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HEATING TOKAMAKS BY PARAMETRIC DECAY OF INTENSE EXTRAORDINARY MODE RADIATION

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HEATING TOKAMAKS BY PARAMETRIC DECAY OF INTENSE EXTRAORDINARY MODE RADIATION

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Abstract. Intense electron beam technology has developed coherent, very high power (350 megawatts) microwave sources at frequencies which are a modest fraction of the electron cyclotron frequency in tokamaks. Propagation into a plasma occurs via the extraordinary mode which is subject to parametric decay instabilities in the density range $\omega_o^2 < \omega_{pe}^2 < \omega_o(\omega_o + \Omega_e)$. For an incident wave focused onto a hot spot by a dish antenna of radius ρ, the effective threshold power P_o required to induce effective parametric heating is

$$P_o \simeq 10\ \mathrm{MW}\ \frac{x}{\rho}\ \frac{\Omega_e}{\omega_o} \left(\frac{T_e}{1\ \mathrm{keV}}\right)^{3/2}$$

where x denotes the distance to the hot spot.

1. INTRODUCTION

Most plasma heating methods in the electron cyclotron range of frequencies rely on resonant absorption at either the fundamental or the second harmonic electron cyclotron frequency to bring about plasma heating. But high power tubes become increasingly difficult to build as the frequency increases into the range of the fundamental electron cyclotron frequency (84 - 168 GHz) characteristic of high field tokamaks (30 - 60 kG). On the other hand, tubes capable of high peak power at lower frequencies have already been developed. For example, Hirschfield and Granatstein [1] have reported a peak power of 350 MW at 15 GHz. In this work we ask how can such a fixed-frequency ω_o, high-power tube be used to heat a high-density, high-field tokamak.

Clearly our interest will be in situations when ω_o is less than the electron cyclotron frequency Ω_e throughout the plasma and propagation into the plasma can occur via the extraordinary mode. But the extraordinary mode does not lead to a plasma resonance and, in the absence of collisions, the cold plasma dispersion relation for a normally incident extraordinary mode shows it will be perfectly reflected when

$$\omega_{pe}^2 = \omega_o(\omega_o + \Omega_e) \quad , \qquad (1)$$

where ω_{pe} denotes the electron plasma frequency. As a result, the extraordinary mode has not been considered attractive for heating tokamaks at these frequencies. However, the use of a high power source permits consideration of nonlinear heating processes. It is the object of this paper to calculate when nonlinear heating via the parametric decay of the extraordinary mode radiation into Langmuir and ion-acoustic waves occurs.

Several general remarks can be made directly. The frequency of the ion-acoustic wave is much less than ω_o. Thus, the frequency matching condition for parametric decay [2] becomes

$$\omega_o = \omega_I + \omega_L \simeq \omega_L \quad , \tag{2}$$

where the subscripts I and L denote the ion-acoustic wave and the Langmuir wave respectively. The formula for ω_L is

$$\omega_L^2 = \frac{1}{2}\{\Omega_e^2 + \omega_{pe}^2 - [(\Omega_e^2 - \omega_{pe}^2)^2 + 4\Omega_e^2\omega_{pe}^2 \sin^2\theta]^{1/2}\} \quad , \tag{3}$$

where θ is the angle between the wave vector $\vec{K}_L$ of the Langmuir wave and the magnetic field of the tokamak. Since the value of $\sin^2\theta$ cannot be less than zero, Eqs. (3) and (1) imply that parametric decay instabilities can occur only in the density range

$$\omega_o^2 < \omega_{pe}^2 < \omega_o(\omega_o + \Omega_e) \ . \tag{4}$$

This region of allowed parametric instability is shown on the familiar CMA diagram in Fig. 1. Inequalities (4) demonstrate that the parametric instabilities can occur only in the interior of the plasma; propagation through the low density peripheral plasma will not be subject to parametric decay. Figure 2 recasts the information of Fig. 1 into practical units which show the regions in density and magnetic field for which parametric decay instabilities are allowed for a sequence of pump frequencies which are of practical interest.

We consider extraordinary mode radiation that is introduced through a wave guide or a dish antenna with a phase variation across the aperture appropriate to focus the radiation onto an area in the plasma interior where Inequalities (4) are satisfied (Fig. 3). As we will show, it is necessary for the electric field associated with the extraordinary wave to exceed a threshold value for parametric decay to take place. Therefore, the area in which instabilities occur is limited to where E_o^2 exceeds the threshold (the hot spot).

In a thermonuclear plasma, the velocity distribution is Maxwellian and T_i is of the same order as T_e. Consequently, the ion-acoustic response does not propagate due to heavy Landau damping. We envision a situation in which the Langmuir wave will propagate across the hot spot, locally exciting an ion-acoustic response. As the Langmuir wave moves through the region, its growth can be described by the convective amplitude amplification

$$E_L = E_{oL}\exp(A) \equiv E_{oL}\exp[\int \mathrm{Im}(\vec{K}_L)\,d\ell] \quad . \tag{5}$$

A is a measure of the energy transferred from the extraordinary wave to the plasma, for the growth of the Langmuir wave is driven by the extraordinary wave energy (αE_o^2) while it is in the hot spot and after it has passed out of the interaction region it will be electron Landau damped, thereby transferring energy from the pump to the electrons. A's of six lead to over a 10^5 amplification of the Langmuir wave energy and therefore represent a good coupling of the pump energy to the plasma. We will refer to the "effective threshold" as the extraordinary mode power required to have A's of six or more, while the threshold power is simply that power required to make A positive.

Section 2 derives a wave propagation equation governing the propagation of extraordinary mode radiation in a nonuniform plasma, and solves the equation to provide a quantitative description of its intensity and spatial variation. In Section 3 we use the standard parametric decay dispersion relation to evaluate $\mathrm{Im}(\vec{K}_L)$, and then estimate A. The frequencies and wave vectors that will maximize A are then found, and these results are used to predict the necessary power for A's of six. Section 4 applies our results to a couple of test cases, while Section 5 discusses the results.

2. PROPAGATION AND FOCUSING OF EXTRAORDINARY MODE RADIATION

We will consider the situation portrayed in Fig. 3. A slab geometry will be used with the plasma density increasing from zero (at $x > 0$) in the positive $\hat{x}$ direction. The magnetic field is a constant and is in the positive $\hat{z}$ direction. The source of the extraordinary mode radiation will be an aperture at $x = 0$. It will be assumed that the phase distribution across the aperture is such to focus the radiation onto a hot spot in the region of allowed parametric instabilities. Our calculation of the wave propagation will be based on the approximation that the spatial variations in the $\hat{y}$ and $\hat{z}$ directions will be on scales large compared to the wavelength in the $\hat{x}$ direction.

The propagation of the extraordinary mode is described by the cold plasma conductivity tensor $\overline{\overline{\sigma}}$ used in connection with the wave propagation equation

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \frac{\omega_o^2}{c^2} \vec{E} + \frac{4\pi\omega_o i}{c^2} \overline{\overline{\sigma}} \cdot \vec{E} = 0 \qquad (6)$$

where

$$\frac{4\pi\omega_o i}{c^2} \overline{\overline{\sigma}} = \begin{bmatrix} \dfrac{\omega_{pe}^2 \omega_o^2}{c^2(\Omega_e^2 - \omega_o^2)} & \dfrac{-i\omega_{pe}^2 \omega_o \Omega_e}{c^2(\Omega_e^2 - \omega_o^2)} & 0 \\ \dfrac{i\omega_{pe}^2 \omega_o \Omega_e}{c^2(\Omega_e^2 - \omega_o^2)} & \dfrac{\omega_{pe}^2 \omega_o^2}{c^2(\Omega_e^2 - \omega_e^2)} & 0 \\ 0 & 0 & -\dfrac{\omega_{pe}^2}{c^2} \end{bmatrix} \qquad (7)$$

and an $\exp(-i\omega_o t)$ time dependence has been assumed.

It is the goal of this section to find a wave propagation equation in an approximation where

$$\frac{\omega_o}{c} >> \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \qquad E_x \sim E_y >> E_z \tag{8}$$

and solve it to find the intensity of the electric field of the wave in the hot spot. For the first step $\partial/\partial y$, ∂/∂_z, and E_z will be considered ignorable. Then the $\hat{x}$ component of Eq. (6) becomes

$$E_x = \frac{i\Omega_e\omega_{pe}^2}{\omega_o(\Omega_e^2 + \omega_{pe}^2 - \omega_o^2)} E_y . \tag{9}$$

Using this in the $\hat{y}$ component of Eq. (6), one directly obtains the wave propagation equation appropriate to a normally incident, plane-polarized extraordinary wave,

$$\frac{\partial^2}{\partial x^2} E_y = -\frac{\omega_o^2\Omega_e^2 - (\omega_{pe}^2 - \omega_o^2)^2}{c^2(\Omega_e^2 + \omega_{pe}^2 - \omega_o^2)} E_y \equiv -k_{ox}^2 E_y . \tag{10}$$

The behavior of the expression for k_{ox}^2 can be made clearer by writing ω_{pe} in terms of the cutoff density. Defining

$$\eta = \frac{\omega_{pe}^2}{\omega_o(\omega_o + \Omega_e)} , \tag{11}$$

and using it in Eq. (10) yields

$$k_{ox}^2 = \frac{\omega_o^2}{c^2}(1-\eta)\frac{\Omega_e(1+\eta) - \omega_o(1-\eta)}{\Omega_e - \omega_o(1-\eta)} , \tag{12}$$

which serves as a useful definition for k_{ox}^2. As is expected, $k_{ox}^2 = \omega_o^2/c^2$ in the limit of $\eta \to 0$ (i.e., outside the plasma), *while* k_{ox}^2 *goes to zero as the cutoff density is approached* $(\eta \to 1)$.

The next approximation is to retain both y and z derivatives as well as the parallel electric field E_z. Because our slab model varies only in the $\hat{x}$ direction, we can introduce a Fourier transform in the $\hat{y}$ and $\hat{z}$ directions with the corresponding wave numbers being considered small,

$$E(\vec{x}) = \int E(\vec{x},\vec{k}) \exp[i(k_y y + k_z z) - i\omega_o t] \frac{dk_y dk_z}{2\pi} , \tag{13}$$

$$k_y \sim k_z << \frac{\omega_o}{c} . \tag{14}$$

Therefore to second order in the small quantities, the $\hat{x}$ and $\hat{y}$ components of Eq. (6) become

$$[-k_y^2 - k_z^2 + \frac{\omega_o^2}{c^2}(1 + \frac{\omega_{pe}^2}{\Omega_e^2 - \omega_o^2})]E_x = i\left(k_y \frac{\partial}{\partial x} + \frac{\omega_{pe}^2 \omega_o \Omega_e}{c^2(\Omega_e^2 - \omega_o^2)}\right)E_y + ik_z \frac{\partial}{\partial x} E_z , \tag{15}$$

$$[-k_z^2 + \frac{\partial^2}{\partial x^2} + \frac{\omega_o^2}{c^2}(1 + \frac{\omega_{pe}^2}{\Omega_e^2 - \omega_o^2})]E_y = i\left(k_y \frac{\partial}{\partial x} - \frac{\omega_{pe}^2 \omega_o \Omega_e}{c^2(\Omega_e^2 - \omega_o^2)}\right)E_x , \tag{16}$$

respectively. Since E_z only enters Eqs. (15 - 16) through the combination $k_z E_z$, it is only necessary to carry the z component of Eq. (6) to first order yielding

$$-ik_z \frac{\partial}{\partial x} E_x + \left(\frac{\partial^2}{\partial x^2} + \frac{\omega_o^2 - \omega_{pe}^2}{c^2}\right) E_z = 0 \ . \tag{17}$$

Let us first consider Eq. (17) and use it to solve for E_z. In order to do this it is necessary to estimate the order of any corrections to the zeroth order wave propagation equation [Eq. (10)]. If Eq. (15) is used to eliminate E_x in Eq. (16), it is apparent that

$$\left(\frac{\partial^2}{\partial x^2} + k_{ox}^2\right) E_y = O\left(\frac{k_y}{\Omega_e^2} \frac{\partial \omega_{pe}^2}{\partial x}, k_y^2, k_z E_z\right) \ . \tag{18}$$

Next we will assume that the variations in the plasma density occur on scales L long enough that

$$k_y L \gtrsim \frac{\omega_{pe}^2}{\Omega_e^2} \ , \tag{19}$$

so that all the corrections on the right hand side of Eq. (18) are at least second order in the small quantities.

Now since E_z need only be to first order, $\partial^2/\partial x^2$ can be replaced by $-k_{ox}^2$. Combining Eq. (17) with the polarization of the extraordinary mode [Eq. (9)] and the definition of k_{ox}^2 yields

$$E_z = \frac{c^2}{\omega_o \Omega_e}\left(\frac{\Omega_e^2 + \omega_{pe}^2 - \omega_o^2}{\omega_{pe}^2}\right) k_z \frac{\partial}{\partial x}\left(\frac{\omega_{pe}^2}{\Omega_e^2 + \omega_{pe}^2 - \omega_o^2} E_y\right) \ . \tag{20}$$

In most large tokamaks the scale length L will be much greater than $1/k_{ox}$ except at points very close to the cutoff. If the distance from the cutoff r satisfies the inequality

$$r >> \frac{c^2}{2L\omega_o^2} , \tag{21}$$

then $k_{ox}L >> 1$, as can be shown by expanding k_{ox}^2 in a Taylor series near the cutoff and comparing it to L^2. In this case the derivative in Eq. (20) can be moved through the plasma frequencies to give

$$E_z = k_z \frac{c^2}{\omega_o \Omega_e} \frac{\partial}{\partial x} E_y \tag{22}$$

without introducing any error at this order.

The expression for E_z can be used in Eq. (15) to replace the E_z term with one of the form $k_z^2(\partial^2/\partial x^2)E_y$. Once again it is possible to replace this $\partial^2/\partial x^2$ with $-k_{ox}^2$ without introducing any error at this order of the calculation. Then Eq. (15) and Eq. (16) can be combined to give the desired wave propagation equation

$$\frac{\partial^2}{\partial x^2} E_y + (k_{ox}^2 - k_y^2 - g(x)k_z^2)E_y = 0 \tag{23}$$

where

$$g(x) = \frac{\omega_o^2\Omega_e^2 - \omega_{pe}^4 + \omega_o^4}{\omega_o^2(\Omega_e^2 + \omega_{pe}^2 - \omega_o^2)} \quad (24)$$

and we have made use of Inequality (19).

Our principal interest lies in the amplitude of the electric field, and an equation for this amplitude can be obtained following the methods appropriate to the parabolic approximation [3] for wave propagation. Accordingly, we write

$$E_y = E(x)\exp[i\int^x k_{ox}(x')dx'] \quad (25)$$

and recognize that the spatial variations of E will occur on scales characteristic of the size of the plasma or of the length $\ell \sim k_{xo}/\langle k_y^2 + k_z^2\rangle$. Physically, if ℓ is shorter than the size of the plasma, then diffraction is the dominant process governing the electric field amplitude. Employing Eq. (25) in the wave propagation equation yields an equation for $E(x)$

$$\frac{\partial^2}{\partial x^2}E + 2ik_{ox}\frac{\partial E}{\partial x} + iE\frac{\partial k_{ox}}{\partial x} - [k_y^2 + g(x)k_z^2]E = 0 . \quad (26)$$

The second order derivative can be compared to the k_y^2 term, and in the case of diffraction

$$\frac{\partial^2 E/\partial x^2}{k_y^2 E} \sim \frac{1}{\ell^2 k_y^2} \simeq \frac{\langle k_y^2 + k_z^2\rangle}{k_{ox}^2} << 1 . \quad (27)$$

An additional condition on the distance from the cutoff r is necessary if this inequality is to be satisfied. Using the definition of k_{ox}^2, it is simple to show that the second derivative can be dropped as long as

$$\frac{r}{L} >> \frac{c^2 k_y^2}{2\omega_o^2} . \tag{28}$$

One final simplification can be made by substituting

$$E = E_o \left(\frac{\omega_o}{ck_{ox}}\right)^{1/2} . \tag{29}$$

This puts the parabolic approximation in the form

$$2ik_{ox} \frac{\partial E_o}{\partial x} - [k_y^2 + k_z^2 g(x)]E_o = 0 . \tag{30}$$

Finally, doing the inverse Fourier transform yields a first order equation in x for the electric field

$$2ik_{ox} \frac{\partial E_o}{\partial x} + [\frac{\partial^2}{\partial y^2} + g(x)\frac{\partial^2}{\partial z^2}]E_o = 0 . \tag{31}$$

The Green's function solution to Eq. (31) for an aperture in the $x = 0$ plane is

$$E_o(x,y,z) = \frac{-i}{2\pi (\xi_1 \xi_2)^{1/2}} \int_s dz'dy' E_a(y',z')$$

$$\times \exp[\frac{i(y-y')^2}{2\xi_1} + \frac{i(z-z')^2}{2\xi_2}] , \tag{32}$$

where

$$\xi_1 = \int_0^x \frac{dx'}{k_{ox}(x')} , \qquad (33)$$

$$\xi_2 = \int_0^x \frac{dx' g(x')}{k_{ox}(x')} . \qquad (34)$$

E_a is the electric field at the aperture, and s is the surface defined by the aperture. If $k_{ox} = \omega_o/c$ and $g(x) = 1$ (i.e., the plasma density is zero), then Eq. (32) is equivalent to the Fresnel-Kirchoff diffraction integral in the limit of x (the focal length) being much greater than y, z, y'_{max}, and z'_{max} [4]. The interpretation of formulae (32 - 34) is straightforward: the focusing of a wave obeys the usual formula for optics provided the distance to the focal spot is calculated on the basis of the virtual distances

$$x_1 = \frac{\omega_o}{c}\int_0^x \frac{dx'}{k_{ox}(x')} , \qquad x_2 = \frac{\omega_o}{c}\int_0^x \frac{dx' g(x')}{k_{ox}(x')} . \qquad (35)$$

The phase variation across the aperture perpendicular and parallel to the magnetic field is determined by x_1 and x_2 respectively.

For the purposes of estimating the intensity and the extent of the pump electric field at a distance x into the plasma, let us choose a uniformly illuminated square aperture of side a and a phase distribution

$$E_a(y',z') = E_a \exp\left(-\frac{iy'^2}{2\bar{\xi}_1} - \frac{iz'^2}{2\bar{\xi}_2}\right) . \tag{36}$$

Using Eq. (36) in Eq. (32) yields

$$E = E_z\left(\frac{\omega_o}{ck_{ox}}\right)^{1/2} \frac{-i\omega_o \exp(iy^2/2\xi_1 + iz^2/2\xi_2)}{2\pi c(x_1x_2)^{1/2}}$$

$$\times \iint_{a/2}^{-a/2} dy'dz' \exp\left(-\frac{iyy'}{\xi_1} - \frac{izz'}{\xi_2} - \frac{iy'^2(\xi_1 - \bar{\xi}_1)}{2\bar{\xi}_1\xi_1}\right.$$

$$\left. - iz'^2 \frac{(\xi_2 - \bar{\xi}_2)}{2\bar{\xi}_2\xi_2}\right) . \tag{37}$$

The integral in Eq. (37) can be performed numerically to find the value of E at any point near the geometric focus. For our purposes it suffices to find some approximate expressions for the extent of the hot spot in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions, along with the magnitude of E at the center. First we let $y = z = 0$, then if x is chosen so that $\xi_1 = \bar{\xi}_1$ and $\xi_2 = \bar{\xi}_2$, Eq. (37) becomes

$$E = \frac{a^2}{\lambda_o (x_1x_2)^{1/2}}\left(\frac{\omega_o}{ck_{ox}}\right)^{1/2} E_a . \tag{38}$$

This is the maximum value of E and occurs at the geometric focus. The terms which are first order in y' and z' will cause an appreciable decrease in E^2 in the distance

$$\Delta y = \frac{x_1}{2a} \lambda_o \qquad \Delta z = \frac{x_2}{2a} \lambda_o \ , \tag{39}$$

as can be seen by considering the integral in the focal plane (i.e., $\xi_1 = \bar{\xi}_1$, $\xi_2 = \bar{\xi}_2$). If one considers the variation of E along the mirror axis ($x = y = 0$, $\xi_1 \neq \bar{\xi}_1$), then the distance along the beam necessary for an appreciable change in E can be seen to be

$$\Delta x = 2 \frac{x_1^2}{a^2} \lambda_o \ . \tag{40}$$

These results indicate that the hot spot is a rectangular solid, centered on the geometric focus, and is about $4(x_1/a)$ times larger along the beam than it is in the plane perpendicular to it.

The phase distribution given by Eq. (36) could be created by a phased array of wave guides, however, the use of a properly shaped dish antenna would probably be easier to design. In this case we would want to use a circular aperture in Eq. (37). Born and Wolf [4] have worked out the case of a circular aperture in detail. Their results are approximately given by replacing a in Eqs. (38 - 40) by 2ρ, where ρ is the radius of the aperture.

3. DETERMINATION OF THE CONVECTIVE AMPLIFICATION

We now consider the parametric decay of the extraordinary mode radiation into ion-acoustic and Langmuir waves. As was shown in the introduction, this can occur only when the plasma density satisfies Inequalities (3). Furthermore, we will show that E_o^2 must exceed some threshold value for parametric decay to occur at all. Therefore, the region in which parametric decay takes place is determined by the focusing of the extraordinary wave as calculated in Section 2, expecially Eqs. (38 - 40). In this section the convective amplification of the Langmuir wave will be determined, and the values of ω_I and $\vec{k}_L$ will be found that maximize A.

Since the hot spot is a small region in the plasma, it is possible to choose $\bar{\xi}_1$ and $\bar{\xi}_2$ (i.e., the phase distribution across the aperature) in such a manner that the hot spot is close to the extraordinary mode cutoff. In this region k_{ox} tends towards zero and, therefore,

$$|\vec{k}_o| << |\vec{k}_L| \; . \tag{41}$$

In this case the dipole approximation can be used for the electric field of the pump. Furthermore, the wave vector matching condition for parametric decay is [2]

$$\vec{k}_o = \vec{k}_L + \vec{k}_I \; , \tag{42}$$

so we conclude that $\vec{k}_L = -\vec{k}_I$. The dispersion relation for the parametric decay of such a pump into electrostatic modes is given by Kaw [5]

$$\frac{1}{\chi_i(\omega_I)} + \frac{1}{\chi_e(\omega_I)+1} + \frac{\mu^2}{4}\left(\frac{1}{\varepsilon(\omega_I-\omega_o)} + \frac{1}{\varepsilon(\omega_I+\omega_o)}\right) = 0 \quad , \quad (43)$$

where

$$\varepsilon(\omega) = 1 + \chi_i(\omega) + \chi_e(\omega) \quad , \qquad (44)$$

$$\mu = \left(\frac{(k_{Lx}^2 v_{eox}^2 + k_{Ly}^2 v_{eoy}^2)}{\omega_o^2}\right)^{1/2} << 1 \ . \qquad (45)$$

χ_s is the susceptibility of specie s, and $\vec{v}_{eo}$ is the velocity of the electrons due to the pump field E_o. We will assume that the electrons and ions have a Maxwellian distribution with temperatures T_e and T_i respectively, so that χ_s and ε can be evaluated explicitly. Since μ^2 is very small, it is necessary for at least one of the $\varepsilon(\omega_I \pm \omega_o)$ terms to be nearly zero. For the case we are considering ($\omega_I \neq 0$ and $\omega_L < \omega_o$), the resonant term is $\varepsilon(\omega_I - \omega_o)$, while $\varepsilon(\omega_I + \omega_o)$ is off resonant and can be ignored.

The imaginary part of $\vec{k}_L$ can be found by dropping the non-resonant term and rewriting Eq. (43) as

$$0 = \varepsilon(\omega_I) + \chi_i(\omega_I)\,[1+\chi_e(\omega_I)]\frac{\mu^2}{4}\,\frac{1}{\varepsilon(-\omega_L)} \equiv D(\vec{k}_L,\omega_L,\omega_I) \quad . \qquad (46)$$

Expanding $D[\vec{k}_L + i\mathrm{Im}(\vec{k}_L)]$ and solving for $\mathrm{Im}(\vec{k}_L)$ yields

$$\mathrm{Im}(\vec{k}_L) = \frac{(\mu^2/4)\,\mathrm{Im}\{\chi_i(\omega_I)[1+\chi_e(\omega_I)]/\varepsilon(\omega_I)\} + \mathrm{Im}\varepsilon(\omega_L)}{\partial\varepsilon(\omega_L)/\partial\vec{k}_L} \quad . \tag{47}$$

The first term in the numerator is obviously the driving term due to the transfer of energy from the extraordinary mode to the Langmuir wave, while the second term is the natural Landau damping of the Langmuir wave. Since the $\mathrm{Im}[\varepsilon(\omega_L)]$ is negative, it is clear that

$$\mu^2 > \mu^2_{\mathrm{Thres}} \equiv \left| \frac{4\,\mathrm{Im}[\varepsilon(\omega_L)]}{\mathrm{Im}\{\chi_i(\omega_I)[1+\chi_e(\omega_I)]/\varepsilon(\omega_I)\}} \right| \tag{48}$$

for $\mathrm{Im}(k_L)$, and therefore A, to be greater than zero. Since μ^2 is proportional to E_o^2, Eq. (48) determines the threshold value of E_o^2.

The next step is to evaluate and maximize $\mathrm{Im}(k_L)$. We observe that

$$\frac{\mathrm{Im}[1+\chi_e(\omega_I)]}{\mathrm{Im}[\chi_i(\omega_I)]} \simeq 9\,\frac{T_i}{T_e}\,\frac{k_L}{k_{L\parallel}}\,\frac{m}{M} \ll 1 \ ; \tag{49}$$

that is, the electron damping is negligible compared to the ion damping of the ion-acoustic response. Therefore,

$$\mathrm{Im}\left(\frac{\chi_i(\omega_I)[1+\chi_e(\omega_I)]}{\varepsilon(\omega_I)}\right) = -\{\mathrm{Re}[1+\chi_e(\omega_I)]\}^2\,\mathrm{Im}\left(\frac{1}{\varepsilon(\omega_I)}\right). \tag{50}$$

Using the explicit expressions for χ and ε allows the driving term in Eq. (47) to be written as

$$\frac{\mu^2}{4}\,\frac{m\omega_{pe}^2}{k_L^2 T_e}\,B\left(\frac{\omega_I}{k_I v_{Thi}}\right) \;, \tag{51}$$

where

$$B(s) = -\mathrm{Im}\,\frac{1}{1 + (T_e/T_i)W(s)} \;, \tag{52}$$

and

$$W(s) = \int \frac{\vec{k}\cdot v f_i}{\vec{k}\cdot v - s|k_I|v_{Thi}}\, d^3v \;, \tag{53}$$

where $s = \omega_I/|k_I|v_{Thi}$. From Eq. (51) it is clear that to maximize $\mathrm{Im}(k_L)$ we need to pick ω_I so that $B(\omega_I/k_I v_{Thi})$ is a maximum. The value of B, parameterized by T_e/T_i, has been compiled in many places [6]. For the case of $T_i = T_e$, B has a maximum value of about 0.6 at $\omega_I^2 \approx (3T_e/M)k_L^2$. Therefore, the decay which has an ion response at this frequency will grow the fastest and dominate the decay process.

Next it is necessary to find the real part of $\vec{k}_L$. However,

$$\varepsilon(-\omega_L) \approx \varepsilon(\omega_o) = 1 + \frac{k_{L\perp}^2}{k_L^2}\,\frac{\omega_{pe}^2}{\Omega_e^2 - \omega_o^2} - \frac{k_{L\parallel}^2}{k_L^2}\,\frac{\omega_{pe}^2}{\omega_o^2} \quad 0 \;, \tag{54}$$

which is the dispersion relation for the Langmuir wave. It can be used to relate $|\vec{k}_{L\perp}|$ to $|\vec{k}_{L\parallel}|$, giving

$$k_{L\perp}^2 = k_{L\parallel}^2\,\frac{(\Omega_e^2 - \omega_o^2)(\omega_{pe}^2 - \omega_o^2)}{\omega_o^2(\omega_{pe}^2 + \Omega_e^2 - \omega_o^2)} \;. \tag{55}$$

Thus it is necessary only to find the magnitude of $\vec{k}_{L\parallel}$ and the direction of $\vec{k}_{L\perp}$ to specify $\vec{k}_L$ completely. The direction of $\vec{k}_{L\perp}$ affects the convective amplification in two ways: first, if $|v_{eox}| > |v_{eoy}|$, then $\vec{k}_{L\perp} = k_{L\perp}\hat{x}$ will maximize μ^2 and vice-versa [see Eq. (45)]; second, the choice effects the length of time the wave remains in the hot spot. Figure 4a shows the hot spot in the xy-plane. It is apparent from the figure and the expressions for Δx and Δy found in Section 2 that the wave remains in the hot spot for the longest period if $\vec{k}_{L\perp}$ is solely in the $\hat{x}$ direction. In order to evaluate μ^2 it is necessary to calculate the electron velocities due to the pump field,

$$v_{eox} = \frac{\Omega_e \omega_o}{\Omega_e^2 - \omega_o^2} \frac{eE_y}{m\omega_o} + \frac{i\omega_o^2}{\Omega_e^2 - \omega_o^2} \frac{eE_x}{m\omega_o} , \tag{56}$$

$$v_{eoy} = -\frac{\Omega_e \omega_o}{\Omega_e^2 - \omega_o^2} \frac{eE_x}{m\omega_o} + \frac{i\omega_o^2}{\Omega_e^2 - \omega_o^2} \frac{eE_y}{m\omega_o} . \tag{57}$$

E_x can be eliminated from these expressions by using the extraordinary mode polarization [Eq. (9)]. As a result,

$$|v_{eox}| = \frac{eE_y}{m\Omega_e} \frac{\Omega_e^2}{(\Omega_e + \omega_o)[\Omega_e - \omega_o(1-\eta)]} \geq |v_{eoy}| \tag{58}$$

where equality is true only when $\eta = 1$. Both considerations indicate that $\vec{k}_{L\perp}$ will be only in the x direction, therefore,

$$\vec{k}_L = k_{L\parallel}\hat{z} + k_{L\perp}\hat{x} . \tag{59}$$

Using this result and Eqs. (58) and (55) to evaluate μ^2 results in

$$\mu^2 = \left(\frac{eE_y}{m\Omega_e}\right)^2 \frac{k_{\|}^2}{\omega_o^2} \frac{\omega_{pe}^2 - \omega_o^2}{\omega_o^2} \frac{\Omega_e^4(\Omega_e^2 - \omega_o^2)}{(\Omega_e^2 + \omega_{pe}^2 - \omega_o^2)^3} \quad . \tag{60}$$

Appropriately, the coupling μ^2, which is due to the perpendicular velocities, vanishes when $\omega_{pe} = \omega_o$ and the daughter wave is propagating parallel to the magnetic field.

In order to proceed, the derivative $\partial\varepsilon/\partial\vec{k}_L$ must be found, and using Eq. (54) gives

$$\frac{\partial\varepsilon}{\partial\vec{k}} = \frac{2\omega_{pe}^2\Omega_e^2}{(\Omega_e^2 - \omega_o^2)\omega_o^2} \frac{k_{L\perp}k_{L\|}}{k_L^3}\left(-\frac{k_{L\|}}{k_L}\hat{x} + \frac{k_{L\perp}}{k_L}\hat{z}\right) \quad . \tag{61}$$

The group velocity of the Langmuir wave is proportional to Eq. (61), so it can be used to determine the path that the Langmuir wave takes through the hot spot. This path is needed to evaluate the integral in the definition of A. If

$$R \equiv \frac{(\Delta z)/(\partial\varepsilon/\partial k_{\|})}{(\Delta x)/(\partial\varepsilon/\partial k_{\perp})} < 1 \ , \tag{62}$$

then the wave will convect out of the interaction region in the $\hat{z}$ direction before it does in the $\hat{x}$ direction. Using the expressions for Δx and Δz, along with Eq. (61), R can be written as

$$R \simeq \frac{x_2}{x_1}\frac{1}{4}\frac{a}{x_1}\frac{\omega_o}{(\omega_{pe}^2 - \omega_o^2)^{1/2}} \quad . \tag{63}$$

Except near $\omega_{pe} = \omega_o$, R is much less than one, and the wave will propagate out of the hot spot in the $\hat{z}$ direction first. Therefore, we can approximate the integral for A by

$$\int \mathrm{Im}(\vec{k}_L)\,d\vec{\ell} \simeq \mathrm{Im}(k_{L\|}) \cdot 2\Delta z \quad . \tag{64}$$

This will lead to a slight underestimate of A in the region near the cutoff since the chord across the hot spot in the $\hat{z}$ direction is shorter than that actually traversed (see Fig. 4b). On the other hand, in the region where $\omega_{pe} = \omega_o$ this approximation leads to an overestimate of A, since $\partial\varepsilon/\partial k_{\|}$ becomes zero when $\omega_{pe} = \omega_o$. The expression for A will remain finite in this limit since the coupling constant (μ^2) also goes to zero. Therefore, A remains finite, although it is not correct. The value of ω_{pe} such that $R = 1$ can be used as a limit to the applicability of our expression for A.

Combining all the results so far yields

$$A = \frac{P_a}{P_o}\frac{B}{V}\frac{\rho}{x_1}\frac{\omega_o}{ck_{ox}}\frac{\omega_{pe}^2}{\Omega_e^2}\,C(\Omega_e, \omega_{pe}, \omega_o)$$

$$- \left(\frac{\pi^{1/2}\omega_o\omega_{pe}^2 x_2 \lambda_o}{8v_{The}(\omega_{pe}^2 - \omega_o^2)\rho}\right) V^2 \exp(-V^2) \quad , \tag{65}$$

where

$$P_o = \frac{2^{1/2} m^{1/2} c^2 T_e^{3/2}}{e^2} \simeq 1.0\,[T_e(\text{keV})]^{3/2}\ \text{MW}\ , \qquad (66)$$

$$C(\Omega_e, \omega_{pe}, \omega_o) = \frac{\Omega_e^4(\Omega_e^2 - \omega_o^2)}{(\Omega_e^2 + \omega_{pe}^2 - \omega_o^2)^3} \simeq 1\ , \qquad (67)$$

the standard expression for Landau damping has been used [5], P_a is the power applied at the aperture, $V = (\omega_o/k_{L\parallel})(m/2T_e)^{1/2}$, and ρ is the radius of a circular antenna.

The only parameter left to be determined is the magnitude of $k_{L\parallel}$, which now enters only through V. As can be seen from Eq. (65), the parametric decay driving term increases with $k_{L\parallel}$. This is because a larger $k_{L\parallel}$ implies a smaller group velocity, as can be seen in Eq. (61), and the daughter wave spends a greater amount of time in the interaction region. However, if $k_{L\parallel}$ becomes too large, the Landau damping term will overwhelm the driving term causing A to become negative. The value of V which causes A to be a maximum can be found numerically and is plotted as a function of $[(2\rho/x_1)(v_{The}/c)(AV)]$ in Fig. 5. To use the figure one calculates the value of AV using Eq. (68) and multiplies it by $(2\rho/x_1)(v_{The}/c)$, then V can be read off the solid line (see the examples in Section 4 for more details on the use of Fig. 5). The percentage contribution of the Landau damping term to the value of A has a maximum value of about 15% for the range of interest, its contribution to Eq. (65) can be dropped. Therefore, we conclude that

$$A \simeq \frac{P_a}{P_o} \frac{\omega_o}{ck_{ox}} \frac{1}{V} \eta B\left(\frac{\omega_I}{v_{Thi}k_L}\right) C(\Omega_e, \omega_{pe}, \omega_o) \frac{\rho}{x_1} \frac{\omega_o}{\Omega_e}\left(1 + \frac{\omega_o}{\Omega_e}\right) . \qquad (68)$$

We have considered only the decay into ion-acoustic and Langmuir waves. However, we would expect that the decay into any other waves, at the same power, would have lower A's associated with it. The only other possible electron response in this frequency range would be a whistler wave. However, the group velocity of the whistler wave is greater than that of the Langmuir wave, and therefore it will convect out of the hot spot more quickly. This would imply that for the same power in the extraordinary mode the A due to whistler wave decay would be less than the A we have calculated. As far as other ion modes go, in the range of frequencies and wave numbers required by the Langmuir electron response, there are no other candidates.

4. APPLICATIONS

As an example of how Eq. (68) can be used to obtain A, we will consider the parametric decay heating of a PDX discharge (T_e = 1 keV, B_o = 25 kG, n = $3 \cdot 10^{13}$ cm^{-1}, r = 40 cm). The first step is to estimate the value of ω_o/ck_{ox}. This term appears in Eq. (68) as a result of the increased intensity of the extraordinary mode near the cutoff necessary for energy flux conservation. Near the cutoff the density can be approximated by a linear function of r (the distance from the cutoff), in which case the zeroth order wave propagation equation found in

Section 2 [Eq. (10)] becomes the Airy equation. The solution to this equation indicates that the $1/k_{ox}$ scaling is valid until the inverse wave number is on the order of the distance to the cutoff ($r \sim 1/k_{ox}$), while E_o^2 decreases for smaller values of r. Using this expression for r and the definition of k_{ox}^2 in the limit of $\eta \to 1$, yields a formula for the maximum value of ω_o/ck_{ox},

$$\left(\frac{\omega_o}{ck_{ox}}\right)_{max} \simeq \left(\frac{L\omega_o}{2c}\right)^{1/3} \simeq 3 \ . \qquad (69)$$

Our calculation has been for a monochromatic (temporally coherent) source of extraordinary radiation. In order to estimate the allowable bandwidth, we note that the most sensitive element of A to small changes in the frequency is $B(\omega_I/k_L v_{Thi})$. If B is to be reduced by half, then the argument of B must also be reduced by about half [6]. Thus

$$\Delta\omega_o \simeq \frac{1}{2}|k_L| v_{Thi} \ ,$$

$$\simeq \frac{1}{2V}\left(\frac{\Omega_e}{\omega_o}\right)^{1/2}\left(\frac{m}{M}\right)^{1/2}\omega_o \ ,$$

$$\simeq 3 \cdot 10^{-3}\omega_o \ . \qquad (70)$$

Spatial coherence, which has been demonstrated experimentally [1], is also required so that focusing is possible.

If the 15-GHz, 350-MW source reported by Hirschfield and Granatstein [1] were used to heat PDX, the extraordinary mode radiation would penetrate approximately 30 cm into the plasma (assuming a parabolic density profile). A reasonable sized antenna would have a radius of 10 cm, so that $\rho/x_1 = 1/3$. Equation (68) can be written in the form of AV equal to a function of experimental parameters. Using the evaluations given so far, along with ρ_o, it is possible to evaluate AV. One finds that AV = 39. Now one calculates $AV(2v_{The}/c)(\rho/x_1)$, finding that it is approximately 0.7. Finally from Fig. 5 we see that V = 2.6, and therefore (using AV = 39), A = 15, well above the effective threshold of A = 6. For a prototype reactor with a peak density of 10^{14} and $B_o = 60$ kG, it is necessary to use a 40-GHz source (see Fig. 2). Using $T_e = 10$ keV, A is approximately given by $A \approx (P_o/500)$ MW. The power requirements are markedly higher in this case due to the unfavorable $T_e^{3/2}$ scaling of P_o [Eq. (66)].

Of all the terms in Eq. (68), the only ones which are very sensitive to the actual experimental parameters are ρ/x_1, ω_o/Ω_e, P_o/P_a. Therefore, it is possible to express the power needed to exceed the effective threshold roughly as

$$P_o = 10 \text{ MW} \frac{x}{\rho} \frac{\Omega_e}{\omega_o} \left(\frac{T_e}{1 \text{ keV}}\right)^{3/2} . \tag{71}$$

If the effective threshold is exceeded, then an estimate of the nonlinear absorption of the energy of the extraordinary mode is possible. The estimate follows from the conventional turbulence result [6] that the pump wave will experience an effective damping decrement (γ_{eff}) given by

$$\gamma_{eff} = \frac{\partial \omega_L}{\partial k_{L\parallel}} \frac{1}{2\Delta z} \simeq \frac{\omega_o}{k_{L\parallel} 2\Delta z} \quad . \tag{72}$$

Therefore, the pump wave will be attenuated by a factor of exp(-a) due to parametric decay, where

$$a = \frac{2\Delta x \gamma_{eff}}{(\partial \omega_o / \partial k_o)} \simeq \frac{\Delta x}{\Delta z} V \left(\frac{2T_e}{me^2}\right)^{1/2} \frac{\omega_o}{k_{ox} c} \quad , \tag{73}$$

and the group velocity of the extraordinary mode is

$$\frac{\partial \omega_o}{\partial k_o} = c \frac{k_{ox} c}{\omega_o} \quad . \tag{74}$$

Using the expressions for Δx and Δz yields

$$a \simeq \frac{1}{15} \frac{x}{\rho} V \frac{\omega_o}{ck_{ox}} \left(\frac{T_e}{1 \text{ keV}}\right)^{1/2} > 1 \ . \tag{75}$$

For the cases we considered a is greater than one and, therefore, there is almost complete nonlinear absorption of the extraordinary radiation.

5. DISCUSSION

High power microwave sources with sufficient power, temporal coherence, and spatial coherence tc bring about nonlinear heating of modest temperature tokamaks ($T_e \lesssim 1$ keV) have already been developed. The power required can be estimated by using Eq. (71) and is on the order of hundreds of megawatts for a 1-keV plasma. At the present these sources are single pulse sources (pulse lengths are about 50 ns), and therefore do not supply enough energy to make an effective heating source. However, they do provide the possibility of confirming our results. If a source with a duty cycle of about 1% could be developed, then it would serve as a powerful source of auxilary heating for present-day tokamaks.

The key ability which makes this heating scheme possible is the ability to focus the extraordinary mode radiation onto a hot spot near the extraordinary mode cutoff. Without this ability, the threshold for parametric decay instabilities would not be exceeded. Being able to focus near the cutoff also allows the use of the increasing field intensity of the extraordinary mode due to the low group velocity in the cutoff region. It should be noted that the focusing across the magnetic field is more important than that along the field. This can be seen by observing that only x_1 enters the expression for A [Eq. (68)] and that from its definition x_1 refers to the phase variation perpendicular to the magnetic field [see Eq. (35)]. Although we assumed that the magnetic field was a constant, as long as its variation is also on the scale of L, we would expect the focusing to still be possible. Finally, estimate (75) clearly shows that the nonlinear absorption of the extraordinary mode radiation is appreciable if the effective threshold is exceeded.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

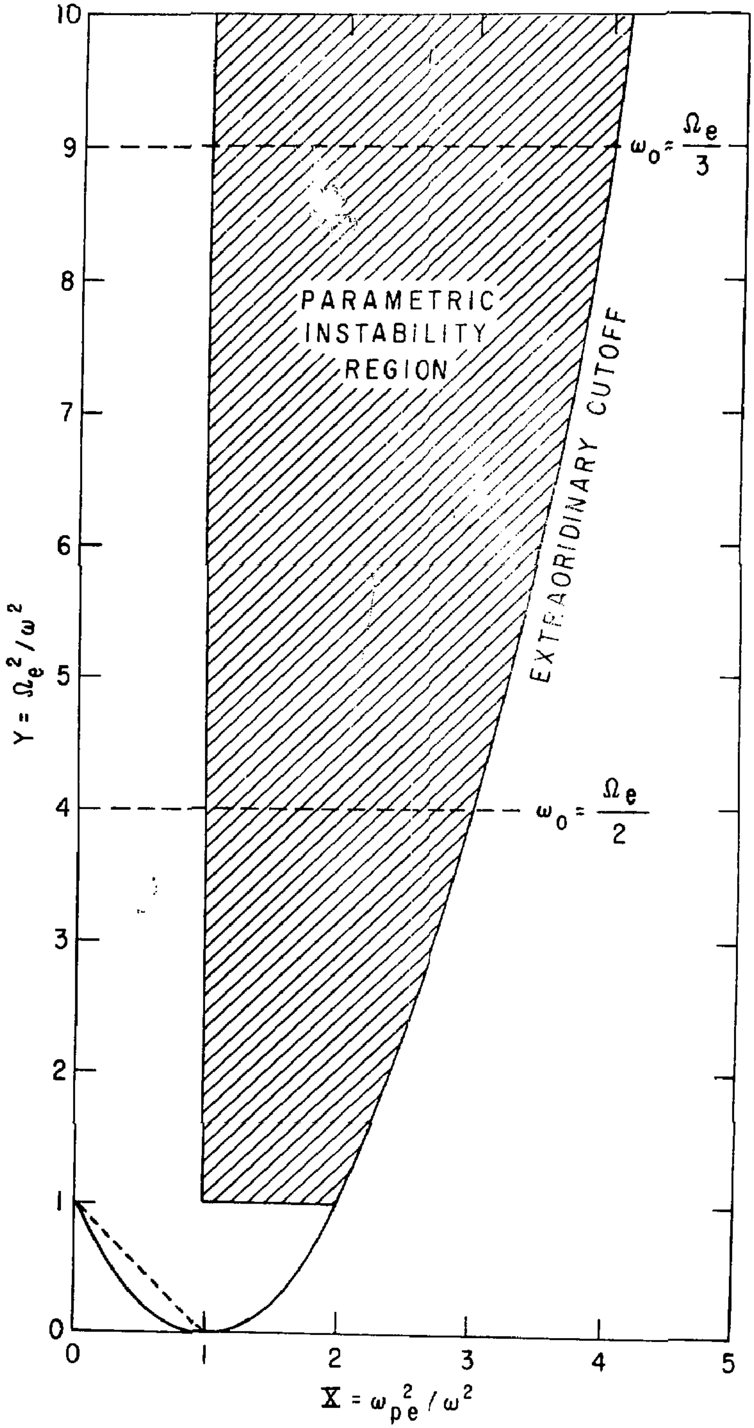
Fig. 1. The CMA diagram for extraordinary mode radiation showing the extraordinary cutoff (solid line), the upper hybrid resonance (dashed line), and the region where parametric decay is possible (shaded region). (PPL 792109)

Fig. 2. Parameters for parametric decay at a variety of pump frequencies. Curves labeled X give the extraordinary mode cutoff, while L refers to the minimum density for Langmuir waves. (PPL 772108)

Fig. 3. The envisioned experimental arrangement. (PPL 792107)

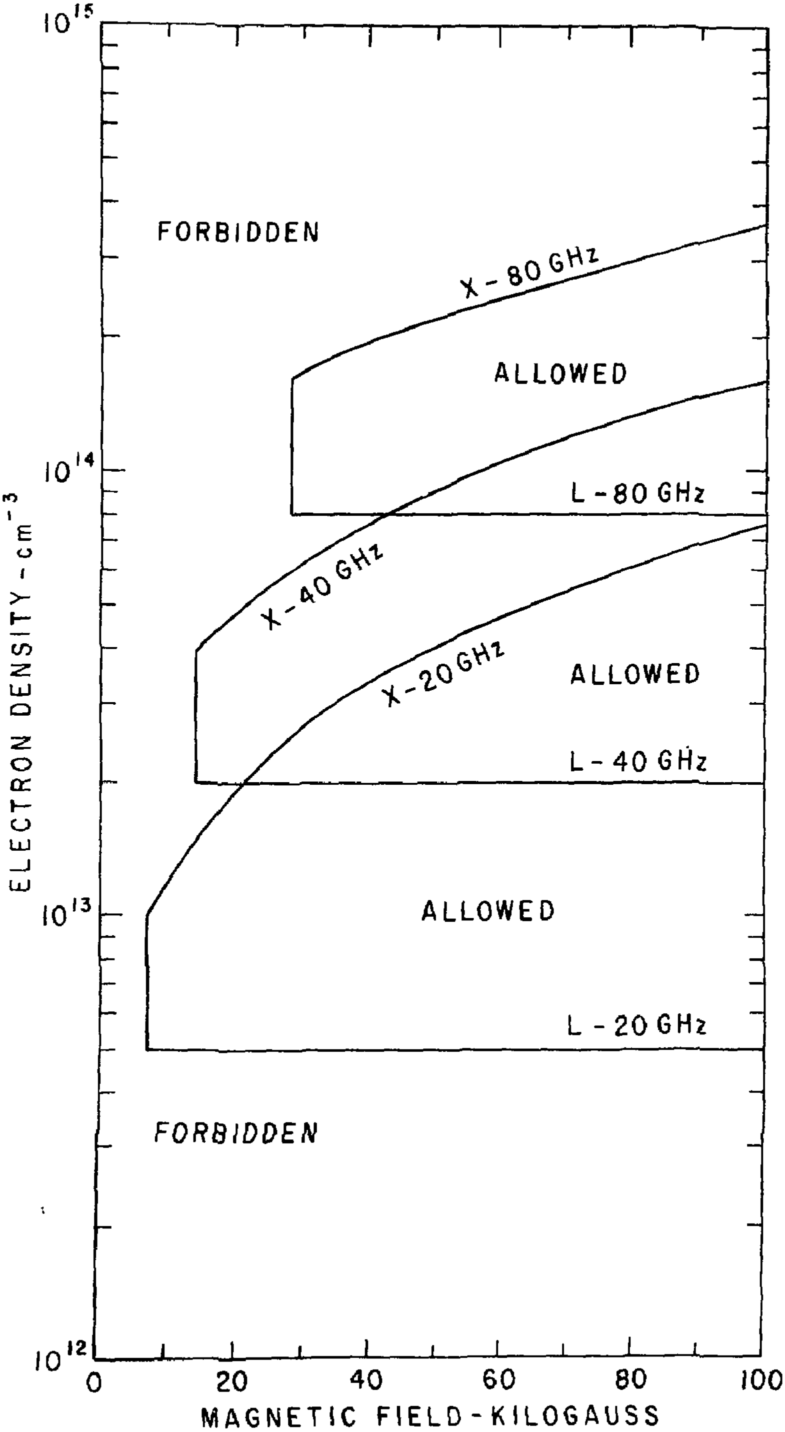
Fig. 4. A schematic diagram of the hot spot (solid lines are lines of constant intensity normalized by the intensity at the geometric focus). (a) The cross section of the hot spot in the xy-plane with an arbitrary perpendicular wave vector drawn in. (b) The dashed line shows the path followed by the Langmuir wave as it convects through the hot spot in the xz-plane. The solid line shows the approximation to this path which is used in calculating the convective amplification. (PPL 792343)

Fig. 5. The ratio of the parallel phase velocity to the electron thermal speed which maximizes A (solid line). The hashed area is the region where, for a 1-keV plasma with $\rho/x \sim 1/3$, A is between 1 and 20. (PPL 792344)



PARAMETRIC
INSTABILITY
REGION
EXTRAORDINARY CUTOFF
$\omega_0 = \frac{\Omega_e}{3}$
$\omega_0 = \frac{\Omega_e}{2}$
$Y = \Omega_e^2/\omega^2$
$X = \omega_{pe}^2/\omega^2$
0
1
2
3
4
5
6
7
8
9
10

Fig. 1. 792108



10^{15}
10^{14}
10^{13}
10^{12}
FORBIDDEN
X - 80 GHz
ALLOWED
L - 80 GHz
X - 40 GHz
X - 20 GHz
ALLOWED
L - 40 GHz
ALLOWED
L - 20 GHz
FORBIDDEN
ELECTRON DENSITY - cm^{-3}
0
20
40
60
80
100
MAGNETIC FIELD - KILOGAUSS

Fig. 2. 792109

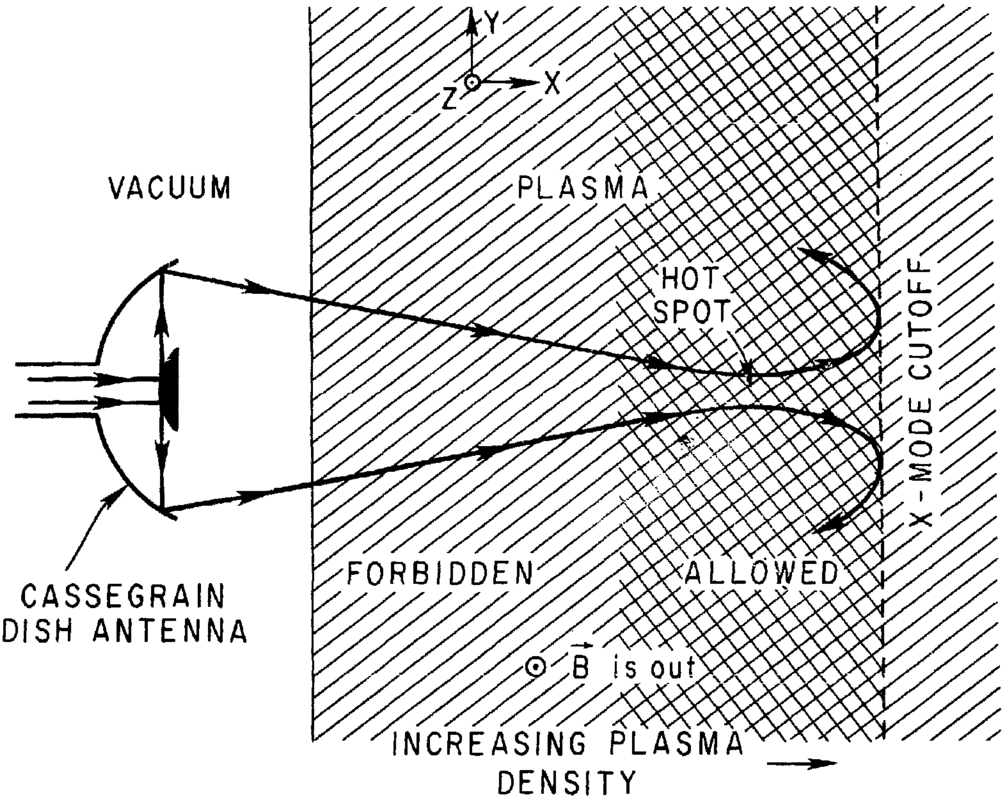

Y
X
Z
VACUUM
PLASMA
HOT
SPOT
X-MODE CUTOFF
CASSEGRAIN
DISH ANTENNA
FORBIDDEN
ALLOWED
⊙ B is out
INCREASING PLASMA
DENSITY

Fig. 3. 792107

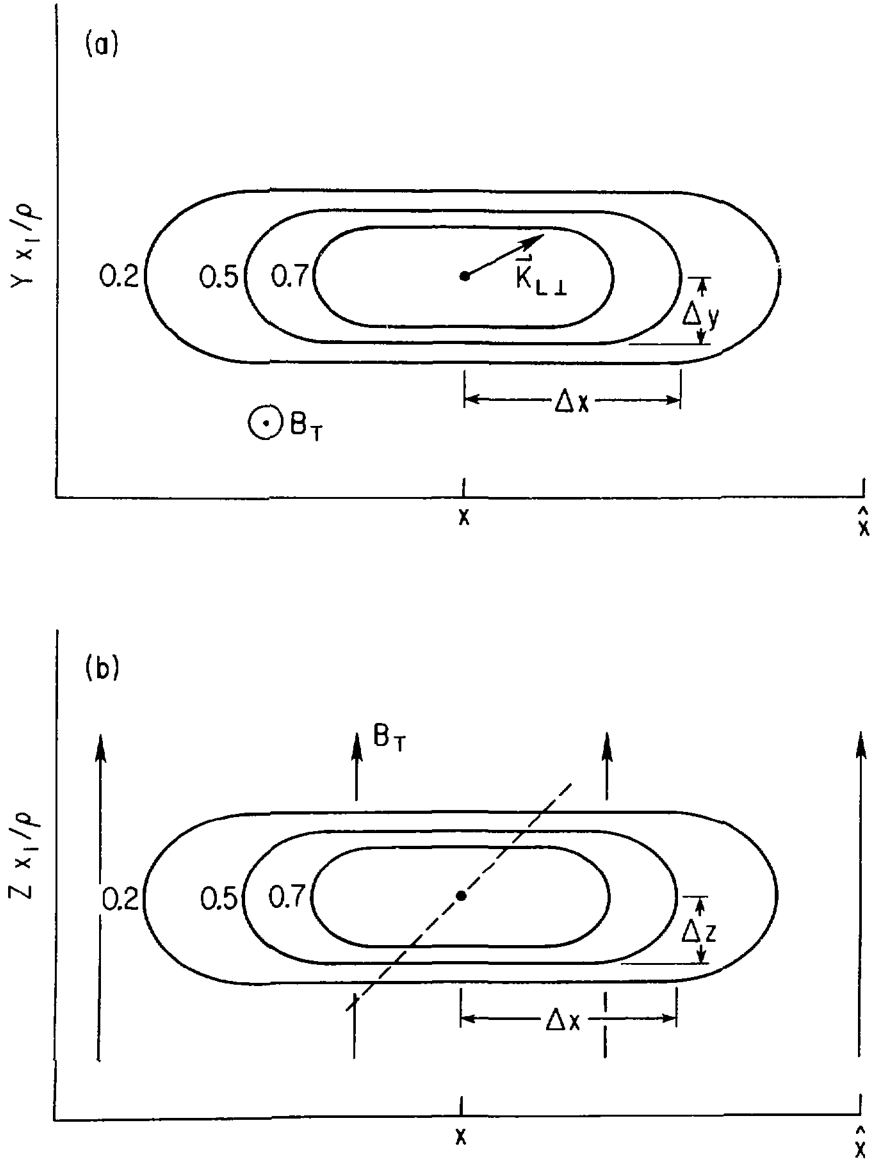


Fig. 4. 792343

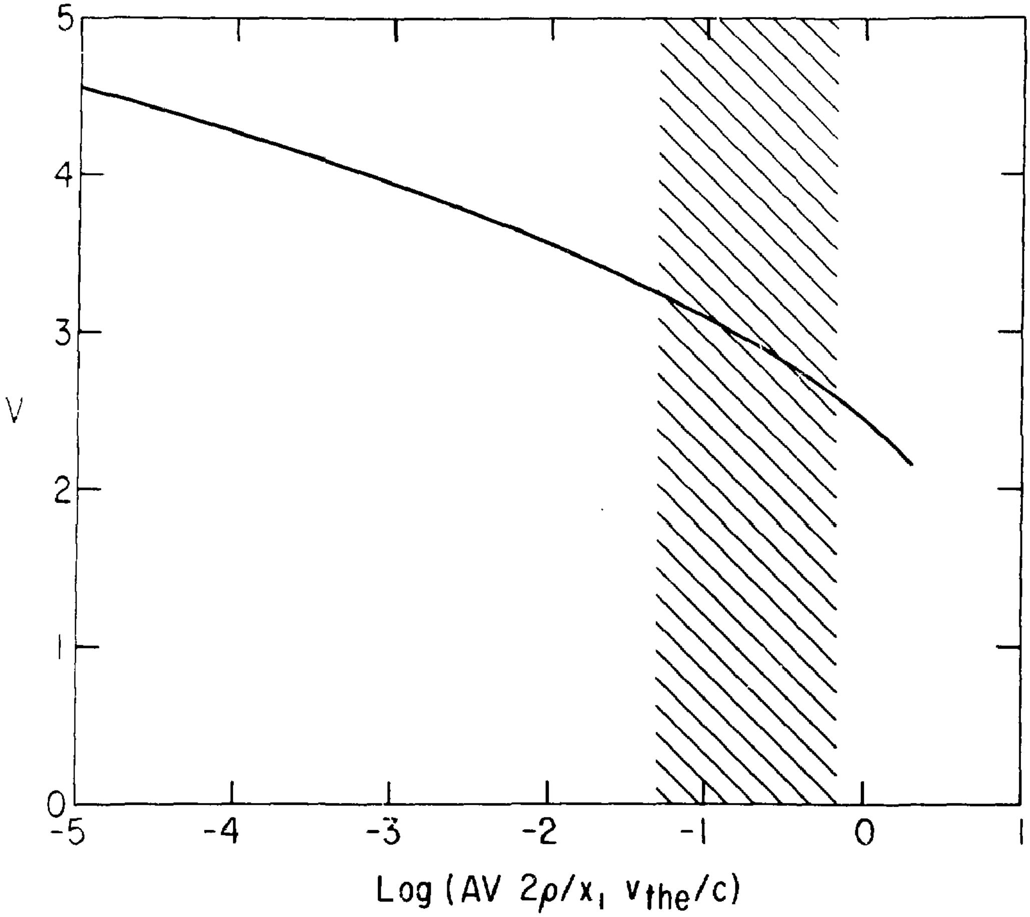


Fig. 5. 792344