CNRS-CPT-79/P-1097

FR 7901721

The $O(\epsilon^2)$ SCALING LAW FOR $d\sigma/dt$ IN THE REGGEON FIELD THEORY

Jan W. DASH *

Thierry GRANDOU**

ABSTRACT : We calculate the two loop contributions within the ϵ - expansion to the Reggeon Field Theory scaling law for $d\sigma/dt$, derived using the renormalization group and a general renormalization point for the Pomeron propagator. This generalizes the $O(\epsilon)$ work of Abarbanel, Bartels, Bronzan, and Sidhu. The invariance of the results under certain coupling constant rescalings is demonstrated. We also make some qualitative comments regarding phenomenological applications. Our amplitude in a certain limit approximates the form of the low energy diffractive amplitude advocated by Kane.

APRIL 1979

79/P.1097

* Centre de Physique Théorique, CNRS Marseille

** Allocataire DGRST, and Centre de Physique Théorique, CNRS Marseille

POSTAL ADDRESS : C.N.R.S. - LUMINY - CASE 907
Centre de Physique Théorique
F-13288 MARSEILLE CEDEX 2 (France)

I. INTRODUCTION

The critical Reggeon Field Theory (RFT) is an attractive way of imagining how truly asymptotic hadron-hadron scattering cross sections could behave [1]. However, serious objections have been raised concerning the applicability of the RFT at presently available (or even conceivably available) energies [2]. This pessimism is partially countered by the surprisingly successful calculation of $d\sigma/dt$ using the renormalization group (RG) at the one loop level in the ϵ-expansion. A quite sophisticated treatment was given by Abarbanel, Bartels, Bronzan, and Sidhu (hereafter referred to as ABBS) [3]. These authors derived a certain system of equations in two variables related to j and t, based on the most general form of various quantities in the presence of a zero at $g = g_1$ in the Gell-Mann-Low β function defining the critical theory. These equations enabled ABBS to generalize the straightforward RG one-loop $O(\epsilon)$ results in a non-trivial way. Indeed, if only the $O(\epsilon)$ RG results are used, the critical Pomeron inverse propagator $\Gamma^{(1,1)}$ is basically constant in t. Ignoring, as did ABBS, the t-dependence of external particle couplings and the signature factor, the resulting $d\sigma/dt$ would also be roughly constant in t in this approximation. The ABBS equations allow a (correct) determination of the 2-loop $O(\epsilon^2)$ t-dependence of g_1^2 and further exponentiate the one-loop results to all orders in ϵ. In this way a form for $d\sigma/dt$ is produced with a reasonable dip-bump structure.

This paper is devoted to a generalization of the ABBS results by explicit calculation of the two loop $\Gamma^{(1,1)}$ graphs using the t $\neq$ 0 renormalization point of the inverse propagator required as input to the ABBS formalism. Previous $O(\epsilon^2)$ results [4,5] were based on the use of a t = 0 renormalization point of $\Gamma^{(1,1)}$. Both t = 0 and t $\neq$ 0 renormalizations are compatible with the dictum that the physics should not be changed by the convention of where the theory is renormalized. We shall show that the properly renormalized theory has critical exponents independent of t in $O(\epsilon^2)$. The general scaling law in s and t is also true regardless of the renormalization point of $\Gamma^{(1,1)}$. However, the t = 0 renormalization point does not lead to the detailed information concerning the scaling law that can be obtained by using a t ≠ 0 renormalization point.

The results that we obtain are the following. We verify the ABBS prescription when both the bare triple Pomeron coupling r_0 and the bare slope α_0' are properly renormalized. If the α_0' renormalization is not performed, the ABBS equations do not have a well-defined ϵ - expansion and the critical exponents are not independent of t. (One might have hoped to avoid the α_0' renormalization and thus avoid a complicated inversion formula for $\alpha' = \alpha'(\alpha_0')$ which would have made the phenomenology simpler). We perform a consistency check of our $O(\epsilon^2)$ results for g_1^2 with that generated by the ABBS equations from the $O(\epsilon)$ results. Our $O(\epsilon^2)$ calculations, coupled with the ABBS equations, lead to a prediction of the $O(\epsilon^3)$ t-dependence of g_1^2.

Although the RG critical exponents are independent of t, the exponentiated results for $\Gamma^{(1,1)}$ have a highly non-trivial (j,t) dependence. One piece agrees with the ABBS results, with an appropriate generalization. The amplitude in the limit $-\alpha_0' t \ln s \ll 1$ is associated with a Bessel function whose argument has a form suggested by Kane long ago on phenomenological grounds [6]. We regard this as highly interesting.

In the $O(\epsilon^2)$ work of Ref. 4, hereafter referred to as BD, the invariance of the critical exponents with respect to a certain coupling constant rescaling was demonstrated. The critical coupling g_1 is not, however, invariant. For example, the dependence of g_1 on $\ln \pi$ and the Euler constant can be changed (the critical exponents are independent of these quantities). We extend this idea to show that the t-dependence of g_1 can be changed without changing either the critical exponents or the overall scaling law for $\Gamma^{(1,1)}$.

Unlike the critical exponents, the final result for $\Gamma^{(1,1)}$

that we obtain seems to depend on $\ln \pi$ and the Euler constant. We do not fully understand this point.

Finally, a question is raised relative to the α_0' renormalization of singular objects. A higher order calculation may be necessary to fully resolve this issue.

This paper deals mainly with the theoretical aspects of the calculation. A future publication will treat the phenomenology more fully.

The organization of the rest of this paper is as follows. In Section II we set out the calculation of the $t \neq 0$ two-loop graphs in $\Gamma^{(1,1)}$ and the perturbative expressions of the renormalization functions Z, Z_2, and Z_3. In Section III, we calculate the functions needed for the renormalization group, the critical coupling, and the critical exponents. Section IV is devoted to the ABBS equations. In Section V we comment on the singular α_0' renormalization. Section VI exhibits our main results. The renormalization group scaling law for $\Gamma^{(1,1)}$ is exhibited, generalizing the results of ABBS. Section VIIA has some general remarks on phenomenology, and the connection of Kane's Pomeron to our results is exhibited in Section VIIB. The Appendix contains miscellaneous expressions, some of which correct those in Ref. 4 .

II. THE TWO-LOOP GRAPHS AT $t \neq 0$

In order to utilize the renormalization group, a perturbative input is required. We need the Pomeron inverse propagator $\Gamma^{(1,1)}$ renormalized at external "energy" $-E_N < 0$ (generally $E = 1 - j$) and also renormalized at non-zero transverse momentum $\vec{k}_N$ in $D = 4 - \epsilon$ dimensions (generally at $D = 2$, $k^2 = -t$). Our notation follows that of ABBS [3] and BD [4] closely. We shall reproduce enough of their formulae to be self-contained. The base Pomeron parameters are the bare coupling r_0 and the bare slope α_0'. The bare intercept α_0 is imagined to be at the critical value α_{0c} required to produce the critical theory. It does not appear in the ϵ-expansion of quantities we shall consider. Actually α_{0c} is non-analytic in ϵ and $\alpha_{0c} - 1 = 0$ to any order in ϵ. At $\epsilon = 0$ the unrenormalized dimensionless coupling $g_0 = r_0 (\alpha_0')^{-D/4} E_N^{-\epsilon/4}$ becomes independent of E_N. We must eventually set $\epsilon = 2$.

The two-loop $\Gamma^{(1,1)} = \Gamma_a^{(1,1)} + \Gamma_b^{(1,1)}$. As in Fig.1 of BD, $\Gamma_a^{(1,1)}$ contains a bubble on an internal line, and $\Gamma_b^{(1,1)}$ has an internal line joining the two sides of the one loop graph.

The easier graph to evaluate is $\Gamma_a^{(1,1)}$. Define, following ABBS

$$X_0 = \alpha_0' k_N^2 / E_N \qquad (2.1)$$

and

$$R = -r_0^4 \, \Gamma(3-D) \, E_N^{D-3} \, (4\pi\alpha_0')^{-D} \qquad (2.2)$$

Then

$$-i\Gamma_a^{(1,1)}(-E_N, X_0) = \frac{R}{2}\int_0^1 dy\,(1-y)\left[y(4-y)\right]^{\frac{\epsilon}{2}-2}\left[1+\frac{X_0(2-y)}{4-y}\right]^{1-\epsilon} \qquad (2.3)$$

Using the same techniques as BD, we determine the $1/\epsilon^2$ and $1/\epsilon$ terms of $\Gamma_a^{(1,1)}$ as

$$-i\Gamma_a^{(1,1)}(-E_N,X_0) = -i\Gamma_a^{(1,1)}(-E_N,0) - \frac{r_0^4 E_N}{2(8\pi d_0^2)^4}\frac{1}{\epsilon^2}\left[\frac{3X_0}{4} + \epsilon\,\tilde{\Gamma}_a\right] \tag{2.4}$$

where $\Gamma_a^{(1,1)}(-E_N,0)$ is given in Eq.38 of BD, and

$$\tilde{\Gamma}_a = \frac{3X_0}{4}\left[\frac{55}{36} + 4\ln 2 - \frac{1}{2}\ln 3 + \delta - f(X_0) + \ln\frac{d_0'}{E_N}\right] - f(X_0) \tag{2.5}$$

Here

$$\delta = \ln\pi - \gamma_{Euler} \tag{2.6}$$

where γ_{Euler} is the Euler constant, and

$$f(X_0) = \ln\left(1 + \frac{X_0}{2}\right) \tag{2.7}$$

Notice that $f(X_0)$ and δ appear in the combination $X_0(\delta - f)$ in $\tilde{\Gamma}_a$. Further the $1/\epsilon^2$ term in $-i\Gamma_a^{(1,1)}(-E_N,0)$ has the same coefficient as does the $-f/\epsilon$ term of $\tilde{\Gamma}_a/\epsilon$. These symmetries will eventually produce critical exponents independent of f (and δ). An overall $\delta - f$ dependence will turn up in the differential equations giving the scaling law for $\Gamma^{(1,1)}$, however.

The calculation of $\Gamma_b^{(1,1)}$ is more difficult. Here

$$-i\Gamma_b^{(1,1)}(-E_N,X_0) = R\int_0^1 dv\int_v^1 du\left[1 + \frac{X_0}{2}\left(\frac{2-u^2-v^2}{3-2u-v^2}\right)\right]^{1-\epsilon}(3-2u-v^2)^{\frac{\epsilon}{2}-2} \tag{2.8}$$

$$= A_b + B_b + C_b + D_b \tag{2.9}$$

This decomposition arises from integrating by parts once with respect to u and expanding the bracket containing X_0 in ϵ. The terms are

$$A_b = \frac{R}{2-\epsilon}\int_0^1 dv\,(1-v^2)^{\frac{\epsilon}{2}-1}\left(1+\frac{x_0}{2}\right)^{1-\epsilon} \tag{2.10}$$

$$B_b = \frac{-R}{2-\epsilon}\int_0^1 dv\,(3-2v-v^2)^{\frac{\epsilon}{2}-1}\left[1+\frac{x_0(1+v)}{3+v}\right]^{1-\epsilon} \tag{2.11}$$

$$C_b = \frac{-R}{2-\epsilon}(1-\epsilon)x_0\int_0^1 dv\int_v^1 du\,(3-2u-v^2)^{\frac{\epsilon}{2}-3}\left[2-u^2-v^2-u(3-2u-v^2)\right] \tag{2.12}$$

$\mathcal{D}_b$ is the same as C_b except for an extra factor of $-\epsilon\ln\left[1+\frac{x_0}{2}\frac{2-u^2-v^2}{3-2u-v^2}\right]+O(\epsilon^2)$ in the integrand.

We want the $1/\epsilon^2$ and $1/\epsilon$ pieces of $\Gamma_b^{(1,1)}$. One pole in ϵ occurs in R. One can show that $C_b = O(1/\epsilon)$ and, after some algebra, that $\mathcal{D}_b$ is not singular in ϵ. We obtain to this order

$$-i\Gamma_b^{(1,1)}(-E_N,x_0) = -i\Gamma_b^{(1,1)}(-E_N,0) + \frac{4r_0^4E_N}{(8\pi\alpha_0')^4}\frac{1}{\epsilon^2}\left[\frac{x_0}{2}+\epsilon\widetilde{\Gamma}_b\right] \tag{2.13}$$

where

$$\widetilde{\Gamma}_b = \frac{x_0}{2}\left[\frac{19}{12}+\ln 6+\delta-f(x_0)+\ln\frac{\alpha_0'}{E_N}\right]-f(x_0) \tag{2.14}$$

and $\Gamma_b^{(1,1)}(-E_N,0)$ is given in Eq.4D of BD. The remarks for $\Gamma_a^{(1,1)}$ after Eq.2.7 also hold for $\Gamma_b^{(1,1)}$.

To the above we must add the one-loop $\Gamma^{(1,1)}$ at $t\neq 0$ expanded to $O(1/\epsilon)$ and $O(\epsilon^0)$. This is given in Eq.A.7.

We now define the renormalization function $Z_3(g_0,x_0)$. Replacing r_0 by the dimensionless coupling g_0 we have

$$Z_3^{-1}(g_0,x_0) = \frac{\partial}{\partial E_N}\left[-i\Gamma^{(1,1)}(-E_N,k_N^2)\right]_{r_0,\alpha_0',k_N^2} \tag{2.15}$$

$$= 1 + a_2(x_0)\frac{g_0^2}{\epsilon} + a_4(x_0)\frac{g_0^4}{\epsilon^2} \tag{2.16}$$

We also define $Z_2(g_0, x_0)$ by

$$(Z_2 Z_3)^{-1}(g_0, x_0) = \frac{1}{\alpha_0'} \frac{\partial}{\partial k_N^2} \left[-i \Gamma^{(1,1)}(-E_N, k_N^2) \right]_{r_0, \alpha_0', E_N} \tag{2.17}$$

$$= 1 + c_2(x_0) \frac{g_0^2}{\epsilon} + c_4(x_0) \frac{g_0^4}{\epsilon^2} \tag{2.18}$$

The renormalized slope parameter α' is given by $\alpha' = \alpha_0' Z_2^{-1}$. To $O(\epsilon)$ we write

$$a_2(x_0) = 2c_2(x_0) = a_2 + \epsilon\, \alpha_2(x_0) \tag{2.19}$$

$$a_4(x_0) = a_4 + \epsilon\, \alpha_4(x_0) \tag{2.20}$$

$$c_4(x_0) = c_4 + \epsilon\, \gamma_4(x_0) \tag{2.21}$$

The constants a_2, a_4, and c_4 can be read off Eqns. 63, 65 of BO. The x_0 -dependent functions are

$$(8\pi)^2 \alpha_2(x_0) = \frac{1}{2} f(x_0) \tag{2.22}$$

$$(8\pi)^4 \alpha_4(x_0) = -\frac{7}{2} f(x_0) + \frac{x_0}{4} f'(x_0) \tag{2.23}$$

$$(8\pi)^4 \gamma_4(x_0) = \frac{-13}{8} f(x_0) + \frac{x_0}{8} f'(x_0) \tag{2.24}$$

It is useful to note the simple identity $1 - x_0 f' = 2f'$ in deriving these results.

Next we need the vertex graphs $\Gamma^{(1,2)}$ in one and two loops. We adopt the same convention as ABBS, using a k_N -independent renormalization point (i.e. $\vec{k}_i = 0$ for all three legs of $\Gamma^{(1,2)}$). Actually, any renormalization point can be chosen. We are not sure if our results are independent of this or not, but the extension to $\vec{k}_i \neq 0$ involves too much complexity. Even if there would be some dependence on this convention, one might take some comfort from calculations in the loop expansion (with $D = 2$ kept fixed). There it was shown that this dependence is weak. Specifically one observes "twisted fans" which give critical exponents within a factor of 2 of those of the ϵ-expansion [7].

So we use the results for $\Gamma^{(1,2)}$ from BD, normalized at N.P. $E_1 = -E_N = 2E_{2,3}$. Graph $\Gamma_j^{(1,2)}$ must be corrected by a factor 1/2 from Eq.(57) in BD. This yields the function $Z_1(g_0)$ as

$$Z_1^{-1}(g_0) \doteq r_0^{-1}(2\pi)^{\frac{D+1}{2}} \left.\Gamma^{(1,2)}\right|_{N.P.} \tag{2.25}$$

$$= 1 + d_3 \frac{g_0^2}{\epsilon} + d_5 \frac{g_0^4}{\epsilon^2} \tag{2.26}$$

Here d_3 and d_5 are constants given in Eqs. A1, A8. Since the renormalized coupling r is defined as

$$r = (2\pi)^{\frac{D+1}{2}} \left.\Gamma_R^{(1,2)}\right|_{N.P.} \tag{2.27}$$

we obtain $Z_1^{-1} = Z_3^{-3/2}\, r/r_0$. Using the definitions of g_0 and $g = r(\alpha')^{-D/4} E_N^{-\epsilon/4}$ along with $\alpha' = \alpha_0' Z_2^{-1}$ yields $Z = Z_1^{-1} Z_2^{D/4} Z_3^{3/2}$ where we have defined

$$Z(g_0, x_0) = \frac{g}{g_0} \tag{2.28}$$

$$= 1 + W(x_0)\frac{g_0^2}{\epsilon} + W_4(x_0)\frac{g_0^4}{\epsilon^2} \tag{2.29}$$

We obtain

$$W(x_0) = W - \epsilon\,\alpha_2(x_0) \tag{2.30}$$

$$W_4(x_0) = W_4 + \epsilon\,\hat{W}_4(x_0) \tag{2.31}$$

Here W and W_4 are given by Eq.69 of BD and Eq. A2. Also to this order

$$\hat{W}_4(x_0) = \alpha_2'\left(-d_3 + \frac{7a_2}{4}\right) - \frac{\alpha_4}{2} - \gamma_4' \tag{2.32}$$

$$= (8\pi)^{-4}\left[\frac{9}{2}f(x_0) - \frac{x_0}{4}f'(x_0)\right]$$

(2.33)

If a $\vec{k}_i \neq 0$ renormalization point for $\Gamma^{(1,2)}$ were chosen, the k-dependence of d_5 would enter in $\hat{w}_4$. The inverse relation $g_0 = g_0(g)$ is given by

$$g_0 = g - W(x_0)\frac{g^3}{\epsilon} + \left[3w^2(x_0) - W_4(x_0)\right]\frac{g^5}{\epsilon^2} \tag{2.34}$$

Once this relation is inserted, the renormalization of g_0 has been accomplished.

The renormalization of X_0 to $X = \frac{\kappa' k_N^2}{\mathcal{E}_N}$. is given by

$$X_0 = X Z_2 \tag{2.35}$$

$$= X\left[1 - \frac{g_0^2}{2(8\pi)^2}\left(\frac{1}{\epsilon} + z_2(x_0)\right) + O\left(g_0^4\epsilon^{-M}, \epsilon g_0^2\right)\right] \tag{2.36}$$

where $M \leq 2$ and

$$2 z_2(x_0) = 3\ln 2 + \delta - f(x_0) \tag{2.37}$$

To this order we can replace g_0, X_0 by g, X to get

$$X_0 = X\left[1 - \frac{g^2}{2(8\pi)^2}\left(\frac{1}{\epsilon} + z_2(x)\right)\right] \tag{2.38}$$

We will need the renormalization of $f(x_0)$ and $x_0 f'(x_0)$. These are given by

$$f(x_0) = f(x) - \frac{1}{2}x f'(x)\frac{g^2}{(8\pi)^2}\left[\frac{1}{\epsilon} + z_2(x)\right] \tag{2.39}$$

and the formula

$$x_0 f'(x_0) + x_0 f^{(2)}(x_0)\frac{g^2}{\epsilon(8\pi)^2} = x f'(x) - x f^{(2)}(x) z_2(x)\frac{g^2}{(8\pi)^2} \tag{2.40}$$

This ends the perturbative calculations. We now proceed to the renormalization group, which will elevate these results to a non-perturbative status.

III. THE RENORMALIZATION GROUP FUNCTIONS

Since there are two variables (E_N and k_N^2) which are being treated on an equal footing, we shall have to take partial derivatives with respect to both of them. We note the identities

$$E_N \frac{\partial g_0^\rho}{\partial E_N} = -\frac{\epsilon\rho}{4} g_0^\rho \tag{3.1}$$

and

$$\left(E_N \frac{\partial}{\partial E_N} + k_N^2 \frac{\partial}{\partial k_N^2} \right) F(x_0) = 0 \tag{3.2}$$

where F is any function of $x_0 = \alpha_0' k_N^2 / E_N$.
Define, following ABBS,

$$\beta_E = E_N \frac{\partial g}{\partial E_N} \tag{3.3}$$

$$\beta_k = k_N^2 \frac{\partial g}{\partial k_N^2} \tag{3.4}$$

where the derivatives are to be taken at fixed bare parameters r_0, α_0' .
Then

$$\beta_E + \beta_k = \left(E_N \frac{\partial}{\partial E_N} + k_N^2 \frac{\partial}{\partial k_N^2} \right) g\,(g_0, x_0) \tag{3.5}$$

$$= E_N \frac{\partial}{\partial E_N} g(g_0, x_0) \Big|_{x_0} \tag{3.6}$$

Thus $\beta_E + \beta_k$ has the same functional form as the $\hat{\beta}$ function of the $k_N = 0$ calculation, except that X_0 is held fixed at a nonzero valve. This means that Eq.72 of BD is valid with W and W_4 replaced by $W(X_0)$ and $W_4(X_0)$, i.e.

$$\beta_E + \beta_k = -\frac{\epsilon g}{4}\left\{1 + 2W(x_0)\frac{g^2}{\epsilon} + \left[4W_4(x_0) - 6W^2(x_0)\right]\frac{g^4}{\epsilon^2}\right\} \tag{3.7}$$

$$= \beta(g) + \tfrac{1}{2}\alpha_2(x_0)\epsilon g^3 - \left[\hat{W}_4(x_0) + 3W\alpha_2(x_0)\right]g^5 \tag{3.8}$$

Here $\beta(g)$ is the Gell-Mann-Low function at $X_0 = 0$, given in Eq. A3. The above formula has been renormalized in g but not in X. Nevertheless, there are no singularities in ϵ for fixed g in $\beta_E + \beta_k$. This must be the case for the β function of any renormalized theory, and the $g_0 \to g$ renormalization is enough to ensure it for $\beta_E + \beta_k$.

If we were to imagine using g and X_0 as our independent variables instead of g and x, then $\beta_E + \beta_k$ as given above would be the appropriate β function. However the results for the critical coupling $g_1(X_0)$ at which $\beta_E + \beta_k = 0$ would contain a dependence on $X_0 f'(x_0)$. This would also appear in the critical exponents which, however, should be independent of X_0 [3] (otherwise the general form for the scaling law is violated). The $X_0 \to X$ renormalization just compensates for this unwanted dependence as we shall see.

Performing the $X_0 \to X$ renormalization yields

$$\beta_E + \beta_k = \beta(g) + \frac{1}{4}f(x)\frac{\epsilon g^3}{(8\pi)^2} + \frac{1}{8}xf'(x)\frac{g^5}{(8\pi)^4} \tag{3.9}$$

Two points are noteworthy. The $g_0 \to g$ renormalization removed the $O(fg^5)$ term and the $X_0 \to X$ renormalization of the $O(fg^3)$ term halved the $O(g^5)$ coefficient of $\beta_E + \beta_k$.

The behaviour of either β_E or β_k separately is more singular than that of the sum $\beta_E + \beta_k$. This is because the derivative $\partial/\partial E_N$ of $\hat{W}_4(x_0) g_0^5/\epsilon$ contains $\hat{W}_4'(x_0) g_0^5/\epsilon$. The extra ϵ arising from the derivative of g_0^5 is lost (cf. Eq.3.1), and the result is singular as $\epsilon \to 0$ for fixed g. Specifically,

$$\beta_k = x_0 \left[-\alpha_2'(x_0) g_0^3 + \hat{W}_4'(x_0) g_0^5/\epsilon \right] \tag{3.10}$$

where, using the identity $(x_0 f')' = 2f'^2$ we have

$$(8\pi)^4 \hat{W}_4'(x_0) = \frac{9}{2} f'(x_0) - \frac{1}{2} f'^2(x_0) \tag{3.11}$$

Substituting in g_0 and x_0 as functions of g and x then gives

$$\beta_k = -\tfrac{1}{2} x f'(x) \left\{ \frac{g^3}{(8\pi)^2} + \frac{3}{2}\left[\frac{15}{4} + 5\ln 2 + 3\delta + f(x) - \frac{2}{3} z_2(x) f'(x)\right] \frac{g^5}{(8\pi)^4} \right\} \tag{3.12}$$

The singularities of β_k have cancelled as follows : the $g_0 \to g$ renormalization removes the f' term in $\hat{W}_4'$ and the $x_0 \to x$ renormalization removes the f'^2 term in $\hat{W}_4'$. The $O(g^5)$ term of β_k arises from the $O(\epsilon)$ term in $W(x_0)$ via the g_0^3 renormalization (Eq. 2.34) and from the $O(z_2 g^5)$ term in the $x_0 \to x$ renormalization of $x_0 f'(x_0) g^3$.

We now turn to the critical exponent functions. As in ABBS, we define

$$\gamma_E = \frac{\partial \ln Z_3}{\partial \ln E_N} \tag{3.13}$$

$$\tau_E = - \frac{\partial \ln Z_2}{\partial \ln E_N} \tag{3.14}$$

with γ_k and τ_k defined similarly by replacing $E_N \to k_N^2$. Derivatives are as usual at fixed r_0, α_0'. Inserting the perturbative results for Z_2 and Z_3 yields γ_E, γ_k, τ_E, and τ_k. First,

$$\gamma_E + \gamma_k = \gamma(g^2) + \tfrac{1}{2}\alpha_2(x_0)\epsilon g^2 + \left[\alpha_4(x_0) - w\alpha_2(x_0)\right] g^4 \tag{3.15}$$

$$= \gamma(g^2) + \tfrac{1}{4}f(x)\frac{\epsilon g^2}{(8\pi)^2} + \left[-2f(x) + \tfrac{1}{8}xf'(x)\right]\frac{g^4}{(8\pi)^4} \tag{3.16}$$

Here $\gamma(g^2)$ is the $x_0 = 0$ critical exponent function in Eq.75 of BD. Note that $\gamma_E + \gamma_k$ has no singular term in ϵ for fixed g, x_0. However γ_E or γ_k separately does have such a term, which only disappears after the $x_0 \to x$ renormalization. Specifically,

$$\gamma_k = -x_0\alpha_2'(x_0)g^2 + x_0\left\{-\alpha_4'(x_0) + \left[a_2 + 2w - \epsilon\alpha_2(x_0)\right]\alpha_2'(x_0)\right\} g^4/\epsilon \tag{3.17}$$

$$= -\tfrac{1}{2}xf'(x)\left\{\frac{g^2}{(8\pi)^2} + \tfrac{1}{4}\left[15 + 26\ln 2 + 14\delta + 2f(x) - 4g_2(x)f'(x)\right]\frac{g^4}{(8\pi)^4}\right\} \tag{3.18}$$

The singularities in γ_k cancel in a similar way to those of β_k, with the $x_0 \to x$ renormalization eliminating the $O(x_0 f'^2 g^4/\epsilon)$ term. The $O(g^4)$ term in γ_k arises from the $O(\epsilon)$ term in $a_2 + 2w - \epsilon\alpha_2$ and from the $g_2 g^4$ term arising from the renormalization of $x_0 f'(x_0) g^2$.

The results for $\tau_E + \tau_k$ are

$$\tau_E + \tau_k = \frac{\gamma}{\alpha'}(g^2) + \tfrac{1}{4}\alpha_2(x_0)\epsilon g^2 + \left[-\gamma_4(x_0) + d_4(x_0) - \tfrac{1}{4}(a_2 + 2w)\alpha_2(x_0)\right] g^4 \tag{3.19}$$

$$= \frac{\gamma}{\alpha'}(g^2) + \tfrac{1}{8}f(x)\frac{\epsilon g^2}{(8\pi)^2} + \left[-f(x) + \tfrac{1}{16}xf'(x)\right]\frac{g^4}{(8\pi)^4} \tag{3.20}$$

where $\gamma/\alpha'(g^2)$ is given in Eq.78 of BD. Note that the x-dependent terms of $\tau_E + \tau_k$ are just 1/2 those of $\gamma_E + \gamma_k$. Finally

$$\tau_k = -\frac{1}{2} x_0 \alpha_2'(x_0) g^2 + x_0 \left\{ \gamma_4'(x_0) - \alpha_4'(x_0) + \left[\frac{3}{4} a_2 + W - \frac{1}{4} \epsilon \alpha_2(x_0) \right] \alpha_2'(x_0) \right\} \frac{g^4}{\epsilon} \tag{3.21}$$

$$= -\frac{1}{4} x f'(x) \left\{ \frac{g^2}{(8\pi)^2} + \frac{1}{4} \left[15 + 29 \ln 2 + 15\delta + f(x) - 4 z_2(x) f'(x) \right] \frac{g^4}{(8\pi)^4} \right\} \tag{3.22}$$

Now we proceed to the quantities $\bar{\gamma}$, $\bar{\tau}$, and $\bar{\beta}$ which contain information we need to utilize the renormalization group. We use g, x as independent variables. We shall point out as we go along what would have happened if g and x_0 were chosen as independent instead of g and x .

By using the chain rule for γ_E

$$\gamma_E = \frac{\partial \ln Z_3}{\partial \ln g} \frac{\partial \ln g}{\partial \ln \epsilon_N} + \frac{\partial \ln Z_3}{\partial \ln x} \frac{\partial \ln x}{\partial \ln \epsilon_N}$$

and inserting $x = x_0 Z_2^{-1}$ one obtains

$$\gamma_E = \frac{\beta_E}{g} \frac{\partial \ln Z_3}{\partial \ln g} + (\tau_E - 1) \frac{\partial \ln Z_3}{\partial \ln x} \tag{3.23}$$

A similar expression for γ_k then produces

$$\frac{\partial \ln Z_3}{\partial \ln g} = \frac{g \bar{\gamma}}{\bar{\beta}} \tag{3.24}$$

where

$$\bar{\beta} = \beta_E + \beta_k + (\beta_E \tau_k - \beta_k \tau_E) \tag{3.25}$$

$$\bar{\gamma} = \gamma_E + \gamma_k + (\gamma_E \tau_k - \gamma_k \tau_E) \tag{3.26}$$

Hence $\bar{\beta}$ is the β-function for the (g,x) formalism. The cross term $\beta_E \tau_k - \beta_k \tau_E$ is not present in the (g, x_0) formalism.

It does what is needed to make the critical exponents independent of t.

The function $\bar{\gamma}$ simplifies in this order since the cross term involves only the $O(g^2)$ terms in γ_E, γ_k, τ_E, and τ_k. Since to that order $\tau_i = \frac{1}{2}\gamma_i$ $(i = E, k)$ we get

$$\bar{\gamma} = \gamma_E + \gamma_k \tag{3.27}$$

Further, one can show that the relevant function involving τ_E and τ_k is

$$\bar{\tau} = \tau_E + \tau_k \tag{3.28}$$

for which

$$\frac{\partial \ln Z_2}{\partial \ln g} = -\frac{g\bar{\tau}}{\bar{\beta}} \tag{3.29}$$

The functions $\bar{\gamma}$ and $\bar{\tau}$ have already been evaluated in Eqs. 3.16, 3.20. The $\bar{\beta}$ function is given by

$$\bar{\beta} = -\frac{\epsilon g}{4}\left\{1 + 2\bar{W}(x_0)\frac{g^2}{\epsilon} + \left[4\bar{W}_4(x_0) - 6\bar{W}^2(x_0)\right]\frac{g^4}{\epsilon^2}\right\} \tag{3.30}$$

where to $O(\epsilon)$

$$\bar{W}(x_0) = W - \frac{\epsilon}{2(8\pi)^2}\left[f(x_0) + \frac{1}{4}x_0 f'(x_0)\right] \tag{3.31}$$

$$\bar{W}_4(x_0) = W_4 + \frac{\epsilon}{(8\pi)^4}\left[\frac{9}{2}f(x_0) + \frac{11}{8}x_0 f'(x_0) - \frac{1}{16}x_0 f'^2(x_0)\right] \tag{3.32}$$

Upon renormalization, the $x_0 f'^2$ term in $\bar{W}_4$ is cancelled and the $x_0 f'$ term changed. After renormalization, Eq. 3.30 remains valid if $\bar{W}(x_0)$ is replaced by $\bar{W}(x)$ and if $\bar{W}_4(x_0)$ is replaced by $\bar{W}_4(x)$ which is now redefined as

$$\bar{W}_4(x) = W_4 + \frac{\epsilon}{(8\pi)^4}\left[\frac{9}{2}f(x) + \frac{3}{2}x f'(x)\right] \tag{3.32}$$

Note that $4\overline{w}_4 - 6\overline{w}^2$ is actually $O(\epsilon)$ so that $\overline{\beta}$ is not singular in ϵ.

We are now ready to calculate the critical coupling $g_1(x)$ at which $\overline{\beta} = 0$. We obtain

$$\frac{g_1^2(x)}{(8\pi)^2} = \frac{g_1^2}{(8\pi)^2} + \frac{\epsilon^2}{(12)^3}\left[-48 f(x)\right] \tag{3.34}$$

Here g_1 is the zero of $\beta(g)$ (cf. Eqs. A.3, A.4). If we had used the $(g_1\ x_0)$ formalism without the cross term in $\overline{\beta}$, $g_1^2(x_0)$ would have been given by Eq. 3.34 with $x \to x_0$ and with an extra term $-8 x_0 f'(x_0)$ in the bracket.

We are now ready to evaluate $\overline{\gamma}$ and $\overline{\tau}$ at $g = g_1(x)$; we call these quantities $\overline{\gamma}_1$ and $\overline{\tau}_1$. We obtain

$$\overline{\gamma}_1 = \gamma + \frac{\epsilon^2}{(12)^3}\left[6x f'(x)\right] \tag{3.35}$$

$$\overline{\tau}_1 = \gamma/\alpha' + \frac{\epsilon^2}{(12)^3}\left[3x f'(x)\right] \tag{3.36}$$

Here γ and γ/α' are given in Eqs. A.5, A.6. They are critical exponents. The x-dependence in $\overline{\gamma}_1$ and $\overline{\tau}_1$ is correct ; these quantities are not critical exponents.

We also need $\overline{\beta}_1' \equiv (\partial\overline{\beta}/\partial g)_{g=g_1(x)}$. It is

$$\overline{\beta}_1' = -\overline{w}(x) g_1^2(x) - \left[4\overline{w}_4(x) - 6\overline{w}^2(x)\right] g_1^4(x)/\epsilon \tag{3.37}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon^2}{(12)^3}\left[-471 - 318 \ln\frac{4}{3} - 36 x f'(x)\right] \tag{3.38}$$

We are finally ready to obtain the quantities C, C_2, and c_3 which will appear in the scaling law for $\Gamma^{(1,1)}$. These are defined as in ABBS,

$$c_3 = \frac{\bar{\gamma}_1}{\bar{\beta}_1'} \quad . \tag{3.39}$$

$$c_2 = \frac{-\bar{T}_1}{\bar{\beta}_1'} \tag{3.40}$$

and

$$c = \frac{\epsilon}{4\,\bar{\beta}_1'}\,(1 + T_h)_1 \tag{3.41}$$

In order to get c to $O(\epsilon)$ we need $(T_h)_1$ to $O(\epsilon)$.

$$(T_h)_1 = -\frac{\epsilon}{24}\,x f'(x) + O(\epsilon^2) \tag{3.42}$$

We obtain to $O(\epsilon)$

$$c_3 = -\frac{1}{6} + \frac{\epsilon}{(12)^3}\left[-194 - 428 \ln 4/3\right] \tag{3.43}$$

$$c_2 = \frac{1}{12} + \frac{\epsilon}{(12)^3}\left[118 + 112 \ln 4/3\right] \tag{3.44}$$

$$c = \frac{1}{2} + \frac{\epsilon}{(12)^3}\left[471 + 318 \ln 4/3\right] \tag{3.45}$$

We have also

$$\frac{\epsilon c_3}{4c} = \gamma \tag{3.46}$$

$$\frac{\epsilon c_2}{4c} = -\varphi/\alpha' \tag{3.47}$$

These results are what we have been looking for. The $O(\epsilon^0)$ pieces of c , c_2 , and c_3 agree with ABBS and the $O(\epsilon)$ terms are new. No x-dependence is present. This must be the case, as demonstrated in the next section. If we had used the (g, x_0) formalism, some x_0 -dependence would have been present in the c's.

We shall also need $(\gamma_\epsilon)_1$, $(T_\epsilon)_1$, $(\beta_\epsilon/g)_1$, and $(\partial\beta_\epsilon/\partial g)_1$ to $O(\epsilon^2)$ and $(g\,\partial T_\epsilon/\partial g)_1$ to $O(\epsilon)$.

These are given by

$$\left(\gamma_\epsilon\right)_1 = \gamma + \frac{\epsilon}{12} x f'(x) + \frac{\epsilon^2 x f'(x)}{(12)^3}\left[k_\gamma - 12 f(x) - 24 z_2(x) f'(x)\right] \tag{3.48}$$

$$\left(\tau_\epsilon\right)_1 = \varphi/\omega' + \frac{\epsilon}{24} x f'(x) + \frac{\epsilon^2 x f'(x)}{(12)^3}\left[k_\tau - 9 f(x) - 12 z_2(x) f'(x)\right] \tag{3.49}$$

where

$$k_\gamma = 142 \ln 2 - 53 \ln 3 + \frac{169}{2} + 12\delta \tag{3.50}$$

$$k_\tau = 80 \ln 2 - \frac{53}{2} \ln 3 + \frac{169}{4} + 9\delta \tag{3.51}$$

Also

$$\left(\frac{\beta_\epsilon}{g}\right)_1 = \frac{\epsilon}{12} x f'(x) + \frac{\epsilon^2 x f'(x)}{(12)^3}\left[k_\beta + 12 f(x) - 24 z_2(x) f'(x)\right] \tag{3.52}$$

$$\left(\frac{\partial \beta_\epsilon}{\partial g}\right)_1 = \epsilon(1-c) + \frac{\epsilon}{4} x f'(x) + \frac{\epsilon^2 x f'(x)}{(12)^3}\left[k'_\beta + 108 f(x) - 120 z_2(x) f'(x)\right] \tag{3.53}$$

where

$$k_\beta = 166 \ln 2 - 53 \ln 3 + \frac{259}{2} + 36\delta \tag{3.54}$$

$$k'_\beta = 858 \ln 2 - 159 \ln 3 + \frac{1341}{2} + 324\delta \tag{3.55}$$

Finally

$$\left(g \frac{\partial \tau_\epsilon}{\partial g}\right)_1 = -\frac{\epsilon}{6} f'(x) + O(\epsilon^2) \tag{3.50}$$

We are now ready to examine the ABBS equations and the scaling law for $\Gamma^{(1,1)}$.

IV. THE ABBS EQUATIONS

The ABBS equations are algebraic identities. They assume only that $\bar{\beta}$ has a zero $g_1(x)$. Consider the form for $Z_3(g,x)$ obtained from Eq. 3.24.

$$\ln Z_3(g,x) = \int_0^g \frac{\bar{\gamma}(g',x)}{\bar{\beta}(g',x)}\, dg' \tag{4.1}$$

Expanding $\bar{\gamma}$ and $\bar{\beta}$ around $g_1(x)$, Eq. 4.1 yields

$$\ln Z_3(g,x) = \frac{\bar{\gamma}_1}{\bar{\beta}_1'} \ln\left(1 - \frac{g}{g_1(x)}\right) + \sum_{n=1}^{\infty} a_n^{(3)} \left[(g-g_1(x))^n - (-1)^n g_1^n(x)\right] \tag{4.2}$$

where (derivatives are with respect to g)

$$a_1^{(3)} = \frac{\bar{\gamma}_1'}{\bar{\beta}_1'} - \frac{\bar{\beta}_1''\, \bar{\gamma}_1}{2\,\bar{\beta}_1'^{2}} \tag{4.3}$$

Thus $c_3 = \bar{\gamma}_1/\bar{\beta}_1'$ is the coefficient of $\ln(1-g/g_1)$. The fact that c_3 is independent of x is established using the identity analogous to Eq. 3.24.

$$\frac{\partial \ln Z_3}{\partial \ln x} = \frac{\gamma_k \beta_E - \gamma_E \beta_k}{\bar{\beta}} \tag{4.4}$$

which has no $\ln(1-g/g_1)$ term.

The derivative of Eq. 4.2 with respect to $\ln x$ is equal to Eq. 4.4 expanded around $g = g_1(x)$. Equating the pole $(g-g_1)^{-1}$ pieces yields

$$\frac{d \ln g_1(x)}{d \ln x} = \left[\frac{-\beta_E}{g(1-\tau_E)}\right]_1 \tag{4.5}$$

The $O(g-g_1)$ term involves $a_1^{(3)}$. This is eliminated using Eq. 4.3. After some algebra we obtain the ABBS equation for $\bar{Z}_3(x)$

$$\frac{d\ln \bar{Z}_3(x)}{d\ln x} = c_3\left[\frac{\partial}{\partial g}\left(\frac{\beta_E}{1-\tau_E}\right)\right]_1 - \left(\frac{\gamma_E}{1-\tau_E}\right)_1 \tag{4.6}$$

where $\bar{Z}_3(x)$ is defined as

$$\ln \bar{Z}_3(x) = -c_3 \ln g_1(x) - \sum_{n=1}^{\infty} a_n^{(3)} (-1)^n g_1^n(x) \tag{4.7}$$

Similarly defined quantities $\bar{Z}_2(x)$ and $\bar{Z}(x)$ obey similar equations (cf. Eq. 6a of ABBS). These equations yield the following sum rules

$$\frac{d}{d\ln x} \ln\left[\bar{Z}_3(x)\left(\frac{g_1(x)}{\bar{Z}(x)}\right)^{c_3/c}\right] = \left(\frac{\gamma-\gamma_E}{1-\tau_E}\right)_1 \tag{4.8}$$

$$\frac{d}{d\ln x} \ln\left[\bar{Z}_2(x)\left(\frac{g_1(x)}{\bar{Z}(x)}\right)^{c_2/c}\right] = \left(\frac{\tau_E - \gamma/\mu'}{1-\tau_E}\right)_1 \tag{4.9}$$

These equations enter directly in the final results for $\Gamma^{(1,1)}$.

Now consider Eq. 4.5 evaluated to $O(\epsilon)$. It is useful to write $d\ln g_1(x)/d\ln x = (x/2g_1^2(x))\,dg_1^2(x)/dx$. The right hand side (RHS) of Eq. 4.5 gets a contribution of $O(\epsilon)$ from the one-loop graphs. However the one-loop expression for $g_1(x)$ does not depend on x at all, so the left hand side (LHS) of Eq. 4.5 is zero to this approximation. This apparent dilemma is actually the secret of why the ABBS equations are useful. If we simply set (as did ABBS) the LHS of Eq. 4.5 equal to the $O(\epsilon)$ RHS and then solve for $g_1(x)$, a non-perturbative result in ϵ is obtained of the form $g_1^2(x) = O\left[\epsilon\,(1+x/2)^{-\epsilon/6}\right]$. Expanding this yields the first nontrivial x-dependence at the $O(\epsilon^2)$ level for $g_1^2(x)$. But we have just finished calculating this quantity explicitly (cf. Eq. 3.34). This leads to a consistency check on Eq. 4.5.

LHS of Eq 4.5 $= -\frac{\epsilon}{12}\, x f'(x)$ from the two-loop $g_1^2(x)$

RHS of Eq 4.5 $= -\frac{\epsilon}{12}\, x f'(x)$ from the one-loop $[-\beta_\epsilon/g]_1$

Indeed, the two sides of Eq. 4.5 are equal in $O(\epsilon)$ so the consistency check is satisfied.

Since we have evaluated the two-loop $(\beta_\epsilon/g)_1$, we actually predict the 3-loop x-dependence of $g_1(x)$. We obtain

$$\frac{d \ln g_1(x)}{dx} = -\frac{\epsilon f'}{12} + \frac{\epsilon^2 f'}{(12)^3}\left[-k_\beta - 12 f + 12 f'(\nu - f)\right] \qquad (4.10)$$

where, we remind the reader, k_β is given in Eq. 3.54, f and f' are $\ln(1+x/2)$ and $(2+x)^{-1}$, and

$$\nu = 1 + 3\ln 2 + \delta \qquad (4.11)$$

We can integrate this equation simply. Using $f'' = -f'^2$ we have

$$\int_0^x dx\, f'^2 = \frac{1}{2} - \frac{1}{2+x} = f f' + \int_0^x dx\, f f'^2$$

We obtain

$$g_1(x) = g_1\,(1+x/2)^{-\frac{\epsilon}{12} + \frac{\epsilon^2}{(12)^3}\left[-k_\beta - 6\ln(1+\frac{x}{2})\right]} \exp\left[\frac{\epsilon^2}{(12)^3} K(x)\right] \qquad (4.12)$$

where

$$\frac{1}{12} K(x) = (\nu - 1)\left[\frac{1}{2} - \frac{1}{2+x}\right] + \frac{\ln(1+\frac{x}{2})}{2+x} \qquad (4.13)$$

The first factor in Eq. 4.12 is the ABBS term. The other pieces are new.

The ABBS *equations for the* $\bar{Z}_i$ *(meaning* $\bar{Z}_3$, $\bar{Z}_2$, and $\bar{Z}$) are of the form Eq. 4.10 for $g_1(x)$. We get

$$\frac{d\ln \bar{Z}_i(x)}{dx} = \epsilon \ell_i^{(1)} f' + \frac{\epsilon^2}{(12)^3}\left[\ell_i^{(2)} f' + \ell_i^{(3)} ff' + a_i f'^2 + b_i ff'^2\right] \tag{4.14}$$

The numbers $\ell_i^{(1)}$, $\ell_i^{(2)}$, $\ell_i^{(3)}$, a_i, and b_i are given in Table 1. Defining

$$\mathcal{K}_i(x) = (a_i + b_i)\left[\frac{1}{2} - \frac{1}{2+x}\right] - \frac{b_i}{2+x}\ln\left(1+\frac{x}{2}\right) \tag{4.15}$$

we obtain the solution for $\bar{Z}_i(x)$ as

$$\bar{Z}_i(x) = \bar{Z}_i(0)\left(1+\frac{x}{2}\right)^{\epsilon \ell_i^{(1)} + \frac{\epsilon^2}{(12)^3}\left[\ell_i^{(2)} + \frac{1}{2}\ell_i^{(3)} \ln\left(1+\frac{x}{2}\right)\right]} \exp\left[\frac{\epsilon^2}{(12)^3} K_i(x)\right] \tag{4.16}$$

These results generalize Eq. 74 of ABBS. The x-dependence is quite complex. We call the attention of the reader to the presence of both δ and f in Eq. 4.16. No symmetry of the form $\delta - f$ is present, although it is there in the sum rules Eqns. 4.8, 4.9. The presence of δ at all is quite surprising. The critical exponents γ and γ/ν (or c, c_2, c_3) do not depend on δ. In BD, this was shown to be a consequence of an invariance of these quantities under a certain coupling constant rescaling. We shall examine this in Sect. VI. The result is that the δ dependence seems to remain. We do not understand this point fully. It may be simply giving the δ dependence of the as yet uncalculated $O(\epsilon^3)$ terms. It may be connected with the $x_0 \to x$ renormalization of singular objects like γ_k, which we shall discuss in the next section. Finally, the δ dependence may be illusory and cancel out of the scaling law for $\Gamma^{(1,1)}$ once the inversion formula for $X = X(x_0)$ is inserted (cf. Sect. VI below).

We close this section with the remark that the leading approximation to $\bar{Z}_i(x)$

$$\bar{Z}_i(x) \approx \left[g_1(x)\right]^{-c_i} \tag{4.17}$$

does not satisfy the ABBS equations. Thus

$$\left[g_1(x)\right]^{-c_3} \approx g_1^{1/6}\left(1+\frac{x}{2}\right)^{-\epsilon/72} \tag{4.18}$$

has the $O(\epsilon)$ exponent off by a factor 9 from that of $\bar{Z}_3(x)$. Thus the non leading terms from the sum in Eq. 4.7 are quite important in satisfying the ABBS equations.

V. A POTENTIAL PROBLEM

We now raise an issue that we are unable to resolve. We do not know whether it poses a real problem or not.

Consider the expressions for β_k/g , γ_k , and T_k before the $X_0 \to X$ renormalization (cf. Eqs. 3.10, 3.17, 3.21). These all contain a singular term of $O(g^4/\epsilon)$. Now this term does disappear when the X_0 renormalization is performed. However, it is also clear that terms of $O(\epsilon)$ in $\hat{W}_4'$, α_4' , and γ_4' will contribute to $O(g^4)$ in these quantities. Such terms have not been included. They correspond to terms of $O(\epsilon^2)$ in $W_4(x_0)$ and arise from terms of $O(\epsilon^0)$ in the two-loop $\Gamma^{(1,1)}$, e.g. from D_b in $\Gamma_b^{(1,1)}$ (cf. remarks after Eq. 2.12). Ordinarily, these quantities would never enter in an $O(\epsilon^2)$ calculation of critical exponents but only in $O(\epsilon^3)$ where three loop graphs must also be included. In fact, for the well behaved quantities $\bar{\beta}$, $\bar{\gamma}$, and $\bar{T}$ this is the case, and no problem exists. Thus this issue does not arise for γ , γ/α' , c , c_2 and c_3 (note only the $O(g^2)$ term of T_k is needed to get c). The question only arises for the $O(g_1^4)$ terms in the ABBS equations due to the singular quantities β_k , γ_k , and T_k .

If the ABBS equations are well defined in the same sense as the critical exponents, then things like D_b cannot enter at all to $O(\epsilon^2)$. The resolution of this may be that, since the ABBS equations always involves setting $g = g_1(x)$, there could be singular three-loop contibutions of $O(g_1^6/\epsilon) = O(\epsilon^2)$ which cancel the $O(\epsilon^2)$ contribution from quantities like D_b . Clearly we cannot test this conjecture without performing the herculean task of going to three loops, and we have no intention of doing this. However there is no reason that forces us to be pessimistic about this possibility.

There are two points which we can invoke to be optimistic. First, the ABBS equations have already overcome a substantial hurdle in being potentially well defined at all. Consider what would have happened in the (g_1, x_*) formalism. Similar equations can be derived ; they differ from the ABBS equations in having the $(1-T_E)_1$ denominators replaced by 1. However they also involve the x_0 -unrenormalized quantities like γ_k, which have terms of $O(g_1^4/\epsilon) = O(\epsilon)$. That is, the two-loop graphs would produce $O(\epsilon)$ terms through the singular objects β_k, δ_k, T_k comparable to the one-loop graphs in the (g, x_0) equations. In fact, one would expect loops of arbitrary complexity to enter at $O(\epsilon)$. The ABBS (g, x_0) equations are thus quite sick, even though they are completely finite in ϵ. The $x_0 \rightarrow x$ renormalization cures this problem by eliminating the singular g_1^4/ϵ terms from β_k/g, γ_k, and T_k.

We can express this in a slightly different way. Suppose that ABBS, instead of replacing $x_0 f'(x_0)$ by $x f'(x)$ as they did in the $O(g^2)$ term of γ_k, had used the more complete renormalization formula Eq. 2.40. Then they would have generated an $O(g_1^4/\epsilon)$ term in $(\gamma_k)_1$, which would have modified their $O(\epsilon)$ results. In fact this does not happen, since as we have seen the singular $O(g_1^4/\epsilon)$ term from the two-loop $\Gamma^{(1,1)}$ just cancels this term. Thus the one-loop ABBS results awaited the two-loop calculation to be verified. Similarly, our two-loop calculation for the final scaling law for $\Gamma^{(1,1)}$ in principle awaits a three-loop calculation. In the absence of that we shall assume that the ABBS equations do give well-defined results, and that our retention of only the $O(\epsilon)$ terms in $W_4(x_0)$ is correct.

Finally, due to the sum rules Eqs. 4.8, 4.9, β_k does not enter the scaling law for $\Gamma^{(1,1)}$. Thus if there is a problem, it is confined to γ_k and T_k (or γ_E and T_E) when physical quantities are considered.

VI. THE SCALING LAW FOR $\Gamma^{(1,1)}$

We begin by writing $g = Z g_0$ near $g \approx g_1(x)$ with $Z(g,x) \approx \bar{Z}(x)[g_1(x)-g]^c$ approximating the RG solution for Z. We get

$$g_1(x) - g = E_N^{\epsilon/4c} \left[\frac{g_1(x) A}{\bar{Z}(x)} \right]^{1/c} \qquad (6.1)$$

where

$$A = (\alpha_0')^{D/4} \; r_0^{-1} \qquad (6.2)$$

This shows that we can send g to g_1 by sending $E_N \to 0$ if $c > 0$, i.e. by letting $g_0 \to \infty$. Also sending $g_0 \to g_1(x)$ produces $g \to 0$.

Eq. 6.1 can be inserted in the RG solutions for Z_2 and Z_3 with the result (i = 2,3)

$$Z_i(g,x) = E_N^{\epsilon c_i/4c} \; \bar{Z}_i(x) \left[\frac{g_1(x) A}{\bar{Z}(x)} \right]^{c_i/c} \qquad (6.3)$$

Actually this equation has a factor $\exp\left[\sum_{n=1}^{\infty} O(E_N^{\epsilon n/4c}) \right]$. This should be ignored. Recall that e^{-bj} has a Mellin transform proportional to $\theta(\ln s - b)$. Hence exponentials in E_N translate into finite energy effects. One would be suspicious of the validity of such terms in an asymptotic theory such as the one we are dealing with. ABBS also ignore these terms. Nevertheless, as noted at the end of Section IV, the nonleading corrections to the $[g_1(x)]^{-c_i}$ approximation of $\bar{Z}_i(x)$ are kept.

Now we shall reconstruct $\Gamma^{(1,1)}(E, k^2)$ through its derivatives expressed in terms of the scaling law forms for the Z_i which we have found. Writing the fixed-x integral

$$\Gamma^{(1,1)}(E,k^2) = \int_0^{-E} dE_N \left[\frac{\partial}{\partial E_N} \Gamma^{(1,1)}(-E_N, k_N^2) \Big|_{k_N^2} + \frac{\partial}{\partial k_N^2} \Gamma^{(1,1)}(-E_N, k_N^2) \Big|_{E_N} \frac{\partial k_N^2}{\partial E_N} \Big|_x \right] \tag{6.4}$$

evaluated at $k_N^2 = k^2$, inserting the definitions of Z_3 and Z_2 (Eqs. 2.15, 2.17) along with their scaling law forms eq. 6.3 implying the relations

$$\frac{\partial \ln Z_2}{\partial \ln E_N}\Big|_x = -\frac{\gamma}{\alpha'} \quad \text{and} \quad \alpha_0' \frac{\partial k_N^2}{\partial E_N}\Big|_x = x Z_2 \, (1 - \gamma/\alpha')$$

then produces

$$-i\Gamma^{(1,1)}(E,k^2) = [1 + x(1-\gamma/\alpha')] \int_0^{-E} dE_N \, Z_3^{-1}(g, x)$$

$$= \frac{(-E)^{1-\gamma}}{1-\gamma} [1 + x(1-\gamma/\alpha')] \, \overline{Z}_3^{-1}(x) \left[\frac{g_1(x) A}{\overline{Z}(x)} \right]^{-c_3/c} \tag{6.5}$$

where now x is redefined by replacing $k_N^2 \to k^2$ and $E_N \to -E$ viz

$$x = \frac{\alpha' k^2}{-E} \tag{6.6}$$

These equations must be supplemented by the inversion formula $X_0 = x Z_2$ which follows from Eqs. 6.3 and 3.47,

$$X_0 = x \, (-E)^{-\gamma/\alpha'} \, \overline{Z}_2(x) \left[\frac{g_1(x) A}{\overline{Z}(x)} \right]^{c_2/c} \tag{6.7}$$

where X_0 is now $-\alpha_0' k^2/E$.

We see that the sum rules Eq. 4.8, 4.9 are the relevant quantities needed to evaluate Eqs. 6.5 and 6.7. The scaling law and inversion formula thus depend on $(\gamma_E)_1$ and $(\tau_E)_1$, which are given in Eqs. 3.48

and 3.49. Integrating Eqs. 4.8 and 4.9 yields

$$\bar{Z}_3(x)\left[\frac{g_1(x)A}{\bar{Z}(x)}\right]^{c_3/c} = k_3\left(1+\frac{x}{2}\right)^{-\frac{\epsilon}{12}+\frac{\epsilon^2}{(12)^3}\left[-k_\gamma+6\ln\left(1+\frac{x}{2}\right)\right]} \exp\left[\frac{\epsilon^2}{(12)^3}\mathcal{K}(x)\right] \quad (6.8)$$

(6.8)

where k_γ is given in Eq. 3.50 and $\mathcal{K}(x)$ in Eq. 4.13. Also

$$k_3 \equiv \bar{Z}_3(0)\left[\frac{g_1 A}{\bar{Z}(0)}\right]^{c_3/c} \approx (g_1 A)^{c_3/c} \quad (6.9)$$

where we used $\bar{Z}_3(0) \approx g_1^{-c_3}$ and $\bar{Z}(0) \approx g_1^{-c}$ in Eq. 6.9. See the footnote. Further

$$\bar{Z}_2(x)\left[\frac{g_1(x)A}{\bar{Z}(x)}\right]^{c_2/c} = k_2\left(1+\frac{x}{2}\right)^{\frac{\epsilon}{24}+\frac{\epsilon^2}{(12)^3}\left[k_\gamma-\frac{9}{2}\ln\left(1+\frac{x}{2}\right)\right]} \exp\left[\frac{-\epsilon^2}{2(12)^3}\mathcal{K}(x)\right] \quad (6.10)$$

where k_γ is given in Eq. 3.51 and

$$k_2 \equiv \bar{Z}_2(0)\left[\frac{g_1 A}{\bar{Z}(0)}\right]^{c_2/c} \approx (g_1 A)^{c_2/c} \quad (6.11)$$

where we used $\bar{Z}_2(0) \approx g_1^{-c_2}$.

Equations 6.6-6.11 give the final results for the partial wave amplitude $[-i\Gamma^{(1,1)}(E,k^2)]^{-1}$ with $E=1-j$, $k^2=-t$, and $\epsilon=2$. The full amplitude $T(s,t)$ is then the integral of this over the Sommerfeld-Watson contour for $\mathrm{Re}\, j > 1$ as usual. We leave the numerical details for a future publication.

Our results agree with ABBS and generalize them to include $O(\epsilon^2)$ exponents. The $\delta\cdot f(x)$ dependence in the unintegrated sum rules Eqs. 4.8, 4.9 becomes $\delta-\frac{1}{2}f(x)$ in the exponents of Eqs. 6.8, 6.10. The function $\mathcal{K}(x)$ of Eq. 4.13 arises from the product

$(\gamma_E \tau_E)_1$, when $(1-\tau_E)_1^{-1} \approx 1+(\tau_E)_1$ is expanded in the sum rules. It is new, and we will examine it a little further in Section VII when we discuss the phenomenology.

We turn now to the question of rescaling the coupling constant. In BD it was shown that δ (Eq. 2.6) can be removed from $\beta(g)$ and $\gamma(g^2)$ by a redefinition of g, without changing the critical exponents γ and Υ/α' . Our $t \neq 0$ calculation has preserved the δ - independence of γ and Υ/α' , although the final results for $\Gamma^{(1,1)}$ do seem to depend on δ through k_γ and k_T .

Consider the following redefinition of g ,

$$g^2 = \frac{(8\pi)^{1-\epsilon/2}}{\Gamma(1+\frac{\epsilon}{2})} \exp\left[\frac{\epsilon\tilde{r}}{2} f(x)\right] \tilde{G}^2 \tag{6.12}$$

The x-independent factor is that of BD and $\tilde{r}$ is a constant. Expanding Eq. 6.12 to $O(\epsilon)$ yields

$$\tilde{G} = \frac{g}{(8\pi)^{1/2}}\left[1+\frac{\epsilon}{4}\left(\delta+3\ln 2-\tilde{r}f(x)\right)\right] \tag{6.13}$$

Here x is again $\alpha' k_N^2/E_N$. We note the identity

$$k_N^2 \frac{\partial F(x)}{\partial k_N^2} = x\, F'(x)\left[1+\tau_k\right] \tag{6.14}$$

for any $F(x)$ at fixed r_0, α_0' . A similar identity holds for $E_N\, \partial F(x)/\partial E_N$. We define

$$\beta_k(\tilde{G},x) = k_N^2 \frac{\partial \tilde{G}}{\partial k_N^2} \tag{6.15}$$

Using Eqs. 6.13 - 6.15 and equivalent equations with E_N produces the β function for the rescaled theory which we call $\tilde{\beta}$, defined as

$$\tilde{\beta}(\tilde{G},x) = \left[\beta_E+\beta_k+\beta_E\tau_k-\beta_k\tau_E\right](\tilde{G},x) \tag{6.16}$$

This function has a zero at $\widetilde{G} = \widetilde{G}_1(x)$, where

$$\frac{\widetilde{G}_1^2(x)}{8\pi} = \frac{G_1^2}{8\pi} + \frac{\epsilon^2}{(12)^3}\left[-48 f(x)\right](1+3\tilde{r}) \tag{6.17}$$

Here G_1 is given in Eq. A.9. It is independent of δ.

One can show that $(\gamma_\epsilon)_1$, $(\gamma_k)_1$, $(\tau_\epsilon)_1$, and $(\tau_k)_1$ are unchanged by rescaling, and $(\partial\tilde{\beta}/\partial\widetilde{G})_{\widetilde{G}_1(x)} = (\partial\bar{\beta}/\partial g)_{g_1(x)}$. Thus C_1, C_2, and C_3 are unchanged. This implies that both the scaling law for $\Gamma^{(1,1)}$ and the inversion formula are unchanged by rescaling.

However, $(\beta_\epsilon/g)_1$ is changed. We obtain

$$\left[\frac{\beta_\epsilon(\widetilde{G},x)}{\widetilde{G}}\right]_{\widetilde{G}_1(x)} = \left[\frac{\beta_\epsilon(g,x)}{g}\right]_{g_1(x)} + \frac{\epsilon}{4}\tilde{r}xf'\left[1-\epsilon\left(3\ln 2+\delta-\tilde{r}f-\frac{f'}{12}\right)\right] \tag{6.18}$$

$$= \frac{\epsilon}{12}xf'(x)(1+3\tilde{r}) + O(\epsilon^2) \tag{6.19}$$

Hence to $O(\epsilon)$ the ABBS equation for $\widetilde{G}_1(x)$

$$\frac{d\ln\widetilde{G}_1(x)}{d\ln x} = \left[\frac{-\beta_\epsilon(\widetilde{G},x)}{\widetilde{G}(1-\tau_\epsilon)}\right]_{\widetilde{G}_1(x)} \tag{6.20}$$

is satisfied regardless of $\tilde{r}$. It is interesting that the value $\tilde{r} = -1/3$ results in an x-independent critical coupling, so that the $O(\epsilon)$ ABBS equation for $\widetilde{G}_1(x)$ becomes trivial.

Finally, we note that $(\partial\beta_\epsilon/\partial g)_1$ is also changed by rescaling. Hence individual ABBS equations for $\Xi_i(x)$ (e.g. Eq. 4.6) change, although as mentioned above the sum rules Eqs. 4.8, 4.9 do not.

VII. PHENOMENOLOGY

A/ GENERAL REMARKS

There are many a-priori reasons why a calculation such as the one we have performed here should not be applicable at present (or even much higher) energies. In the authors' viewpoint, these include three main points. First, dominant short range order in rapidity implies the weakness of all j-plane cut effects at present energies. This is confirmed by the extremely small fraction of "double Pomeron exchange" observed experimentally, which in some sense represents all non-trivial enhanced graphs in $\Gamma^{(1,1)}$. Second, clear threshold rapidity scales exist at present energies. These include a "basic" rapidity scale of, e.g., 2 units for each link in any RFT graph below which the graph makes no sense physically. This means that no two-loop graph (let alone the RG exponentiated form) should be meaningful even at ISR. Actually things are even worse since higher rapidity scales exist due to inelastic $K\bar{K}$, $B\bar{B}$, charm ... production. In principle the excitation of new quantum numbers produces a "flavoring" non-diffractive renormalization of the bare Pomeron pole through s-channel unitarity [2]. Inclusion of these effects taking account of the generally small subenergies extending across the bare Pomerons in any given RFT graph is an a-priori important effect at energies even much higher than ISR. Flavoring is ignored in almost all RFT papers, including this one. Flavoring renormalization renders moot the usual sort of discussion of the energies at which the RFT ought to hold (cf. Ref [3] for example). Finally, the asymptotic RFT predictions of $\sigma_{tot} \approx (\ln s)^{-\gamma}$ and decreasing $\sigma_{el}(s)$ are not observed at present energies.

Nevertheless, it is important to see if any asymptotic RFT predictions work at current energies. The qualitatively successful ABBS calculation of $d\sigma/dt$ may continue to hold up. The remarks we will now make seem to imply that this may happen, and appear to lead to rather more optimistic conclusions. This is connected with the appearance of Bessel functions, to which we now turn.

B/ THE SHAPE OF $d\sigma/dt$, BESSEL FUNCTIONS, AND KANE'S POMERON

We consider the scaling law for $\Gamma^{(1,1)}$ and the inversion formula both for small x . Eqs. 6.7 and 6.10 yield

$$x \simeq x_0 \, (-E)^{\gamma/\alpha'} \, k_2^{-1} \tag{7.1}$$

Next we approximate $f(x) = \ln\left(1+\frac{x}{2}\right) \approx \frac{x}{2}$ and $\mathcal{K}(x) \approx 3\nu x$. Recalling that the scaling law solution instructs us to use x_0 with $k_N^2 = k^2$ and $E_N = -E$, viz

$$x_0 = \frac{\alpha_0' k^2}{-E} \tag{7.2}$$

we find from Eqs. 6.5, 6.8 that the small x partial wave amplitude is

$$\left[-i\Gamma^{(1,1)}(E,k^2)\right]^{-1} \approx k_3(1-\gamma)(-E)^{\gamma}\left[-E+\alpha_0' k^2(-E)^{\gamma/\alpha'} k_2^{-1}\right]^{-1} * \exp\left\{\frac{\alpha_0' k^2}{-E}\,\frac{(-E)^{\gamma/\alpha'}}{2k_2}\left[-\frac{E}{12}+\frac{E^2}{(12)^3}\left(-k_\gamma+6\nu\right)\right]\right\} \tag{7.3}$$

$$\approx \frac{k_3}{j-1}\,\exp\left[\frac{R^2 t}{4(j-1)}\right] \tag{7.4}$$

with

$$R^2 = \frac{2\alpha_0'}{k_2}\left[\frac{E}{12}+\frac{E^2}{(12)^3}\left(k_\gamma-6\nu\right)\right] \tag{7.5}$$

Here k_2 , k_γ , and ν are given in Eqs. 6.11, 3.50, and 4.11. See the footnote.

It is important to note that no essential singularity at $j = 1$ is present in the RFT. Our small x (or x_0) approximation assumes $j \neq 1$. Essentially this means $-\alpha_0' k_\perp^{-1} t \ln s \ll 1$. Note that the Sommerfeld-Watson integral of Eq. 7.4 would be taken for $\mathrm{Re}\, j > 1$.

Now[8] if $\mathrm{Re}\, j > 1$ and $t \leq 0$,

$$\frac{1}{j-1} \exp\left[\frac{R^2 t}{4(j-1)}\right] = \int_1^\infty \mathrm{Im}\left[is\, J_0\left(R\sqrt{-t \ln s}\right)\right] s^{-j-1} ds \qquad (7.6)$$

Note that the $\ln s$ is <u>inside</u> the square root. This equation has purposefully been written as a Froissart-Gribov integral. Non-leading terms involve replacing $\ln s$ by $\ln s - i\pi/2$ in the argument of J_0 , as required by crossing symmetry.

Eq. 7.6 shows that Bessel functions are, in some approximation, buried in the RFT. This seems to us to be a physical picture of the dip-bump structure of the results for $d\sigma/dt$ found by ABBS. The interference of the "pole" term in Eq. 7.3 with the "cut" producing this structure is replaced by the first J_0 oscillation. We do not propose that this is an accurate numerical representation. Nonetheless, it looks rather agreeable physically. An extremely interesting phenomenological question would be whether or not the RFT can reproduce the experimentally observed absence of a second dip in $d\sigma/dt$ out to $t \approx -10\ \mathrm{GeV}^2$.

The general emergence of Eq. 7.6 both in the work of ABBS (for which R^2 in Eq. 7.5 is just the first term) and our results, indicates that a RFT phenomenology of $d\sigma/dt$ could continue to be qualitatively successful.

We close with the following observation, which we find fascinating. Amplitudes involving Bessel functions exactly of the form in Eq. 7.6 have long been advocated by Kane[6] on phenomenological grounds, mainly involving data at low energies (e.g. 6 GeV). Can it be that the

Reggeon Field Theory, a supposedly asymptotic theory to which all the pessimistic artillery of Section VII A seems to apply, actually contains within it a form of super-precocious scaling ?

ACKNOWLEDGMENTS

We thank A. White for suggesting this problem and for emphasizing its importance.

- R E F E R E N C E S -

[1] For reviews, see H.D.I. ABARBANEL, J.B. BRONZAN, R.L. SUGAR, and A.R. WHITE, Phys. Rep. 21C, 120 (1975) ;
M. BAKER and K.A. TER-MARTIROSYAN, Phys. Rep. 28C, 3 (1976) ;
MOSHE MOSHE, Phys. Rep. Vol. 37, n° 3.

[2] J.W. DASH, S.T. JONES, and E.K. MANESIS, Phys. Rev. D18, 303 (1978), Sect. VII ;
J.W. DASH, Phys. Lett. 61B, 53 (1976).

[3] H.D.I. ABARBANEL, J. BARTELS, J.B. BRONZAN, and D. SIDHU, Phys. Rev. D12, 2798 (1975).

[4] J.B. BRONZAN and J.W. DASH, Phys. Rev. D10, 4208 (1974) and D12, 1850 (1975).

[5] M. BAKER, Nucl. Phys. 80B, 62 (1974).

[6] B.J. HARTLEY and G.L. KANE, Nucl. Phys. B57, 157 (1973).

[7] J. DASH and S.J. HARRINGTON, Phys. Lett. 57B, 78 (1975).

[8] Bateman manuscript Project, Tables of Integral Transforms, Vol. I, p. 344, Eq. 3.

FOOTNOTE

Under rescaling, the $O(\epsilon^2)$ term of the critical coupling g_1^2 changes. Therefore the approximations indicated in Eqs. 6.9, 6.11 are not reliable ways to evaluate k_2 and k_3 except in lowest order. The same remark holds for the radius parameter R in Eq. 7.5. We do not know if the exact expressions for k_2 and k_3 are invariant under rescaling or not. In any case, the critical RFT cannot determine the scale in t . The parameter $\alpha_0' k_2^{-1}$ is evidently what sets this scale. Therefore it may be reasonable simply to regard k_2 as a free parameter to be fitted by experiment. We thank A. White for a discussion on this point.

- APPENDIX A -

The expressions for the two-loop $k_N = 0$ quantities d_5 of Z_1^{-1}, w_4 of Z, $\beta(g)$, its zero g_1, and the critical exponents γ and γ/α' are

$$(8\pi)^4 d_5 = 20 + \epsilon(26 + 40\ln 2 + 20\delta) \tag{A.1}$$

$$(8\pi)^4 w_4 = \frac{27}{2} + \epsilon\left(\frac{697}{32} + \frac{233}{8}\ln 2 - \frac{53}{16}\ln 3 + \frac{27}{2}\delta\right) \tag{A.2}$$

$$\beta(g) = -\frac{\epsilon g}{4} + \frac{g^3}{(8\pi)^2}\left[\frac{3}{2} + \epsilon\left(\frac{15}{16} + \frac{5}{4}\ln 2 + \frac{3}{4}\delta\right)\right] - \frac{g^5}{(8\pi)^4}\left(\frac{157}{32} + \frac{53}{16}\ln 4/3\right) \tag{A.3}$$

$$\frac{g_1^2}{(8\pi)^2} = \frac{\epsilon}{6} + \frac{\epsilon^2}{(12)^3}\left[-28\ln 2 - 106\ln 3 - 23 - 144\delta\right] \tag{A.4}$$

$$-\gamma = \frac{\epsilon}{12} + \frac{\epsilon^2}{(12)^3}\left[161\ln 4/3 + \frac{37}{2}\right] \tag{A.5}$$

$$-\gamma/\alpha' = \frac{\epsilon}{24} + \frac{\epsilon^2}{2(12)^3}\left[59\ln 4/3 + \frac{79}{2}\right] \tag{A.6}$$

The propagator to one-loop expanded to $O(1/\epsilon, \epsilon^0)$ is

$$-i\Gamma^{(1,1)}(-E_N, x_0) = E_N(1+x_0) - \frac{r_0^2 E_N}{2(8\pi\alpha_0')^2}\left(1+\frac{x_0}{2}\right)\left[\frac{2}{\epsilon} + 1 + 3\ln 2 + \delta - f(x_0) + \ln\frac{\alpha_0'}{E_N}\right] \tag{A.7}$$

The one-loop quantity d_3 in Z_1^{-1} is

$$(8\pi)^2 d_3 = -4 - \epsilon(2 + 4\ln 2 + 2\delta) \tag{A.8}$$

The rescaled critical coupling G_1 at $x = 0$ is

$$\frac{G_1^2}{8\pi} = \frac{\epsilon}{6} + \frac{\epsilon^2}{(12)^3}\left[404 \ln 2 - 106 \ln 3 - 23\right] \tag{A.9}$$

Eqs. A.1 - A.6, and A.9 correct Eqs. 59, 70, 73, 81-83, and 89 in BD. Also Eq. 88 in BD should have $53/16 \ln 4/3$ instead of $144/16 \ln 4/3$ in the $O(G^5)$ term.

TABLE 1

The coefficients in the ABBS equations for $\bar{Z}_i(x)$ (Eq. 4.14) are given by the following (cf. Eqs. 2.6, 4.11)

	$l_i^{(1)}$	$l_i^{(2)}$	$l_i^{(3)}$	a_i	b_i
$\bar{Z}_3$	$-\frac{1}{8}$	$-499\ln 2+\frac{373}{2}\ln 3-\frac{979}{4}-66\delta$	-6	22ν	-22
$\bar{Z}_2$	$\frac{1}{16}$	$\frac{415}{2}\ln 2-\frac{271}{4}\ln 3+\frac{1021}{8}+36\delta$	0	-11ν	11
$\bar{Z}$	$\frac{1}{24}$	$422\ln 2-106\ln 3+\frac{647}{2}+126\delta$	42	-18ν	18