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ON THE QUANTIZATION OF SPIN SYSTEMS AND FERMI SYSTEMS

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ABSTRACT

It is shown that spin operators and Fermi operators can be interpreted as the Weyl quantization of some functions on a "classical phase space" which is a compact group. Moreover the transition from quantum spin to Fermi operators is an isomorphism of the "classical phase space" preserving the Haar measure.

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1. INTRODUCTION

In a previous paper [1], we intended to treat anticommutation relations of Fermi systems and commutation relations of Quantum spin systems as usual canonical commutation relations over a "classical phase space", which is the phase space of a classical spin system. In particular we showed that one can characterize all the central extensions of the "group of classical spin system", satisfying natural physical requirements, and that they are those which are usually considered. Moreover it was shown that they all lead to the same C^*-algebra, up to isomorphism.

In this paper we continue this program. If $\mathcal{F}_\Lambda$ denotes the group of finite subsets of a countable set Λ with the symmetric difference as group law, we show in section 2 how the C^*-algebra $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi)}$ corresponding to the multiplier ξ (see [1]) is $*$-isomorphic to the U.H.F.-algebra (proposition (2.3)).

In the third section we study the states of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi)}$ and in particular we define an important closed convex subset of states, the "quasi-classical states" (definition (3.2)) and we show that it contains a great number of usual states. These states have a nice characterization in terms of probability on the classical phase space (corollary (3.4)). In some sense they are as close as possible to classical states.

In the fourth section we quantify, according to the Weyl procedure, functions on the "classical phase space". (definition (4.2)). A new situation arises, with respect to the usual C.C.R., for a finite number of degrees of freedom, since there are in our case lots of inequivalent irreducible representations of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi)}$ and the set of functions one can quantize depends on the representation. Finally we derive an important result (theorem (4.17)) which shows that the transition from quantum spin systems to Fermi systems is described by an automorphism of the "classical phase space" which leaves invariant the Haar measure.

Through this paper we follow the definitions and notations of [1].

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2. $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^s)}$ AS AN U.H.F.-ALGEBRA

In reference [1] we showed that the C^*-algebra $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi)}$ built in a canonical way on a central extension of $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$ by a multiplier ξ which is non degenerate and invariant by permutations of point in Λ are independent of ξ up to isomorphism. This algebra is, for Λ countable infinite, isomorphic to the U.H.F. algebra and in this section we want to make this correspondence more explicit.

We shall specialize to $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^s)}$ (*) for the sake of simplicity, but our results are also valid for the other multipliers (see [1] theorem (2.40)).

If $\Lambda_1 \subset \Lambda$ then $\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}$ is a subgroup of $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$ and we shall denote by the same symbol ξ^s the restriction of the multiplier ξ^s of $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$ to $\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}$. It is non degenerate. (See [1] definition (1.11)).

<u>Lemma</u> (2.1). Let $\Lambda_1, \Lambda_2 \subset \Lambda$, $\overline{\Delta(\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}, \xi^s)}$ is $*$-isomorphic to a subalgebra of $\overline{\Delta(\mathcal{F}_{\Lambda_2} \times \mathcal{F}_{\Lambda_2}, \xi^s)}$ if $|\Lambda_1| \leq |\Lambda_2|$ (**), they are $*$-isomorphic if and only if $|\Lambda_1| = |\Lambda_2|$.

By definition if $|\Lambda_1| \leq |\Lambda_2|$ there is an injection j of Λ_1 into Λ_2. j extends to an injective homomorphism $\tilde{j}$ of $\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}$ into $\mathcal{F}_{\Lambda_2} \times \mathcal{F}_{\Lambda_2}$; a proof similar to the one of proposition (1.16) of [1] shows that there exists an injective $*$-homomorphism $\alpha_{\tilde{j}}$ of $\overline{\Delta(\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}, \xi^s)}$ into $\overline{\Delta(\mathcal{F}_{\Lambda_2} \times \mathcal{F}_{\Lambda_2}, \xi^s)}$; moreover an application of proposition (3.11) in [2] shows that

$$\begin{matrix} \Lambda_1 \subsetneq \Lambda_2 \\ \Lambda_1, \Lambda_2 \subset \Lambda \end{matrix} \Longrightarrow \overline{\Delta(\mathcal{F}_{\Lambda_1} \times \mathcal{F}_{\Lambda_1}, \xi^s)} \subsetneq \overline{\Delta(\mathcal{F}_{\Lambda_2} \times \mathcal{F}_{\Lambda_2}, \xi^s)}.$$

(*) Notice that ξ^s is :

$$\xi^s(X, Y; X', Y') = i^{-|X \cap Y| - |X' \cap Y'| + |X \Delta X' \cap Y \Delta Y'| + 2|X' \cap Y|}.$$

(**) $|\Lambda|$ is the cardinality of Λ.

On the other hand j hence $\tilde{j}$ and $\alpha_{\tilde{j}}$, are surjective if and only if $|\Lambda_1| = |\Lambda_2|$.

Lemma (2.2). If $\Lambda_1 \in \mathfrak{p}_\Lambda$ then :

$$\Delta(\mathfrak{p}_{\Lambda_1} \times \mathfrak{p}_{\Lambda_1}, \xi^s) \simeq \bigotimes_{l=1}^{|\Lambda_1|} M_{2\times 2} ,$$

the tensor product of two by two matrices. Consequently

$$\Delta(\mathfrak{p}_{\Lambda_1} \times \mathfrak{p}_{\Lambda_1}, \xi^s) = \overline{\Delta(\mathfrak{p}_{\Lambda_1} \times \mathfrak{p}_{\Lambda_1}, \xi^s)} .$$

In order to prove the first statement let us describe the correspondence explicitly.

Let σ_x and σ_y be the usual two by two Pauli spin matrices, they generate the two by two matrices. Let :

$$\sigma_x^i = \underbrace{\underbrace{1 \otimes \ldots \otimes \sigma_x}_{i^{th}\ place} \otimes \ldots \otimes 1}_{|\Lambda_1|} .$$

$$\sigma_y^j = \underbrace{\underbrace{1 \otimes \ldots \otimes \sigma_y}_{j^{th}\ place} \otimes \ldots \otimes 1}_{|\Lambda_1|} .$$

These matrices satisfy the relations

$$\sigma_x^i\,\sigma_x^j = \sigma_x^j\,\sigma_x^i\;,$$
$$\sigma_y^i\,\sigma_y^j = \sigma_y^j\,\sigma_y^i\;,$$
$$\sigma_x^i\,\sigma_y^j = (1-2\delta_{i,j})\,\sigma_y^j\,\sigma_x^i\;,$$

and generate $\bigotimes_{l=1}^{|\Lambda_1|} M_{2\times 2}$.

Then let us define :

$$\sigma_x^X = \prod_{x_i\in X}\sigma_x^i\;,\quad \sigma_y^Y = \prod_{x_j\in Y}\sigma_y^j\;,\quad X,Y\in\mathcal{F}_{\Lambda_1}\,,$$

the elements $i^{-|X\cap Y|}\,\sigma_x^X\,\sigma_y^Y$, $X,Y\in\mathcal{F}_{\Lambda_1}$ are both unitary and hermitian : they span linearily $\bigotimes_{l=1}^{|\Lambda_1|} M_{2\times2}$ and the correspondence.

$$\delta^s_{X,Y} = i^{-|X\cap Y|}\,\sigma_x^X\,\sigma_y^Y\;,$$

extends to a $*$-isomorphism of $\overline{\Delta(\mathcal{F}_{\Lambda_1}\times\mathcal{F}_{\Lambda_1},\xi^s)}$ to $\bigotimes_{l=1}^{|\Lambda_1|} M_{2\times2}$; but that last algebra is closed hence $\Delta(\mathcal{F}_{\Lambda_1}\times\mathcal{F}_{\Lambda_1},\xi^s)$ is closed for $|\Lambda_1|<\infty$.

Proposition 2.3

$\overline{\Delta(\mathcal{F}_\Lambda\times\mathcal{F}_\Lambda,\xi^s)}$ is isomorphic to the U.H.F.-algebra $\overline{\bigcup_{N\in\mathbb{Z}}\bigotimes_{l=1}^{N} M_{2\times2}}$.

It is clear that

$$\Delta(\mathcal{F}_\Lambda\times\mathcal{F}_\Lambda,\xi^s) = \bigcup_{\Lambda_1\in\mathcal{F}_\Lambda}\Delta(\mathcal{F}_{\Lambda_1}\times\mathcal{F}_{\Lambda_1},\xi^s)$$
$$= \bigcup_{\Lambda_1\in\mathcal{F}_\Lambda}\overline{\Delta(\mathcal{F}_{\Lambda_1}\times\mathcal{F}_{\Lambda_1},\xi^s)}\,.$$

This last term is *-isomorphic to the union of tensor products and the isomorphism is isometric hence it extends to a *-isomorphism of the closures.

3. SOME INTERESTING STATES OF $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^s)}$:

Let Δ be the diagonal of $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$. Since the ξ^s we have considered is such that

$$(3.1) \qquad \xi^s(X,X\,;Y,Y) = 1 \quad , \quad X,Y \in \mathcal{F}_\Lambda \, ,$$

the subalgebra generated by the $\delta^s_{X,X}$, $X \in \mathcal{F}_\Lambda$, is abelian.

Definition (3.2) (cf. [4]) $\mathcal{C}$ is the convex and weakly closed set of states ω of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^s)}$ satisfying

$$\omega\left(\delta^s_{X,Y}\right) = 0 \quad \text{if} \quad X \neq Y .$$

This set has a nice characterization in term of the probability measures on $\mathcal{P}_\Lambda$ the dual of $\mathcal{F}_\Lambda$ namely :

Theorem (3.3)

Let f be a positive type function on $\mathcal{F}_\Lambda$ normalized to 1 , namely $f(\phi)=1$, the correspondence :

$$f \longrightarrow \omega_f : \delta^s_{X,Y} \longrightarrow \omega_f\left(\delta^s_{X,Y}\right) = \begin{cases} 0 \text{ if } X \neq Y \\ f(X) \text{ if } X = Y \end{cases}$$

is a bijection of the positive type function on $\mathcal{F}_\Lambda$ normalized to 1 onto $\mathcal{C}$. It sends characters onto pure states.

Indeed if $\omega \in \mathcal{C}$, the restriction of ω to the subalgebra generated by the $\delta^s_{X,X}$ is still a state on an abelian algebra. $X \in \mathcal{F}_\Lambda \to \omega(\delta^s_{X,X})$ is a positive type function. Vice-versa if f is a positive type function on $\mathcal{F}_\Lambda$ then ω_f defined in (3.3) is a state of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^s)}$ (cf. [3] corollary (2.15)).

Corollary (3.4)

Let $\omega \in \mathcal{C}$. There exists a probability measure μ on $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ such that :

$$X, Y \in \mathfrak{p}_\Lambda \times \mathfrak{p}_\Lambda \longrightarrow \omega(\delta^s_{X,Y}) = \mathcal{F}\mu(X,Y) .$$

Proof is a direct application of [3] theorem (2.14).

As we shall see in section 4, the $\delta^s_{X,Y}$ are the quantized of the functions $\hat{X}, \hat{Y} \in \mathcal{P}_\Lambda \times \mathcal{P}_\Lambda \to (-1)^{|X \cap \hat{Y}| + |Y \cap \hat{X}|}$; hence (cf. [4]) corollary (3.4) justify the name of quasi-classical state for elements of $\mathcal{C}$. Let us remark that we have defined quasi-classical states with respect to the diagonal of $\mathfrak{p}_\Lambda \times \mathfrak{p}_\Lambda$; one can as well choose any maximal subgroup of $\mathfrak{p}_\Lambda \times \mathfrak{p}_\Lambda$ on which ξ^s is one.

According to [3] theorem (2.16) one can write explicitly the G.N.S. representation associated to a classical state.

Proposition 3.5. Let ν be a probability measure on $\mathcal{P}_\Lambda$. Let f be its Fourier transform and ω_f the classical state of $\Delta(\mathfrak{p}_\Lambda \times \mathfrak{p}_\Lambda, \xi^s)$ associated to f by theorem (3.3). Let $\mathcal{H}, \pi, \Omega$ be the Hilbert space, the representation and the cyclic vector of the G.N.S. construction associated with ω_f. Then

$$\mathcal{H} = \ell_2(\mathfrak{p}_\Lambda) \otimes L_2(\mathcal{P}_\Lambda, \nu) ,$$

$$\Omega = \delta_\phi \otimes 1 ,$$

$$\{\pi(\delta^s_{X,Y}) F\}(Z_1, \hat{Z}_2)$$

$$= i^{-|X \cap Y|} i^{|Y|} (-1)^{|Z_1 \Delta \hat{Z}_2 \cap Y|} F(Z_1 \Delta X \Delta Y, \hat{Z}_2)$$

$\mathcal{H}$ is separable.

Proof is an obvious extension of theorem (2.16) of [3] with the special choice of application S and T :

$$S(X,Y) = (X \Delta Y, \emptyset) \, , \tag{3.6}$$

$$T(X,Y) = (Y,Y) \quad , \quad X,Y \in \mathcal{F}_\Lambda \, . \tag{3.7}$$

Let us remark that the diagonal $\mathcal{L}$ of $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$ is maximal abelian with respect to ξ^S hence the pure states on the diagonal correspond to pure states of $\mathcal{L}(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^S)$.
Among classical states, we have the product states $\omega_{\{a_i\}}$ which are defined by

$$\omega_{\{a_i\}}(\delta^S_{X,Y}) = \prod_{x_i \in \Lambda} \left((1 - |\{x_i\} \cap X|)(1 - |\{x_i\} \cap Y|) + |\{x_i\} \cap X| \, |\{x_i\} \cap Y| \, a_i \right) \, , \tag{3.8}$$

where a_i is any function from $x_i \in \Lambda \longrightarrow [-1, +1]$.
The product converges since at most a finite number of terms are different from one. Moreover $\omega_{\{a_i\}} \in \mathcal{C}$ since if $X \neq Y$ there is at least a point $x_i \in X \Delta Y$, hence a factor, which is zero.
$\omega_{\{a_i\}}$ is pure if and only if $|a_i| = 1$; since if it exists an a_i which is different from ± 1 , the corresponding factor

$$(1 - |x_i \cap X|)(1 - |x_i \cap Y|) + |x_i \cap X| \, |x_i \cap Y| \, a_i$$

is a convex combination of two functions of this type. On the other hand if $a_i = \pm 1$ let $\Lambda_1 \subset \Lambda$ be the set of $x_i \in \Lambda$ for which $a_i = -1$. Then

$$\omega_{\{a_i\}}(\delta^S_{X,X}) = (-1)^{|\Lambda_1 \cap X|} = \omega_{\Lambda_1}(\delta^S_{X,X}) \tag{3.9}$$

It is a character of $\mathcal{F}_\Lambda$, hence $\bar{\omega}_{\{a_i\}}$ is pure.

All these states are quite familiar indeed :

$\omega_{\{-1\}} = \omega_{\Lambda}$ is the usual Fock representation of the U.H.F. algebra

$\omega_{\{+1\}} = \omega_{\varphi}$ is the anti Fock representation of the U.H.F. algebra

$\omega_{\{0\}}$ is the central state of the U.H.F. algebra.

In that context the $\omega_{\{a_i\}}$ for $a_i = \pm 1$ are the analogue of the coherent states [x]. Indeed we have the following :

Proposition (3.10) : Let ω_{Λ_1} and ω_{Λ_2} two pure product states. Then

$$\omega_{\Lambda_1}(\delta^{s}_{X,Y}) = \omega_{\Lambda_2}(\delta^{s}_{X,Y})(-1)^{|\Lambda_1 \Delta \Lambda_2 \cap X|} \quad , X, Y \in \mathcal{F}_{\Lambda} .$$

moreover ω_{Λ_1} and ω_{Λ_2} are unitary equivalent if and only if :

$$\Lambda_1 \Delta \Lambda_2 \in \mathcal{F}_{\Lambda} .$$

This proposition for the usual C.C.R. is nothing but the fact that two coherent states differ by a character on the phase space. Moreover they are unitary equivalent if and only if this character is continuous. In our case, if $\Lambda_1 \Delta \Lambda_2 \in \mathcal{F}_{\Lambda}$ then $X \in \widehat{\mathcal{F}}_{\Lambda} \rightarrow (-1)^{|\Lambda_1 \Delta \Lambda_2 \cap X|}$ is a continuous character.

At this stage it is worthwhile to give other explicit realizations of these representations of $\overline{\Delta(\mathcal{F}_{\Lambda} \times \mathcal{F}_{\Lambda}, \xi^{s})}$ which are in a sense closer to the usual form of the Schrödinger x and p representation. For the sake of symmetry, it is easier to diagonalize the $\pi(\delta^{s}_{X,\phi})$ (resp. $\pi(\delta^{s}_{\phi,X})$) than the $\pi(\delta^{s}_{X,X})$; the results are equivalent since it is nothing than a permutation of $\{\sigma_x, \sigma_y, \sigma_z\}$ the familiar Pauli spin matrices.

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[x] Those states are not the coherent states which are usually considered for spin systems, they are the states with a fixed particle number. But they are much more the analogue of the usual coherent states of the C.C.R.

$\mathfrak{F}_\Lambda$ is a discrete countable abelian group. It is locally compact and the Haar measure on $\mathfrak{F}_\Lambda$ is the sum over the points. We consider $\ell_2(\mathfrak{F}_\Lambda)$ which is the space of functions f from $\mathfrak{F}_\Lambda \longrightarrow \mathbb{C}$ such that

(3.11) $$\| f \|_\ell^2 = \sum_{x \in \mathfrak{F}_\Lambda} | f(x) |^2 < \infty .$$

$\mathcal{P}_\Lambda$ its dual group is abelian and compact ; it is isomorphic to the group of subsets of Λ equipped with the symmetric difference. We denote by dx its normalized Haar measure ; moreover $L_2(\mathcal{P}_\Lambda)$ is the set of functions $f : \mathcal{P}_\Lambda \longrightarrow \mathbb{C}$ such that

(3.12) $$\| f \|_L^2 = \int_{\mathcal{P}_\Lambda} | f(x) |^2 dx < \infty .$$

The duality correspondence is given by

(3.13) $$(X, Y) \in \mathfrak{F}_\Lambda \times \mathcal{P}_\Lambda \longrightarrow (X | Y) = (-1)^{|X \cap Y|} .$$

the Fourier transform exchange ℓ_2 and L_2

(3.14) $$\{ \mathcal{F} f \}(Z) = \sum_{X \in \mathfrak{F}_\Lambda} (-1)^{|X \cap Z|} f(X) , \quad f \in \ell_2 ,$$

$$\{ \mathcal{F} f \}(Z) = \int_{\mathcal{P}_\Lambda} dx \, (-1)^{|X \cap Z|} f(X) , \quad f \in L_2 .$$

Definition (3.15) $\pi_1^\ell, \pi_1^L, \pi_2^\ell$ and π_2^L are the four representations of $\overline{\Delta(\mathfrak{F}_\Lambda \times \mathfrak{F}_\Lambda , \xi^s)}$ defined by :

$$\{ \pi_1^\ell (\delta^s_{X,Y}) f \}(Z) = i^{-|X \cap Y|} (-1)^{|X \cap Z|} f(Z \Delta Y) , \quad f \in \ell_2 .$$

$$\{ \pi_1^L (\delta^s_{X,Y}) f \}(Z) = i^{-|X \cap Y|} (-1)^{|X \cap Z|} f(Z \Delta Y) , \quad f \in L_2 .$$

$$\{\pi_2^{\ell}(\delta^s_{X,Y})f\}(Z) = i^{|X\cap Y|}(-1)^{|Y\cap Z|}f(Z\Delta X),\ f\in \ell_2,$$
$$\{\pi_2^{L}(\delta^s_{X,Y})f\}(Z) = i^{|X\cap Y|}(-1)^{|Y\cap Z|}f(Z\Delta X),\ f\in L_2.$$

The operators we have defined are evidently unitary, hence the only thing we have to prove is that they are indeed representations of $\overline{\Delta(\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda, \xi^s)}$. The proof is purely algebraic. The next proposition gives the relation between these representations.

Proposition (3.16). The representations defined in the previous definition are irreducible.

If $|\Lambda| = \infty$, π_1^{ℓ} and π_2^{ℓ} (resp. π_1^{L} and π_2^{L}) are disjoint.

π_1^{ℓ} and π_2^{L} (resp. π_1^{L} and π_2^{ℓ}) are unitarily equivalent through a Fourier transformation.

Irreducibility is obvious. To prove the disjonction of π_1^{ℓ} and π_2^{ℓ} let us consider in ℓ_2 a function δ_{X_0} whose support is $X_0 \in \mathfrak{P}_\Lambda$:

(3.17) $$\{\pi_1^{\ell}(\delta^s_{X,\phi})\delta_{X_0}\}(Z) = (-1)^{|X\cap X_0|}\delta_{X_0}(Z),\ \forall Z\in \mathfrak{P}_\Lambda,$$

hence it is an eigenvector of the $\pi_1^{\ell}(\delta^s_{X,\phi})$. On the other hand in ℓ_2 an eigenvector of $\pi_2^{\ell}(\delta^s_{X,\phi})$ would satisfy :

(3.18) $$\{\pi_2^{\ell}(\delta^s_{X,\phi})f\}(Z) = f(Z\Delta X) = (-1)^{|X\cap X_0|}f(Z).$$

it would imply that $f(Z) = (-1)^{|Z\cap X_0|}$ but this function is not in ℓ_2, if $|\Lambda| = \infty$.

To prove that π_1 and π_2^{L} are disjoint it is sufficient to prove that π_1^{ℓ} and π_2^{L} (resp. π_1^{L} and π_2^{ℓ}) are unitarily equivalent. The unitary equivalence is given by Fourier transformation. It is interesting to remark that the disjonction of π_1^{ℓ} and π_1^{L} can be seen in the following way : $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$ is a subgroup of $\mathcal{F}_\Lambda \times \mathcal{P}_\Lambda$ and ξ^{s} can be extended by continuity to $\mathcal{F}_\Lambda \times \mathcal{P}_\Lambda$ using the same expression :

$$\overline{\xi}^{s}(X,Y\,;\,X',Y') = i^{-|X\cap Y|-|X'\cap Y'|+|X\Delta X'\cap Y\Delta Y'|+2|X'\cap Y|}$$

for $(X,Y),(X',Y') \in \mathcal{F}_\Lambda \times \mathcal{P}_\Lambda$. Consequently for $|\Lambda| = \infty$

$$\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^{s})} \subsetneq \overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{P}_\Lambda, \overline{\xi}^{s})} \;.$$

π_1^{L} can be extended by continuity to a representation of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{P}_\Lambda, \overline{\xi}^{s})}$, but not π_1^{ℓ} .

In that context, let us remark that $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{P}_\Lambda, \xi^{s})}$ has only one representation up to quasi-equivalence according to the Mackey-Von Neumann uniqueness theorem [6].

For Λ such that $|\Lambda| < \infty$, the distinction between $\mathcal{F}_\Lambda$ and $\mathcal{P}_\Lambda$ has no meaning and all the representations defined in (3.15) coincide. Let $\pi = \pi_1$ and define $|X_0\rangle$, $X_0 \in \mathcal{F}_\Lambda$ as the function δ_{X_0} which is an eigenvector of $\pi\,(\delta^{s}_{X,\emptyset})$. In the same way define $|\hat{X}_0\rangle$ as the function $\hat{\delta}_{X_0}(Z) = 2^{-\frac{|\Lambda|}{2}}\,(-1)^{|X_0\cap Z|}$, the Fourier transform of δ_{X_0} . They both form an orthonormal set in $\ell_2(\mathcal{F}_\Lambda)$, moreover :

$$\langle \hat{X} | Y \rangle = 2^{-\frac{|\Lambda|}{2}}\,(-1)^{|X\cap Y|} \tag{3.19}$$

and we have the following decomposition of the identity :

$$I = 2^{-\frac{|\Lambda|}{2}} \sum_{X,Y\in\mathcal{F}_\Lambda} (-1)^{|X\cap Y|}\; |\hat{X}\rangle\langle Y| \,. \tag{3.20}$$

Up to now we dealt only with the Quantum spin commutation rules. But there is a $\ast$-isomorphism of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^S)}$ onto $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^F)}$ (see e.g. [1] theorem (2.36)) which allows to translate the states of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^S)}$ onto the states of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^F)}$. The $\ast$-automorphism $\hat{\tau}_\theta$ satisfies :

$$\hat{\tau}_\theta(\delta^S_{X,Y}) = \delta^F_{X,Y} = \delta^S_{\tau_\theta(X,Y)} \tag{3.21}$$

Notice that for the sake of simplicity we have incorporated the function F of theorem (2.36) of [1] into the definition of δ^F. This changes ξ^F by a trivial multiplier which still satisfies $\xi^F(X,Y;X,Y) = 1$, $X,Y \in \mathcal{F}_\Lambda$, (see [1] remark before proposition (2.32)). This even does not change the generators since $F(\{i\},\phi) = F(\phi,\{i\}) = 1$, $\forall i$. The simplification is especially clear for the classical states previously defined.

Proposition (3.22) : Let ω^S be a state of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^S)}$ then $\omega^S \circ \hat{\tau}_\theta = \omega^F$ is a state of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda, \xi^F)}$. If moreover $\omega^S \in \mathcal{C}$, then :

$$\omega^S(\delta^S_{X,Y}) = \omega^F(\delta^F_{X,Y}) , \qquad X,Y \in \mathcal{F}_\Lambda .$$

The first assertion is trivial. For the second one it is sufficient to remark that $\omega^S(\delta^S_{X,Y})$ is zero if $X \neq Y$ but on the diagonal

$$\tau_\theta(X,X) = (X,X) , \qquad \forall X \in \mathcal{F}_\Lambda .$$

This proposition also shows that the product states à la Power of the Clifford algebra (see [5]) are associated to the product states on the spin algebra . In the next section we shall come back to this isomorphism and we shall describe it as a transformation of the classical phase space which preserves the Haar measure.

4. QUANTIZATION

The formalism which has been developped in the previous sections is intended to quantify functions on the "phase space" $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$. Indeed let f be a function on $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ with Fourier transform

$$(4.1) \qquad \{\mathcal{F}f\}(X,Y) = \int_{\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda} dX'dY' \, (-1)^{|X\wedge Y'|+|X'\wedge Y|} f(X';Y') \, ,$$

where $dX'dY'$ is the normalized Haar measure on the compact group $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$. One is tempted to define the quantized $Q(f)$ of f according to the Weyl prescription

$$(4.2) \qquad Q^S(f) = \sum_{X,Y \in \mathfrak{P}_\Lambda} \{\mathcal{F}f\}(X,Y) \, \delta^S_{X,Y} \, ,$$

whenever it exists.

There are two main differences between our situation and the usual situation of the C.C.R. over a finite number of degrees of freedom, namely :

i) the multiplier we consider is not a bicharacter.

ii) there is a whole set of non equivalent representations of $\overline{\Delta(\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda, \xi^S)}$, which indicates that $Q^S(f)$ can be defined only within some representations, for some f .

Later on, we shall return to the first point ; but now we shall give at least a set of functions which can be quantized within any representation.

Proposition (4.3) : Let f be a ℓ_1 function on $\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda$,

$$A_f = \sum_{X,Y \in \mathfrak{P}_\Lambda} f(X,Y) \, \delta^S_{X,Y} \, .$$

defines an element of $\overline{\Delta(\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda, \xi^S)}$. Moreover :

$$(A_f)^* = A_{f^*} \quad , \qquad f^*(X,Y) = \overline{f(X,Y)} \ ,$$
$$A_{f_1} A_{f_2} = A_{f_1 \times f_2} \ ,$$

where

(4.4) $$f_1 \times f_2 (X,Y) = \sum_{X',Y' \in \mathcal{F}_\Lambda} \xi^s(X,Y;X',Y') f_1(X',Y') f_2(X\Delta X', Y\Delta Y') .$$

Equipped with this product $\ell_1(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda)$ is a Banach $*$-algebra and

$$\| A_f \| \leq \| f \|_1 \quad .$$

We shall not give the proof since it is obvious. Notice that the set of A_f is nothing but the algebra $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda , \xi^s)}^{\|\cdot\|_1}$, (cf. [2] formula (2.10)). In our case this algebra is important since we have the following :

Theorem 4.5. Let $\mathcal{B}_\Lambda$ be the algebra of functions on $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ which are Fourier transforms of an ℓ_1 function on $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$, then

$$Q^s(f) = \sum_{X,Y \in \mathcal{F}_\Lambda} \{\mathcal{F} f\}(X,Y) \ \delta^s_{X,Y} \ ,$$

where $\mathcal{F} f$ is the Fourier transform of f , defines an element of $\overline{\Delta(\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda , \xi^s)}$. Moreover

$$Q^s(f)^* = Q^s(f^*) \quad , \qquad f^*(X,Y) = \overline{f(X,Y)}$$
$$Q^s(f_1)\, Q^s(f_2) = Q^s(f_1 \circ f_2) \, , \qquad f, f_1, f_2 \in \mathcal{B}_\Lambda \, .$$

where $$f_1 \circ f_2 = \mathcal{F}^{-1}(\mathcal{F} f_1 \times \mathcal{F} f_2) \ .$$

$$Q^s(1) = 1 \quad ,$$

$$Q^s(\chi_{X,Y}) = \delta^s_{X,Y} \quad , \qquad X, Y \in \mathfrak{P}_{f_A}$$

where : $\chi_{X,Y}(X',Y') = (-1)^{|X \cap Y'| + |X' \cap Y|}$

Notice that since $\mathcal{P}_A \times \mathcal{P}_A$ is compact any continuous function on $\mathcal{P}_A \times \mathcal{P}_A$ can be approximated uniformly by elements of $\mathcal{B}_A$. Conversely let A be an element of $\overline{\Delta(\mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A}, \xi^s)}$ and ω^s_c be the central state of $\overline{\Delta(\mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A}, \xi^s)}$ then :

$$X, Y \in \mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A} \longrightarrow \omega^s_c(\delta_{X,Y} A) \; .$$

is a bounded (continuous) function on $\mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A}$. If, moreover, it is a ℓ_1 function then it is the Fourier transform of an element of $\mathcal{B}_A$. Since $\mathcal{P}_A \times \mathcal{P}_A$ is compact we can use the inversion theorem to state the following theorem :

<u>Theorem 4.6.</u> Let $A \in \overline{\Delta(\mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A}, \xi^s)}$ be such that

$$X, Y \in \mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A} \longrightarrow \omega^s_c(\delta^s_{X,Y} A)$$

is an ℓ_1 function then :

$$A = Q^s(f_A)$$

where :

$$f_A(X,Y) = \sum_{X',Y' \in \mathfrak{P}_{f_A}} (-1)^{|X' \cap Y| + |X \cap Y'|} \omega^s_c(\delta^s_{X',Y'} A) \; .$$

We recall that ω^s_c , the central state of $\overline{\Delta(\mathfrak{P}_{f_A} \times \mathfrak{P}_{f_A}, \xi^s)}$ is defined by :

$$\omega^s_c(\delta_{X,Y}) = \delta_{X,\phi}\, \delta_{Y,\phi} \; , \qquad X, Y \in \mathfrak{P}_{f_A} \; ,$$

where $\delta_{X,Y}$ is a Kronecker symbol.

The next property has an obvious analogue for usual C.C.R.

<u>Proposition</u> (4.7) : Let f and g be functions in $\mathcal{B}_\Lambda$ such that

$$f(X,Y) = f(X,Y') \quad , \forall X,Y,Y' \in \mathcal{F}_\Lambda \ .$$

(resp. $$f(X,Y) = f(X',Y) \quad , \forall X,X',Y \in \mathcal{F}_\Lambda),$$

(resp. $$f(X,Y) = h(X \Delta Y) \quad , \forall X,Y \in \mathcal{F}_\Lambda),$$

then : $$f \circ g = f \cdot g \ ,$$

the usual product of functions.

These results can be stated in the Fermi case as follows : According to (3.21)

$$\hat{\tau}_\theta\,(\delta^S_{X,Y}) = \delta^F_{X,Y} = \delta^S_{\tau_\theta(X,Y)} \ .$$

the quantized of a function f in $\mathcal{B}_\Lambda$ assumes the same form

(4.8) $$Q^F(f) = \sum_{X,Y \in \mathcal{F}_\Lambda} \{\mathcal{F}f\}(X,Y)\,\delta^F_{X,Y} = \hat{\tau}_\theta(Q^S(f)).$$

and one has the same formulas

(4.9) $$Q^F(f_1)\,Q^F(f_2) = Q^F(f_1 \circ f_2).$$

τ_θ acts on $\mathcal{F}_\Lambda \times \mathcal{F}_\Lambda$, it is possible to induce transformation of $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ according to the :

Proposition (4.10) : Let τ_θ be the automorphisms of $\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda$ defined by (cf. [1] (2.8))

$$\tau_\theta(X,Y) = (X \Delta \theta(X\Delta Y),\ Y \Delta \theta(X \Delta Y)),\quad X,Y \in \mathfrak{P}_\Lambda ,$$

with $\theta(\{x_i\}) = \{x_j \in \Lambda ;\ j < i\}$ for some arbitrary order on Λ ; then there exists a unique automorphism ${}^t\tau_\theta$ of $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ such that, if $b^s(X,Y;X',Y')$ is the bicharacter associated with ξ^s :

$$(4.11)\qquad b^s((X,Y);\tau_\theta(X',Y')) = b^s({}^t\tau_\theta(X,Y);(X',Y')) ,$$

$\forall X,Y \in \mathcal{P}_\Lambda$, $\forall X',Y' \in \mathfrak{P}_\Lambda$. It is defined by

$$(4.12)\qquad {}^t\tau_\theta(X,Y) = (X_\theta(X,Y),\ Y_\theta(X,Y)),\quad X,Y \in \mathcal{P}_\Lambda .$$

where :

$$(4.13)\qquad X_\theta(X,Y) = \left\{ x_i \in \Lambda ;\ (-1)^{|\theta(\{x_i\}) \cap Y| + |\{x_i\} \Delta \theta(\{x_i\}) \cap X|} = -1 \right\} ,$$

$$Y_\theta(X,Y) = \left\{ x_j \in \Lambda ;\ (-1)^{|\theta(\{x_j\}) \cap X| + |\{x_j\} \Delta \theta(\{x_j\}) \cap Y|} = -1 \right\} .$$

The existence and the uniqueness of ${}^t\tau_\theta$ is ensured by the fact that b^s is a bijection of $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ onto $\widehat{\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda}$ the dual of $\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda$. Moreover

$$X'', Y'' \in \mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda \longrightarrow b^s((X,Y);\tau_\theta(X'',Y''))$$

is a character of $\mathfrak{P}_\Lambda \times \mathfrak{P}_\Lambda$, hence it is of the form

$$b^s((X_\theta,Y_\theta);(X'',Y'')) = b^s({}^t\tau_\theta(X,Y);(X'',Y'')) .$$

It is easy to see that X_θ, Y_θ are defined by :

$$X_\theta = \{ x_i \in \Lambda \; ; \; b^S((X,Y) \, ; \, \tau_\theta(\phi, \{x_i\})) = -1 \} \; ,$$

and similarly for Y_θ .

Since τ_θ is a bijective automorphism and b^S a non degenerate bicharacter ${}^t\tau_\theta$ is an injective automorphism ; moreover it retains some properties of τ_θ as exemplified by the following proposition.

<u>Proposition</u> (4.14) : ${}^t\tau_\theta$ is a bijective automorphism of $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ which satisfies :

i) ${}^t\tau_\theta^{\,2} = i$ (the identity automorphism) .

ii) ${}^t\tau_\theta\,(X,X) = (X,X)$, $\forall X \in \mathcal{P}_\Lambda$.

iii) ${}^t\tau_\theta \cdot \tilde{\tau}_{\theta_0} = \tilde{\tau}_{\theta_0} \cdot {}^t\tau_\theta$,

where $\tilde{\tau}_{\theta_0}$ denotes the extension of τ_{θ_0} to $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ (see [1] 2.13).

iv) ${}^t\tau_\theta\,(\{x_i\}, \phi) = (\{x_i\} \Delta \Theta_i \, , \, \Theta_i)$,

where $\Theta_i = \{ x_j \in \Lambda \; ; \; j > i \}$.

All these properties are easy to prove.

Using this automorphism, we can give another expression for $Q^F(f)$ where $f \in \mathcal{B}_\Lambda$. Indeed

$$\begin{aligned} Q^F(f) &= \sum_{X,Y \in \mathcal{P}_\Lambda} \{\mathcal{F}f\}(X,Y)\, \delta^F_{X,Y} \\ &= \sum_{X,Y \in \mathcal{P}_\Lambda} \{\mathcal{F}f\}(X,Y)\, \delta^S_{\tau_\theta(X,Y)} \\ &= \sum_{X,Y \in \mathcal{P}_\Lambda} \{\mathcal{F}f\}(\tau_\theta(X,Y))\, \delta^S_{X,Y} \\ &= \sum_{X,Y \in \mathcal{P}_\Lambda} \{\mathcal{F}(f \cdot {}^t\tau_\theta)\}(X,Y)\, \delta^S_{X,Y} \; . \end{aligned}$$

Consequently the correspondence

$$(4.15) \qquad f \longrightarrow f \circ {}^t\tau_\theta = f^F ,$$

satisfies

$$(4.16) \qquad Q^F(f) = Q^S(f^F) .$$

Hence we can state the following :

Proposition (4.17) : There exists an automorphism ${}^t\tau_\theta$ of $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$, which leaves invariant the Haar measure and such that :

$$Q^S(f) = Q^F(f \circ {}^t\tau_\theta) ,$$

$\forall f \in \mathcal{B}_\Lambda$.

Indeed the group $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ is compact, hence any continuous function on $\mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ can be uniformly approximated by finite linear combinations of characters.

7

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