yang-Mills field is given by (2.11) and the Yang-Mills potential by (2.9) with $a_\mu = o$.

Let us show how this problem can be solved in principle First we define the vector g_i of unit length (2.5) by

$$G = \vec{g}\cdot\frac{\vec{\tau}}{2} \qquad (3.3.a)$$

$$g_i = \mathrm{Tr}(G\,\tau_i) \qquad (3.3.b)$$

where the τ_i are the Pauli matrices. Equation (3.2) is given in terms of the determinant of the three vectors $\vec{g}$, $\partial_i\vec{g}$, $\partial_j\vec{g}$ by

$$f_{ij} = (\vec{g},\ \partial_i\vec{g},\ \partial_j\vec{g}) \qquad (3.4)$$

III/A 't Hooft example

The bare monopole solution first obtained by 't Hooft is of this kind since if

$$\rho^m = 4\pi\ \delta^3(x) \tag{3.5}$$

the inverse problem can be solved by

$$g_i = \frac{x_i}{r} \qquad (r^2 = x_1^2 + x_2^2 + x_3^2) \tag{3.6.a}$$

$$f_i = \frac{x_i}{r^3} = \frac{1}{2}\ \epsilon_{ijk}\ f_{jk} \tag{3.6.b}$$

$$f_{oi} = o \tag{3.6.c}$$

and $f_{\mu\nu}$ is nothing else than the field of a magnetic monopole. The Yang-Mills $F_{\mu\nu}$ associated with it by (2.11) is the 't Hooft monopole.

III/B Geometrical approach

The normalisation condition (2.5) suggests the following change of notation

$$g_1 = \sqrt{1 - g^2}\ \cos\ \omega \tag{3.7}$$

$$g_2 = \sqrt{1 - g^2}\ \sin\ \omega$$

$$g_3 = g$$

which transforms (3.4) into

$$f_{ij} = \partial_i\omega\ \partial_j\ g - \partial_i g\ \partial_j\omega \tag{3.8}$$

According to (3.1) there exists a magnetic-like potencial v(x) such that

$$\vec{f} = - \vec{\nabla}\ v \tag{3.9}$$

so that finally the inverse problem can be stated as follows : given v, find g and such that

$$-\vec{\nabla}\, v = \vec{\nabla}\, \omega \wedge \vec{\nabla} g \tag{3.10}$$

Necessary conditions for having (3.10) are

$$\vec{\nabla} v \,.\, \vec{\nabla} g = \vec{\nabla} v \,.\, \vec{\nabla} \omega = o \tag{3.11}$$

Note also that outside the support of ρ^m (3.1.b),

$$\Delta\, v = o \tag{3.12}$$

Suppose we give on a surface of constant v a system of coordinate lines. By transporting these lines on the other equipotential surfaces along the orthogonal lines generated by $\vec{\nabla} v$, we get a three dimensional system of coordinates v, $\tilde{g}$, $\tilde{\omega}$ satisfying (3.11).

A change of variables $g(\tilde{g},\tilde{\omega}),\omega(\tilde{g},\tilde{\omega})$ enables one to solve (3.10). Indeed by construction

$$\vec{\nabla}\, \tilde{g} \wedge \vec{\nabla}\, \tilde{\omega} = n\, \vec{\nabla}\, v \tag{3.13}$$

By taking the divergence of this equation outside the support of ρ^m one sees that

$$\vec{\nabla} n \,.\, \vec{\nabla} v = o \tag{3.14}$$

which shows that n only depends on $\tilde{g}$ and $\tilde{\omega}$.
Then defining the Jacobian $J(\tilde{g},\tilde{\omega})$ of the change of variables by

$$J(\tilde{g},\tilde{\omega}) = \frac{\partial g}{\partial \tilde{g}}\frac{\partial \omega}{\partial \tilde{\omega}} - \frac{\partial g}{\partial \tilde{\omega}}\frac{\partial \omega}{\partial \tilde{g}} \tag{3.15}$$

one finds

$$\vec{\nabla} g \wedge \vec{\nabla} \omega = J\, n\, \vec{\nabla}\, v \tag{3.16}$$

So that a solution is obtained if

$$J \, \Omega = 1 \tag{3.17}$$

For example if

$$\begin{aligned} g &= \tilde{g} \\ \omega &= \int J(\tilde{\omega}', \tilde{g})\, d\tilde{\omega}' \end{aligned} \tag{3.18}$$

Other changes of variables are then still allowed provided that their Jacobian be one.

III/C The magnetic monopole revisited

Let us now illustrate the geometrical method in the case of the magnetic monopole. The equipotential surfaces are spheres. We choose

$$v = \frac{\mu}{r} \tag{3.19.a}$$

$$\omega = m \tan^{-1} \frac{x_2}{x_1} \tag{3.19.b}$$

$$g = \lambda \frac{x_3}{r} \tag{3.19.c}$$

with $r^2 = x_1^2 + x_2^2 + x_3^2$ and m, λ, μ real numbers. It is clear that (3.11) are satisfied.
We get a solution (3.9)

$$f_i = \frac{\mu x_i}{r^3} \tag{3.20}$$

of (3.10) provided

$$m \, \lambda = \mu \tag{3.21.a}$$

If we require G(or $\vec{g}$) to be single valued functions, we must have

$$m = \text{integer} \tag{3.21.b}$$

In fact for $m = 1$,

$$g_1 = \left(\frac{r^2 - \lambda^2 x_3^2}{x_1^2 + x_2^2}\right)^{\frac{1}{2}} \frac{x_1}{r}$$

$$g_2 = \left(\frac{r^2 - \lambda^2 x_3^2}{x_1^2 + x_2^2}\right)^{\frac{1}{2}} \frac{x_2}{r} \tag{3.22}$$

$$g_3 = \frac{\lambda x_3}{r}$$

The corresponding Yang-Mills field $F_{\mu\nu}$ has singularities not only for $r = o$ but also for $x_1 = x_2 = o$ except if $\lambda^2 = 1$ ('t Hooft monopole with unit charge (3.5), (3.6)). Thus we see that the charge quantization follows if we restrict the singularity to be localized at $r = o$, both for $f_{\mu\nu}$ and G and hence for $F_{\mu\nu}$.

The solution (3.22) calls for another interesting remark : $\vec{g}$ is not real if $r^2 - \lambda^2 x_3^2 < o$. More precisely, for $\lambda^2 \leq 1$, $\vec{g}$ is always real (i.e. G Hermitian) and therefore $F_{\mu\nu}$ is Hermitian everywhere $f_{\mu\nu}$ is defined. If $\lambda^2 > 1$, $F_{\mu\nu}$ is Hermitian in a region smaller than the one where $f_{\mu\nu}$ is defined.

III/D Static magnetic fields with cylindrical symmetry

In the present section we are interested in static solutions associated with magnetic fields f_{ij} invariant under rotations around the third axis. Obviously this generalises the monopole example. From the symmetry of the problem, it is easy to see that one may take

$$\omega = m \tan^{-1} \frac{x_2}{x_1} \tag{3.23}$$

and that g is, as v, a function of x_3 and of the invariant u

$$u = x_1^2 + x_2^2 \tag{3.24}$$

We easily find from (3.8, 3.9)

$$f_{23} = -2x_1 \partial_u v = \frac{m x_1}{u} \partial_3 g$$

$$f_{31} = -2x_2 \partial_u v = \frac{m x_2}{u} \partial_3 g \qquad (3.25)$$

$$f_{12} = -\partial_3 v = -2m \partial_u g$$

The integrability conditions on v and g following from (3.25) are (in some domain)

$$\Delta v = 4 \partial_u (u \partial_u v) + \partial_3^2 v = o$$

$$4 u \partial_u^2 g + \partial_3^2 g = o \qquad (3.26)$$

The function g (or v) can then be obtained from v(or g) by a quadrature.

Let us illustrate this result by four examples

i) set of two monopoles.

Defining the distances d_+ and d_- to the two monopoles situated in the positions + a and - a (resp.) along the third axis

$$d_\pm = (x_1^2 + x_2^2 + (x_3 \pm a)^2)^{1/2} \qquad (3.27)$$

the magnetic potential is

$$v = \frac{\mu_+}{d_+} + \frac{\mu_-}{d_-} \qquad (3.28)$$

giving rise to a f_{ij}

$$f_{23} = x_1 (\frac{\mu_+}{d_+^3} + \frac{\mu_-}{d_-^3})$$

$$f_{31} = x_2 (\frac{\mu_+}{d_+^3} + \frac{\mu_-}{d_-^3}) \qquad (3.29)$$

$$f_{12} = (\frac{\mu_+ (x_3 - a)}{d_+^3} + \frac{\mu_- (x_3 + a)}{d_-^3})$$

The function g is clearly the sum of the g's associated to each monopole

$$g = \frac{\lambda_+(x_3-a)}{d_+} + \frac{\lambda_-(x_3+a)}{d_-} \qquad (3.30)$$

and

$$\mu_\pm = m\ \lambda_\pm \qquad (3.31)$$

ii) Set of N monopoles along the third axis.
It is clear that the result just obtained has a generalisation to N monopoles situated at positions a_i along the third axis. With obvious notations

$$g = \sum_{i=1}^{N} \frac{\lambda_i(x_i-a_i)}{d_i} \qquad (3.32)$$

iii) The magnetic dipole.
The magnetic potential of a dipole oriented along the third axis can be obtained as a limit of two approching monopoles. Since we have shown additivity in v and g, we obtain easily the potential

$$v = \mu\frac{x_3}{r^3} \qquad (3.33)$$

the fields

$$f_{23} = \frac{3\mu\ x_1 x_3}{r^5}$$

$$f_{31} = \frac{3\mu\ x_2 x_3}{r^5} \qquad (3.34)$$

$$f_{12} = \frac{\mu}{r^5}\ (3x_3^2 - r^2)$$

and the function g ($\lambda\ m = \mu$)

$$g = -\lambda\ \frac{u}{r^3} \qquad (3.35)$$

For m = 1, by (3.7)

$$g_1 = x_1(\frac{1}{u} - \frac{\mu^2 u}{r^6})^{1/2}$$

$$g_2 = x_2(\frac{1}{u} - \frac{\mu^2 u}{r^5})^{1/2} \qquad (3.36)$$

$$g_3 = -\mu \frac{u}{r^3} \tag{3.36}$$

We see that, if $f_{\mu\nu}$ is defined everywhere except at the origin $r = o$, G(or g_i) is defined outside the region with boundary

$$r^6 = \mu^2 u^2 \tag{3.37}$$

and away from the third axis ($u = o$ or $x_1 = x_2 = o$).

iv) The cylindrical multipoles.
Due to linearity in v and g, the results of the dipole can be generalized to the cylindrical multipole where we obtain

$$v = \mu\, \partial_3^n \left(\frac{1}{r}\right) \tag{3.38}$$

and

$$g = \lambda \partial_3^{n-1} \left(\frac{u}{r^3}\right) \tag{3.39}$$

with (3.9) and (3.21). The proof of these statements can be traced to the identity

$$\partial_3^2 \left(\frac{1}{r}\right) = 2\, \partial_u \left(\frac{u}{r^3}\right) \tag{3.40}$$

III/E <u>Other examples of static magnetic multipoles</u>

In the preceding section we have solved the general case of a static magnetic field f_i with cylindrical symmetry. We could be more ambitious and try the general static magnetic multipole of arbitrary type, where the potential has the form of a superposition of multipoles.

$$v = \sum_{n_1 n_2 n_3}^{N_1 N_2 N_3} C_{n_1 n_2 n_3} \; \partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3} \left(\frac{1}{r}\right) \tag{3.41}$$

We have unfortunately not succeeded in solving this problem in a closed algebraic form. In specific instances it is possible to give particular solutions as we will show in two examples. The monopole and the dipole have already been treated completely since the dipole can always be put along the third axis by a rotation. The next examples will treat the general quadrupole and an octupole.

i) The general quadrupole.

The most general potential corresponding to a quadrupole is characterized by a symmetric matrix C_{ij} of zero trace (3.2) since

$$v = \sum_{ij} C_{ij} \, \partial_i \, \partial_j (\frac{1}{r}) \tag{3.42}$$

By a space rotation the matrix C_{ij} can be put in diagonal form. Hence, in a domain D excluding the origin,

$$\begin{aligned} v &= \frac{1}{2}(\alpha\partial_1^2 + \beta\partial_2^2 - (\alpha+\beta)\partial_3^2)(\frac{1}{r}) \\ &= (\alpha\partial_1^2 + \beta\partial_2^2)\frac{1}{r} \end{aligned} \tag{3.43}$$

In some suitable domain this is solved by

$$\begin{aligned} g &= -\frac{3\ x_1 x_2 x_3}{r^5} \\ \omega &= \ln\ ((\frac{x_3}{x_2})^{\alpha}\ (\frac{x_3}{x_1})^{\beta}) \end{aligned} \tag{3.44}$$

ii) An octupole

The octupole potential

$$v = \lambda\partial_1\partial_2\partial_3(\frac{1}{r}) = -\ 15\lambda\ \frac{x_1 x_2 x_3}{r^7} \tag{3.45}$$

can be solved easily by noting that

$$\begin{aligned} \tilde{g} &= \frac{(x_2^2 - x_1^2)\ (x_3^2 - x_1^2)}{r^7} \\ \tilde{\omega} &= \frac{x_3^2 - x_1^2}{x_2^2 - x_1^2} \end{aligned} \tag{3.46}$$

have their gradient orthogonal to the gradient of v. The functions g and ω can then be found by simple integration using (3.13-3.18). One particularly symmetrical solution is

$$g = \sqrt{5}\lambda \; \frac{x_2^2 - x_1^2}{r^{7/2}} \qquad \qquad (3.47)$$

$$\omega = \sqrt{5} \; \frac{x_3^2 - x_1^2}{r^{7/2}}.$$

IV. U(1) - symmetric potentials associated with non static magnetic fields. Ephemerons as typical examples.

In the preceeding section we have been concerned with the pure magnetic potentials corresponding to a static magnetic source ($j_4^m = \rho^m$, $j_i^m = o$). We now extend our study to general magnetic sources ($j_\mu^m \neq o$, $j_\mu^e = o$). Before giving a general geometrical approach to the problem, generalizing that of section III.B, we will first give some explicit examples with cylindrical symmetry.

IV/A. The Dipole ephemeron

We will call "ephemerons" gauge potentials corresponding to magnetic sources localized at a point in space-time. The simplest example corresponds to the following magnetic current

$$j_4^m = \mu\ \partial_3(\delta^4(x))$$

$$j_3^m = -\mu\partial_4(\delta^4(x)) \qquad (4.1)$$

$$j_1^m = j_2^m = o$$

We readily note that this current is conserved since

$$\partial_3\ j_3^m + \partial_4\ j_4^m = o \qquad (4.2)$$

It corresponds to a static dipole moment in a fictitious four dimensional space or to a usual magnetic dipole appearing suddenly at time zero and disappearing immediately. This ephemeron would generate an electromagnetic shock wave in the Minkowski space time.

In order to find out what is the field created by the current (4.1) in Euclidean space-time, we use the property of the Laplace-Beltrami operator in four dimensions

$$\Box(\frac{1}{R^2}) = 2\pi^2\ \delta^4(x) \qquad (4.3.a)$$

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad (4.3.b)$$

Defining
$$\phi = \frac{\mu}{2\pi^2 R^2} \qquad (4.4)$$

the magnetic Euclidean four potential a_μ^* created by the current density (4.1) is

$$\begin{aligned} a_4^* &= \partial_3 \phi \\ a_3^* &= -\partial_4 \phi \\ a_1^* &= a_2^* = o \end{aligned} \qquad (4.5)$$

and the corresponding field $f_{\mu\nu}$ is given by

$$f_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \partial_\rho a_\sigma^* \qquad (4.6)$$

that is

$$\vec{f} = \begin{cases} f_{23} = f_1 = \partial_1\partial_3 \phi \\ f_{31} = f_2 = \partial_2\partial_3 \phi \\ f_{12} = f_3 = (\partial_3^2 + \partial_4^2)\phi \end{cases} \qquad \vec{e} = \begin{cases} f_{14} = e_1 = -\partial_2\partial_4 \phi \\ f_{24} = e_2 = \partial_1\partial_4 \phi \\ f_{34} = e_3 = o \end{cases} \qquad (4.7)$$

Remark that this field is neither self-dual nor antiselfdual.
It is easy to see that

$$\frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \vec{e}.\vec{f} = o \qquad (4.8)$$

Let us now examine our inverse problem namely : find g and ω such that

$$f_{\mu\nu} = \partial_\mu\omega \, \partial_\nu g - \partial_\mu g \, \partial_\nu\omega \qquad (4.9)$$

A necessary condition obeyed by ω (and by g) which generalizes (3.11) is

$$\varepsilon_{\alpha\beta\gamma\delta} (\partial_\beta\omega) f_{\gamma\delta} = o \qquad (4.10)$$

which is equivalent to

$$\vec{f} \cdot \vec{\nabla} \omega = 0 \tag{4.11}$$

$$\vec{e} \wedge \vec{\nabla} \omega = (\partial_4 \omega) \vec{f} \tag{4.12}$$

Note that (4.12) is compatible with (4.8). Since ϕ is a scalar function, $\partial_3 \phi$ is invariant under rotations in the x_1, x_2 plane. This suggests to take as ω the function (3.19)

$$\omega = m \tan^{-1} \frac{x_2}{x_1} \tag{4.13}$$

which satisfies (4.11) and (4.12).
Writing (4.9) in the form

$$\begin{aligned} \vec{f} &= \vec{\nabla} \omega \wedge \vec{\nabla} g \\ \vec{e} &= (\partial_4 g) \vec{\nabla} \omega \end{aligned} \tag{4.14}$$

it is now easy to find for g, using (3.24)

$$g = -\frac{\lambda}{\pi^2} \frac{u}{R^4} \tag{4.15}$$

with as usual the relation (3.21).
Therefore, for m=1, one obtains

$$\vec{g} = \begin{cases} g_1 = \left(\frac{1}{u} - \frac{\mu^2}{\pi^4} \frac{u}{R^8}\right)^{1/2} x_1 \\ g_2 = \left(\frac{1}{u} - \frac{\mu^2}{\pi^2} \frac{u}{R^8}\right)^{1/2} x_2 \\ g_3 = -\frac{\mu}{\pi^2} \frac{u}{R^4} \end{cases} \tag{4.16}$$

The functions g_i (and therefore G and the Yang-Mills fields $F_{\mu\nu}$ (2.11)) are defined everywhere except in the 2-plane $u = 0$ (i.e. $x_1 = x_2 = 0$) and inside the surface

$$R^4 = \frac{|\lambda|}{\pi^2} u \tag{4.17}$$

IV/B Other types of Ephemerons with cylindrical symmetry.

Let us now consider a generalized ephemeron with cylindrical symmetry defined by the following magnetic sources (sec. (4.1))

$$\hat{j}_4^m = P\ j_4^m = \partial_3\ (P\ \delta^4(x))$$

$$\hat{j}_3^m = P\ j_3^m = -\partial_4 (P\ \delta^4(x)) \qquad (4.18)$$

$$\hat{j}_1^m = \hat{j}_2^m = o$$

Where P is an arbitrary polynomial in the derivatives with respect to x_3 and x_4 :

$$P = \sum_{a,b} \mu_{ab}\ \partial_3^a \partial_4^b \qquad (4.19)$$

It is clear that formulae (4.5), (4.6) are still valid provided a_μ^*, $f_{\mu\nu}$ and ϕ are replaced by

$$\hat{a}_\mu^* = P\ a_\mu^* \qquad (4.20.a)$$

$$\hat{f}_{\mu\nu}^* = P\ f_{\mu\nu} \qquad (4.20.b)$$

$$\hat{\phi} = P\ \phi \qquad (4.20.c)$$

The problem is readily solved by applying the operator P on equation (4.9) and by noting that ω does not depend on x_3 and x_4

$$P\ \omega = o \qquad (4.21)$$

Hence

$$\hat{\omega} = \omega \qquad (4.22.a)$$

$$\hat{g} = P\ g \qquad (4.22.b)$$

is a solution of the inverse problem. Again as is required by (4.9)

$$\frac{1}{4}\ \epsilon_{\mu\nu\rho\sigma}\hat{f}_{\mu\nu}\ \hat{f}_{\rho\sigma} = o \qquad (4.23)$$

IV/C <u>Geometrical approach</u>.

Suppose that we want to solve the general inverse problem for $\alpha_\mu = o$ and $f_{\mu\nu} = h_{\mu\nu}$ deriving from a general magnetic source. We have to find g and ω such that

$$f_{\mu\nu} = \partial_\mu \omega \partial_\nu g - \partial_\mu\ g\ \partial_\nu\ \omega \qquad (4.24)$$

in some domain D of space-time where the sources are zero. As we have remarked earlier eq. (4.24) implies that

$$\epsilon_{\mu\nu\rho\sigma}\ f_{\mu\nu}\ f_{\rho\sigma} = o \qquad (4.25)$$

so that not all magnetic sources are allowed.

Geometrical arguments will tell us that once (4.25) is satisfied there exists g's and ω's satisfying (4.24) in some D.

Indeed, given a $f_{\mu\nu}$ field satisfying (4.25), elementary properties of the four dimensional Euclidean space imply that a 2-plane T_x can be associated with every point x_μ. The 2-plane T_x is spanned by vectors δx_μ, such that

$$\delta\ x_\mu\ f_{\mu\nu} = o \qquad (4.26)$$

Starting from a point x_μ^o in D the aquations

$$\frac{\partial x_\mu}{\partial t}\ \frac{\partial x_\nu}{\partial s} - \frac{\partial x_\mu}{\partial s}\ \frac{\partial x_\nu}{\partial t} = \epsilon_{\mu\nu\rho\sigma}\ f_{\rho\sigma} \qquad (4.27)$$

generate a two dimensional surface $x_\mu(t,s)$ such that

$$x_\mu(o,o) = x_\mu^o \qquad (4.28)$$

which we will call the surface generated by $f_{\mu\nu}$ and x_μ^o. Tangent vectors to this 2-surface satisfy (4.26).

Now let us consider an (almost) arbitrary 2-surface V such that the 2-plane tangent to V at x_μ intersects T_x only at x_μ (if necessary, the domain D can be restricted). Let us also choose on this surface a set of curvilinear coordinates $\tilde{g}_o$, $\tilde{\omega}_o$.

Let $\tilde{g}$ = constant (resp. $\tilde{\omega}$ = constant) be the 3-surface generated by $f_{\mu\nu}$ and the points of the line $\tilde{g}_o$ = constant (resp. $\tilde{\omega}_o$ = constant). Then, for $\tilde{g}$(resp. $\tilde{\omega}$)

$$\epsilon_{\mu\nu\rho\sigma}\, \partial_\nu \tilde{g}\, f_{\rho\sigma} = o \tag{4.29}$$

and hence

$$\partial_\mu \tilde{\omega}\, \partial_\nu \tilde{g} - \partial_\mu \tilde{g}\, \partial_\nu \tilde{\omega} = n\, f_{\mu\nu} \tag{4.30}$$

Taking the exterior derivative of (4.30) and noting that both the left-hand side and the exterior derivative of $f_{\mu\nu}$ are zero inside D, one finds that n satisfies also (4.29) and hence that n only depends on $\tilde{g}$ and $\tilde{\omega}$.

We have now proved that we obtain a situation quite similar to section III. By the change of variables $g(\tilde{g}, \tilde{\omega})$, $\omega(\tilde{g},\tilde{\omega})$ with Jacobian J (3.15) our problem is solved provided

$$J_n = 1 \tag{4.31}$$

which can be integrated for example by (3.18) as in the static case.

Let us once more insist on the fact that the solution of the problem rests on the condition (4.25) which was automatically satisfied in the static case (see 4.8) since ($e_i = f_{oi} = o$). Here however this condition is a non trivial restriction on the theory.

IV/D The Dipole Ephemeron revisited

We can now show that the dipole ephemeron we have presented in section IV/A is the most general one pp to rotations. Indeed the most general dipole current is

$$j^m_\mu = a_{\mu\nu}\partial_\nu(\delta^4(x)) \tag{4.32}$$

The conservation of the current implies that $a_{\mu\nu}$ is antisymmetrical. By rotation all components of $a_{\mu\nu}$ can be put to zero except

$$a_{43} = - a_{34} = \mu$$
$$a_{23} = - a_{12} = \sigma \qquad (4.33)$$

and it is not difficult to show that (4.25) implies

$$\mu\sigma = o \qquad (4.34)$$

i.e. the dipole we have presented in section IV/A.

V. Conclusion

In this paper we have obtained some solutions of pure classical SU(2) Yang-Mills equations which are invariant under a U(1) subgroup of the gauge group, valid in a domain D of the four-dimensional Euclidean space and with sources which are of magnetic type. We now present some positive and negative remarks on our work together with some suggestions for further studies.

i) We have obtained all static magnetic multipoles with cylindrical symmetry, and have given geometrical arguments showing that the solutions exist for all multipoles. Explicit algebraic solutions are however still missing in the general case though we have constructed many examples for the lower multipoles.

ii) Extension of our results to SU(n) (n>2) Yang-Mills theories can be handled by an approach similar to the one presented in this paper.

iii) Not all ephemerons (with sources which are distributions with support at one isolated point of space time) have solutions for the inverse problem since equation (4.25) is a necessary and sufficient condition. Apart from the dipole and other ephemerons with cylindrical symmetry it would be interesting to see if there exist other types of ephemerons.

iv) About the method itself of getting solutions of the Yang-Mills equations by asking the invariance under a U(1) subgroup of the gauge (one generator G) we could ask for invariance under a larger subgroup of the gauge group. It is then easy to show that the solutions correspond essentially to vacuum (pure gauge) solutions.

v) In stating our inverse problem, we have always chosen a_μ and j_μ^e to be zero. It would be possible of course to abandon these restrictions and hence add electric type sources.

vi) Finally, in view of application, it would be necessary to couple the gauge fields to other fields. One of the possibilities consists of using G(x) itself as a scalar field of isospin one. However, as we have seen, G, as a field, has singularities of various kinds. For the solutions which have cylindrical symmetry the interpretation of the solutions could proceed as follows.

Write a sufficiently general <u>ansatz with cylindrical symmetry</u> for the potential A_μ and the scalar field G. Then look for solutions of the restricted set of fields which are smooth and approach asymptotically (i.e. far from the singularities) the solutions we have found. We hope to come back on this problem in a later publication.

vii) We have seen that to a $f_{\mu\nu}$ defined in a domain D there corresponds a G which is usually defined in a domain D' smaller than D (see(3.37)). It would be interesting to study how the extra singularities vary with gauge transformations and if the singularities of $f_{\mu\nu}$ are the only intrinsic ones.

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Reference

1) H. BACRY and J. NUYTS
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