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AND EXPERIMENTAL PHYSICS

D.R. Lebedev

BENNEY'S LONG WAVES EQUATIONS. HAMILTONIAN FORMALISM

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The connection between the Lie algebra of Hamilton plane vector fields and the Benney's momentum equations is shown.

I. Introduction

Benney suggested in[1] the following equations of motion of incompressible nonviscous fluid with free surface in the approximation of long waves:

$$\begin{cases} u_t + u u_x - u_y \int_0^y u_x\big|_{y=\eta}\, d\eta + h_x \\ h_t + \left(\int_0^h u\, dy \right)_x = 0 \end{cases} \tag{1}$$

Here $h = h(x,t)$ denotes the height of free surface over the bottom $y = 0$, $u = u(x,y,t)$ is the horizontal component of velocity, subscript indices denote partial derivatives. We introduce the moment functions $A_n(x,t) = \int_0^h u(x,y,t)^n\, dy,\ n \geq 0$. The moments satisfy the following system of evolution equations (cf.[1])

$$A_{n,t} + A_{n+1,x} + n A_{n-1} A_{0,x} = 0, \quad n \geq 0 \tag{2}$$

Kupershmidt and Manin [2]-[3] have shown that (2) can be rewritten in the form $\vec{A}_t = B\,\delta H/\delta \vec{A}$, where $\vec{A} = (A_0, A_1, \ldots)^t$,

$H = -\frac{1}{2}(A_2 + A_0^2)$, $\delta H/\delta \vec{A} = \left(\delta H/\delta A_0, \ldots \right)^t$ and finally B is a matrix differential operator

$$B = B_1 \partial + \partial \circ B_1^t, \quad (B_1)_{ij} = i A_{i+j-1}, \quad \partial = \partial/\partial x \tag{3}$$

This operator B is Hamiltonian, i.e. the expression $\{\tilde{P}, \tilde{Q}\} = \frac{\delta P}{\delta \vec{A}}^t B \frac{\delta Q}{\delta \vec{A}}$ defines Poisson brackets on the space of polynomials in $A_i^{(j)}$ modulo exact ∂-

derivatives ($\tilde{\rho} = \int \rho\, dx$). The fact that B is Hamiltonian was established in[3] by means of the direct check of the Jacobi identity. Due to the fact that coefficients of B explicitly depend on the unknown functions this check was fairly complicated and did not explain the actual reason.

The aim of this paper is to solve the problem of invariant description of this operator B pointed out in[4].

It turns out that Kupershmidt-Manin Hamiltonian structure is the formal analogue of the Kirillov's structures on the orbits of the coadjoint representation of Lie groups. The role of the Lie algebra in question here plays the differential Lie algebra[5] which is closely connected with the infinite Lie algebra of formal Hamiltonian vector fields on the plane $H_2(\mathbb{R}^2)$.

Lately there have appeared some works showing the connection of the infinite Lie algebras with some "completely integgrable" nonlinear equations (cf. [4], [6]-[8]). This work furnishes one more example of this connection.

2. Lie algebra of Hamiltonian vector fields on the plane.

Let $H_2(\mathbb{R}^2) = \{ D_X \mid D_X = X_\xi(x,\xi)\frac{\partial}{\partial x} - X_x(x,\xi)\frac{\partial}{\partial \xi},$ $X(x,\xi) \in C^\infty(\mathbb{R}^2) \}$. $H_2(\mathbb{R}^2)$ is the Lie algebra with the commutator rule

$$[D_X, D_Y] = D_{\{X,Y\}}, \quad \{X,Y\} = X_\xi Y_x - Y_\xi X_x .$$

The structure of the Lie algebra in $C^\infty(\mathbb{R}^2)$ that is Poisson brackets, is induced by the Lie structure on the $H_2(\mathbb{R}^2)$.

3. Benney's differential algebra $\mathfrak{g}(\mathcal{B})$.

Let $\mathcal{B}$ is a differential ring. Such a ring is an algebra over a field k of characteristic zero endowed with a derivation $\partial : \mathcal{B} \to \mathcal{B}$ which is trivial on k. If $X \in \mathcal{B}$ we write $X^{(j)}$ instead of $\partial^j X$. $\mathcal{B}^\infty$ means the linear space of columns $\vec{X} = (X_0, X_1 \dots)^t$ with $X_i \in \mathcal{B}$.

Let $X, Y \in \mathcal{B}[[\xi]] = \{ X \mid X = \sum_{i \geq 0} X_i \xi^i , X_i \in \mathcal{B} \}$

be a ring of formal power series. There are two derivations on $\mathcal{B}[[\xi]]$ $\partial_\xi : \sum_{i=0}^{\infty} X_i \xi^i \mapsto \sum_{i=1}^{\infty} i X_i \xi^{i-1}$ and $\partial \cdot \sum_{i=0}^{\infty} X_i \xi^i \mapsto \sum_{i=0}^{\infty} X_i^{(1)} \xi^i$

By analogy with n° 2 there is a structure of Lie algebra on $\mathcal{B}[[\xi^{-1}]]$

$$\{X, Y\} = X_\xi Y^{(1)} - Y_\xi X^{(1)}$$

or

$$\{X, Y\} = \sum_{i,j} [(i+1) X_{i+1} Y_i^{(1)} - (i+1) Y_{i+1} X_j^{(1)}] \xi^{i+j} =$$

$$= \sum_{l \geq 0} [\sum_{j=0}^{l} (l-j+1) X_{l-j+1} Y_j^{(1)} - (l-j+1) Y_{l-j+1}] \xi^l = \sum_{l \geq 0} \{X, Y\}_l \xi^l$$

There is an isomorphism

$$\mathcal{B}^\infty \to \mathcal{B}[[\xi]] : \vec{X} = (X_0, X_1, \dots)^t \leftrightarrow X = \sum_{i=0}^{\infty} X_i \xi^i$$

We define the Lie algebra structure on $\mathcal{B}^\infty$

$$[\vec{X} * \vec{Y}] = \vec{Z} \qquad \text{, where} \quad \vec{Z} \leftrightarrow Z = \sum_{\ell \geq 0} \{X, Y\}_\ell \, \xi^\ell$$

and denote this Lie algebra by $\mathfrak{g}(\mathcal{B})$. . This is a differential algebra of infinite rank in the sense of Ritt [5]. We shall denote by the $\mathfrak{g}^{fin}(\mathcal{B})$ subalgebra of $\mathfrak{g}(\mathcal{B})$ consisting of finite columns (that is almost all of X_j's vanish).

4. Linear functions on $\mathfrak{g}(\mathcal{B})$.

If $\vec{X}, \vec{Y} \in \mathcal{B}^\infty$ are two columns we put $\vec{X}^t \vec{Y} = \sum_{i=0}^{\infty} X_i Y_i \in \mathcal{B}$ (the sum is defined if $\mathcal{B}$ is endowed with a topology in which the series converges). We denote the map $\mathcal{B} \to \mathcal{B}/\partial\mathcal{B}$ by $\int$. In these notations a convenient substitute of a linear functional on a Lie algebra in finite dimensional theory is the map

$$\mathfrak{g}(\mathcal{B}) \to \mathcal{B}/\partial\mathcal{B} : \vec{X} = (X_i) \mapsto \int \vec{A}^t \vec{X} , \quad \text{where} \quad \vec{A} = (A_0, \dots)^t \in \mathcal{B}^\infty$$

is a column

<u>Theorem</u>. For all $\vec{X}, \vec{Y} \in \mathfrak{g}(\mathcal{B})$ we have

$$\int \vec{A}^t [\vec{X} * \vec{Y}] = \int \vec{X}^t B \vec{Y},$$

where B is the operator (3).

<u>Proof</u>. L.h.s. of (5) is

$$\int \vec{A}^t [\vec{X} * \vec{Y}] = \int \sum_{\ell \geq 0} A_\ell [\vec{X} * \vec{Y}]_\ell =$$

$$= \int \sum_{\ell \geq 0} \sum_{j=0}^{\ell} [X_{\ell+1-j}(\ell+1-j) A_\ell Y_j^{(1)} - Y_{\ell+1-j}(\ell+1-j) A_\ell X_j^{(1)}].$$

Integrating the sum of second terms by parts and making the substitution $i = \ell + 1 - j$ one finds

$$\int \sum_{i,j} X_i \, i \, A_{i+j-1} Y_j^{(1)} + X_i \partial(j A_{i+j-1} Y_j) =$$

$$= \int \vec{X}^t B_1 \vec{Y}^{(1)} + \vec{X}^t \partial(B_1^t \vec{Y}) = \int \vec{X}^t B \vec{Y}.$$

This completes the proof.

Corollary. The commutator $[\vec{X} * \vec{Y}]$ can be rewritten in the form

$$[\vec{X} * \vec{Y}] = \vec{X}^t \frac{\partial B_1}{\partial \vec{A}} \vec{Y}^{(1)} - \vec{X}^{t(1)} \frac{\partial B_1^t}{\partial \vec{A}} \vec{Y}.$$

Here for instance $\vec{X}^t \frac{\partial B_1}{\partial \vec{A}} \vec{Y}$ is the column in $\mathcal{B}^\infty$ with the element $\vec{X}^t \frac{\partial B_1}{\partial A_\ell} \vec{Y}^{(1)}$ in the ℓ -th place.

5. Kirillov construction

Here we'll briefly review Kirillov construction; cf.[10] for further details.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^*$ the duel space of $\mathfrak{g}$. Take a point $u \in \mathfrak{g}^*$ and denote Ω its orbit reletive to the coadjoint representation. As G is transitive on Ω each element $X \in \mathfrak{g}$ defines a vector field ∂_X on Ω. The map $X \mapsto \partial_X$ is the morphism of Lie algebras. Its image generates the whole Lie algebra of vector fields. The Kirillov symplectic form ω on Ω is defined by the formula

$$\omega(\xi_X, \xi_Y)(u) = \langle u, [X, Y] \rangle,$$

for all $X, Y \in \mathfrak{g}$, $u \in \Omega$.

Let us consider the ring of C^∞ functions on Ω. For any $f, h \in C^\infty(\Omega)$, $u \in \Omega$, $df(u)$ and $dh(u)$ can be considered as elements of $\mathfrak{g}$. Then the Poisson bracket on $C^\infty(\Omega)$ is defined by the formula

$$\{f, h\}(u) = \langle u, [df(u), dh(u)] \rangle.$$

The Hamilton vector field with the Hamiltonian h on Ω is given by $f_t = \{f, h\}$ for any $f \in C^\infty(\Omega)$.

6. Formal analog of Kirillov construction for $\mathfrak{g}(\mathcal{B})$.

Here we take $A_0, A_1, \ldots$ differentially independent over $\mathcal{B}$ (this means that the family $\{A_i^{(j)} \mid i \in I, j \geq 0\}$ is algebraically independent over $\mathcal{B}$). Let $\mathcal{A} = \mathcal{B}[A_i^{(j)}]$, $\tilde{\mathcal{A}} = \mathcal{A}/\partial\mathcal{A}$. The linear space $\tilde{\mathcal{A}}$ is a substitute of the ring of

. C^∞ - functions on the orbit Ω of the point u in the finite dimensional case.

We will consider the Lie algebra $\mathcal{D}ev =$ $= \{ \partial_{\vec{X}} \mid \partial_{\vec{X}} = \sum_{i,j} X_i^{(j)} \frac{\partial}{\partial A_i^{(j)}}, \; X_i \in \mathcal{B}[A_i^{(j)}] \}$ of continuous derivations of $\mathcal{A}$, commuting with ∂ and trivial on $\mathcal{B}$. The subalgebra $\mathfrak{g}_A = \{ \partial_{BX} \mid X \in \mathfrak{g}(\mathcal{B}) \}$ is the formal analog of the vector fields on the orbit Ω. Let us define the 2-form on $\mathfrak{g}_A$

$$\omega(\partial_{B\vec{X}}, \partial_{B\vec{Y}})(\vec{A}) = \int \vec{A}^t [\vec{X} * \vec{Y}] = \int \vec{X}^t B \vec{Y}$$

and construct the Poisson bracket on $\tilde{\mathcal{A}}$:

$$\{ \tilde{P}, \tilde{Q} \}(\vec{A}) = \omega\left(\partial_B \frac{\delta P}{\delta \vec{A}}, \partial_B \frac{\delta Q}{\delta \vec{A}}\right),$$

where $\tilde{P} = \int P$, $\tilde{Q} = \int Q$, $P, Q \in \mathcal{A}$.

__Theorem__. For all $\tilde{P}, \tilde{Q}, \tilde{H} \in \tilde{\mathcal{A}}$ we have

a) $\{ \tilde{P}, \tilde{Q} \} = - \{ \tilde{Q}, \tilde{P} \}$

b) $\{ \tilde{P}, \{ \tilde{Q}, \tilde{H} \} \} + \{ \tilde{H}, \{ \tilde{P}, \tilde{Q} \} \} + \{ \tilde{Q}, \{ \tilde{H}, \tilde{P} \} \} = 0.$

__Proof__. The property a) is evident. b) Let $\vec{X} = \delta P / \delta \vec{A}$, $\vec{Y} = \delta Q / \delta \vec{A}$, $\vec{Z} = \delta H / \delta \vec{A}$. In [3] the following identity was proved:

$$\frac{\delta}{\delta \vec{A}} (\vec{X}^t B \vec{Y}) = \mathcal{D}(\vec{X}) B \vec{Y} - \mathcal{D}(\vec{Y}) B \vec{X} + [\vec{X} * \vec{Y}].$$

Here $\mathcal{D}(\vec{X})$ is an infinite matrix, with the differential operator $\mathcal{D}_{A_j}(X_i) = \sum_{s=0}^{\infty} \frac{\partial X_i}{\partial A_j^{(s)}} \partial^s$ in the (i,j)-th place. Hence

$$\{\tilde{P}, \{\tilde{Q}, \tilde{H}\}\} = \int \vec{X}^t B \mathcal{D}(\vec{Y}) B \vec{Z} - \vec{X}^t B \mathcal{D}(\vec{Z}) B \vec{Y} + \vec{X}^t B [\vec{Y} * \vec{Z}] \quad (6)$$

$$\{\tilde{H}, \{\tilde{P}, \tilde{Q}\}\} = \int \vec{Z}^t B \mathcal{D}(\vec{X}) B \vec{Y} - \vec{Z}^t B \mathcal{D}(\vec{Y}) B \vec{X} + \vec{Z}^t B [\vec{X} * \vec{Y}] \quad (7)$$

$$\{\tilde{Q}, \{\tilde{H}, \tilde{P}\}\} = \int \vec{Y}^t B \mathcal{D}(\vec{Z}) B \vec{X} - \vec{Y}^t B \mathcal{D}(\vec{X}) B \vec{Z} + \vec{Y}^t B [\vec{Z} * \vec{X}] \quad (8)$$

Adding (6), (7), (8) and using formal selfadjointness of B and antiselfadjointness of $\mathcal{D}(\vec{X}), \mathcal{D}(\vec{Y}), \mathcal{D}(\vec{Z})$ (cf.[3]), we shall rewrite l.h.s. of the identity (6) :

$$\int \vec{X}^t B [\vec{Y} * \vec{Z}] + \int \vec{Z}^t B [\vec{X} * \vec{Y}] + \int \vec{Y}^t B [\vec{Z} * \vec{X}] =$$

$$= \int \vec{A}^t \left([\vec{X} * [\vec{Y} * \vec{Z}]] + [\vec{Z} * [\vec{X} * \vec{Y}]] + [\vec{Y} * [\vec{Z} * \vec{X}]] \right) = 0$$

This finishes the proof.

Remark. The statement of the theorem is equivalent to the "Hamiltonian's condition" for the operator B [3] - [4]. The equations $\tilde{F}_t = \{\tilde{F}, \tilde{H}\}$ or $\vec{A}_t = B\, \delta H / \delta \vec{A}$ are the analogs of the Hamiltonian on the finite dimensional orbit Ω.

7. $\pm$ extension of $\mathfrak{g}^{fin}(\mathcal{B})$ and realization the linear functionals on it ($\pm$ structure).

We shall show in this n° that the algebra $\mathfrak{g}^{fin}(\mathcal{B})$ admits an extension of the type which became traditional after the works[6], [7], [9] and deserves a special name.

Let $\mathcal{B}((\xi^{-1})) = \{ X \mid X = \sum_{i=0}^{N} X_i \xi^i + \sum A_j \xi^{-(j+1)}, X_i, A_i \in \mathcal{B} \}$.
There is a Lie structure on $\mathcal{B}((\xi^{-1}))$: $[X, Y] = X_\xi Y^{(1)} - Y_\xi X^{(1)} = \{X, Y\}$. We denote this algebra by $\mathfrak{g}_1$.
Let $\mathfrak{g}_+ = \{ X \in \mathfrak{g}_1 \mid X = \sum_{i=0}^{N} X_i \xi^i \}$, $\mathfrak{g}_- = \{ A \in \mathfrak{g}_1 \mid A = \sum A_j \xi^{-(j+1)} \}$. $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are the Lie subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_1 = \mathfrak{g}_+ \dotplus \mathfrak{g}_-$ as vector space. Let us note that $\mathfrak{g}_+ \cong \mathfrak{g}^{fin}$. For all $X, Y \in \mathcal{B}((\xi^{-1}))$ we define scalar product by $(X, Y) = \int \operatorname{res} XY$. Here res(·) is the coefficient of ξ^{-1}. For $A \in \mathfrak{g}_-$, $X \in \mathfrak{g}_+$ we have $(A, X) = \int \sum_{\ell \geq 0} A_\ell X_\ell$, $(\mathfrak{g}_+, \mathfrak{g}_+) = 0$ and $(\mathfrak{g}_-, \mathfrak{g}_-) = 0$. We shall identify the linear functionals on $\mathfrak{g}^{fin}(\mathcal{B})$ with $\mathfrak{g}_-$ by means of this scalar product (,).

Lemma (on the invariance of the scalar product). Let $\Phi, X, Y \in \mathcal{B}((\xi^{-1}))$, then

$$(\Phi, [X, Y]) = ([\Phi, X], Y).$$

<u>Proof.</u> Since $0 = \operatorname{res} (\Phi X^{(1)} Y)_\xi = \operatorname{res} (\Phi_\xi X^{(1)} Y + \Phi X_\xi^{(1)} Y + \Phi X^{(1)} Y_\xi)$, we have

$$(\Phi, [X, Y]) = \int \operatorname{res} (\Phi X_\xi Y^{(1)} - \Phi Y_\xi X^{(1)}) =$$

$$= \int \operatorname{res} (-\Phi^{(1)} X_\xi Y - \Phi X_\xi^{(1)} Y - \Phi Y_\xi X^{(1)}) =$$

$$= \int \operatorname{res} (-\Phi^{(1)} X_\xi Y - \Phi_\xi X^{(1)} Y) = ([\Phi, X], Y).$$

Let $\Phi = \sum_{i \geq 0} A_i \xi^{-(i+1)} \in \mathfrak{g}_-$, $F_A = \sum_{j \geq 0} \frac{\delta F}{\delta A_j} \xi^j$ for any $F \in \mathfrak{R}$. It is easy to check that main formulae of n°5 can be rewritten in the following way.

Poisson bracket:

$$\{\tilde{P}, \tilde{Q}\}(\Phi) = (\Phi, [P_A, Q_A]) = ([\Phi, P_A]_-, Q_A).$$

The Hamiltonian flow with Hamiltonian $\tilde{H} = \int H,\ H \in \tilde{A}$:

$$\Phi_t = [\Phi, H_A]_- . \tag{9}$$

In particular the equations (2) can be rewritten in the form (9) with the Hamiltonian $H = \frac{1}{2}(A_2 + A_0^2)$.

The question of the choice of Hamiltonians $\tilde{H}$ for which the equation (9) has an infinite series of conservation laws being in involution relative to the Poisson bracket demands special treating which will be attended to in the next work.

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