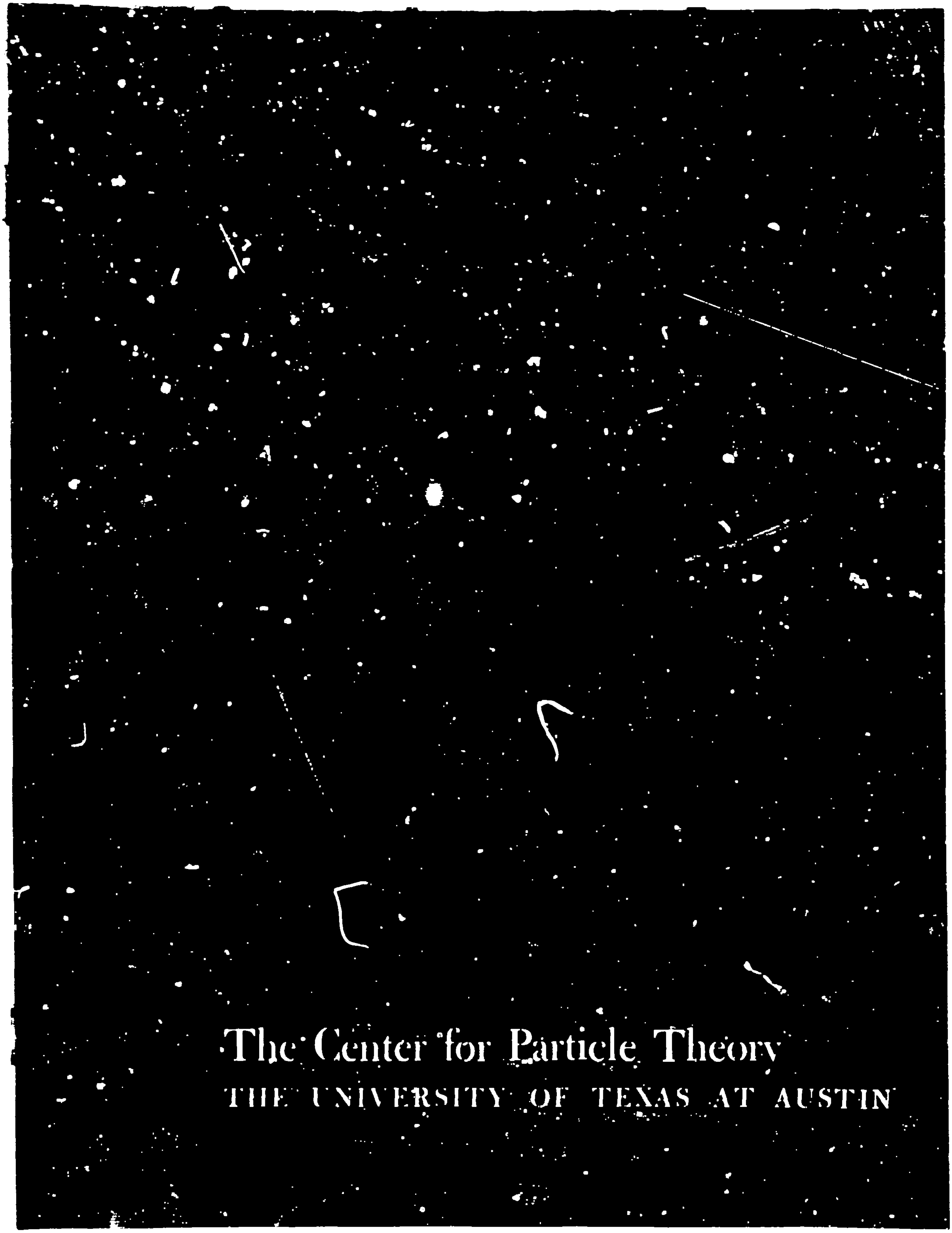

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Re-introducing the Concept of 'Force' into Relativity Theory

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Abstract

We suggest that re-introducing 'forces' into relativity theory may provide new insights and results. A look at the Kerr-Newmann geometry, and special cases of it, from this viewpoint indicates that there can be a short range repulsion in general. This repulsion suggests that 'naked singularities' may be physically feasible. We also find that there is a 'gravito-electric repulsion' which would be important to consider in a grand unification scheme of strong, weak and electromagnetic forces.

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Introduction

The usual presentation of general relativity avoids the concept of 'force' in favour of a completely geometric description. This approach might have been fine if our physical intuition was not so firmly based on the concept of force. Without giving up the program of geometrization, we may get new insights into the workings and predictions of general relativity by re-introducing the concept of force into relativity theory.

There is a general tendency to suppose that an observer in a freely falling frame would not feel any force acting on him and would, therefore, be unaware of the presence of the gravitating source towards which he is falling. It is well known that this view is not really correct because the observer could detect the presence of the gravitating source by measuring tidal forces. The usual view would be correct if the size of the measuring device were negligible compared with its distance from the gravitating source, thus making the tidal force negligible as well. However, near the source, the tidal force can be quite significant. As we show later, an observer equipped with a sufficiently accurate device (to measure the tidal force) would see some interesting changes in the readings on his device. In the next section we shall consider how he would interpret those changes, and in the two sections thereafter, what changes he would see in a Schwarzschild, Reissner-Nordstrom or Kerr-Newmann[1] geometry.

We are thus led to consider the usual relativistic generalization of the Newtonian concept of gravitational force as the inward, radial accelerations (times mass) due to a source. As we shall see in Section 5, we get non-gravitational forces which can be repulsive, apart from the usual gravitational attraction. This brings out the fact that general relativity is a theory of acceleration, and not merely a theory of gravitation. It also demonstrates, as shown in Section 6, that even 'naked singularities' are not all that naked.

Finally, in Section 7, there is a brief summary and discussion of the results obtained. It is hoped that the above-mentioned ideas may help to give a more intuitive understanding of the demonstration by Qadir and Wheeler[2] of the validity of Penrose's conjecture[3] that in a closed cosmology the black hole singularity and the big crunch singularity will amalgamate. This result could become more comprehensible in terms of the usual concept of force, as an observer falling into the hole sees a 'repulsive force' (in some sense), which keeps him from falling into the singularity till the universe collapses. This approach may also help us to answer the question "What happens to the supposed amalgamation of singularities if the black hole breaks up?" The 'repulsion' experienced by the incoming particles may not allow the actual formation of a singularity in a black hole which breaks up -- there never being enough time for its formation. Another question that the present view may help to answer in this connection is "What happens in an open universe?" Since there is no 'big crunch' there can be no amalgamation of singularities. The formation of the black hole singularity, however, seems to be inescapable because it must happen in a finite proper time. Though we are unable to suggest any answer to the above question at present, it is hoped that it could be found in the concept of force.

2. The Concept of Force on a Freely Falling Observer

There is a problem in defining the 'force' acting on a freely falling observer, because, by definition, his frame is the one in which a point observer can not detect any force. However if the spatial extent of the observer is not negligible compared with his distance from the gravitating source, he can easily tell, by measuring the tidal forces, whether his 'free fall' is towards a gravitational source or not. Here we consider what the observer would measure, and how he would interpret his observations, particularly with reference to attractive or repulsive forces.

Let us first consider tidal forces due to attractive or repulsive forces in Newtonian physics. An ideal apparatus for measuring such forces would be an arrangement of a spring of length ℓ connecting two equal masses, one side having a needle whose rotation on a dial measures the tension in the spring, as shown in Fig. 1. Thus the observer can tell whether the spring is being stretched, left neutral, or compressed by observing the movement of the needle. In classical physics, the needle would be in a neutral position in the absence of a central force, indicate stretching (+ve)/compression (-ve) for an attractive/repulsive central force.

Since our freely falling observer is now in a position to perceive 'forces', it is meaningful to ask whether according to relativity theory, he sees anything remarkable as he passes the event horizon of a black hole. Anyhow it is interesting to consider what the tidal forces acting on a freely falling observer actually are, and what he would see as he falls towards a naked singularity.

Let us consider the Kerr-Newmann metric in Boyer-Lindquist coordinates,

$$ds^2 = A\,dt^2 + 2B\,dt\,d\phi - C\,dr^2 - D\,d\theta^2 - E\,d\phi^2 \tag{1}$$

where

$$\left.\begin{aligned}
A &= (r^2 - 2mr + a^2\cos^2\theta + Q^2)/(r^2 + a^2\cos^2\theta) \\
B &= a\sin^2\theta\,(2mr - Q^2)/(r^2 + a^2\cos^2\theta) \\
C &= (r^2 + a^2\cos^2\theta)/(r^2 - 2mr + a^2 + Q^2) \\
D &= r^2 + a^2\cos^2\theta \\
E &= \frac{\sin^2\theta\,[r^4 + a^2(1 + \cos^2\theta)\,r^2 + a^4\cos^2\theta - a^2Q^2\sin^2\theta]}{r^2 + a^2\cos^2\theta}
\end{aligned}\right\} \tag{2}$$

m being the mass of the gravitating point particle, Q its charge and ma its angular momentum (in gravitational units).

The acceleration vector is

$$A^a \equiv \ddot{p}^a \equiv \frac{d^2p^a}{ds^2} = -R^a{}_{bcd}\, t^b p^c t^d \qquad (3)$$

where t^a is the unit time-like vector tangent to a geodesic and p^a the space-like position vector from the one to a neighboring one. If we consider the front end of the measuring device (shown in Fig. 1) to move along the first geodesic and the rear end along the other geodesic, the device being placed in the 'radial' (or r) direction, the acceleration becomes

$$A^a = -\rho R^a{}_{010} \qquad (4)$$

It is of interest to consider the acceleration vector where the rotational effect is maximum, i.e., in the plane of rotation. As shown in Appendix A, the geodesic equation for θ can be solved by $\theta = \pi/2$, $\dot{\theta} = 0$, which means that the motion remains coplanar if it starts off in the plane of rotation. In this case the metric coefficients reduce to

$$\left.\begin{aligned} A &= (1 - 2m/r + Q^2/r^2) \\ B &= a - aA \\ C &= (A + a^2/r^2)^{-1} \\ D &= r^2 \\ E &= r^2 + 2a^2 - a^2A \end{aligned}\right\} \qquad (5)$$

It is easily verified that now the only non-zero component of the acceleration vector is A^1, which is evaluated in Appendix B to yield

$$A^1 = \left[\frac{2m}{r^3} - \frac{4m^2 + 3Q^2}{r^4} + \frac{m(3a^2 + 2Q^2)}{r^5} + \frac{m^2a^2 - 4Q^2a^2 - 3Q^4}{r^6} + \frac{2ma^2(2m^2 - Q^2)}{r^7} + \frac{a^2Q^2(Q^2 - 10m^2)}{r^8} + \frac{4ma^2Q^2}{r^9} - \frac{2a^2Q^6}{r^{10}}\right] \qquad (6)$$

Even though the expression for A^1 is very complicated it is clear that this vector does not change monotonically, in general. To understand the implications of Eq. (6) more fully, we need to consider some special cases, i.e. when one or two of the quantities a, m and Q are zero.

3. Tidal Forces in the Reissner-Nordstrom Geometry

If we consider the case a = 0 in Eq. (6), we get the tidal force in the Reissner-Nordstrom geometry. It is clear from Eqs. (1) and (2) that there is no preferred plane, and any plane can be chosen as the $\theta = \pi/2$ plane without loss of generality. First, we consider the case $Q = 0$, giving the tidal 'force' in the Schwarzschild geometry[8]

$$A^1 = (1 - 2m/r)2m\ell/r^3 \tag{7}$$

For $r \gg 2m$, we get the usual Newtonian expression for the tidal force. However, it appears that for r nearing 2m, the tidal force starts decreasing after reaching a maximum of $27\ell/(1024m^2)$ at $r = 8m/3$ (outside the event horizon), reaching zero at the horizon, and becoming negative inside.

It may be argued that the result obtained is doubtful becuase the coordinates used are not valid at, or beyond, the event horizon. However, since the force goes to zero at the horizon, and not to infinity, it would be odd that were the reason for the 'repulsive' force. In fact, it can be easily verified that the result obtained in Eq. (7) also holds in Eddington-Finklestein coordinates, which are valid for the entire range of r. Thus it is not correct to say that the observer falling into a Schwarzschild black hole can not tell when he crosses the event horizon -- it is that place where his measuring device reads zero. The freely falling observer will then see an increasing compression, which he will interpret as being an ever increasing repulsion.

The previous result is not as surprising as it may seem at first sight. It is clear that an object falling into a non-rotating black hole can not go on getting extended indefinitely and must start getting compressed at some stage. Also, the result of Qadir and Wheeler[2] seems to be in agreement with the above finding. There it was shown that anything falling into a Schwarzschild black hole, in certain closed universe models, will reach the singularity just as the universe collapses. This came about due to a slowing down of the clocks of the freely falling observer as he approaches the singularity. Another way to understand the same result would be if there were a sort of retardation of the freely falling observer, as we just found. The results appear to be consistent with each other.

We can now consider the result for the Reissner-Nordstrom geometry. Equation (6) with a = 0 gives us

$$A^1 = \frac{2m\ell}{r^3}\left[\left(1 - \frac{2m}{r}\right) + \frac{Q^2}{r^2}\left(1 - \frac{3r}{2m} - \frac{3Q^2}{2mr}\right)\right] \qquad (8)$$

It is clear from Eq. (8) that the inclusion of the charge enhances the 'repulsion' found before, for $r > 2m/(3-3Q^2/m)$, and for $r < 3Q^2/m$. Hence there is always a repulsive effect unless $m > 3Q$, in which case, there is a range $[3Q^2/m < r < (2m/3 - 3Q^2/m)]$ over which there is no repulsive effect.

The above result may seem odd in two ways. The first is the existance of a 'force' on a neutral particle due to electric charge and the second is its repulsive nature. The existence of the force is implicit in the fact that the metric is dependent on the charge, and may be understood as being due to the 'energy' stored in the field on account of the presence of the charge. The repulsive nature of the force can be understood by noticing that the 'energy' is stored in the entire field around the point and 'pulls the particle back more than it pulls it forward'. This argument would not apply if the energy distribution acted like a uniform sphere of matter in Newtonian physics, but a slight deviation from this situation could enable the energy to give the 'repulsive' effect noticed above. Let us consider the effect in more detail.

It is easily seen from Eq. (8) that for $Q \ll m$ the 'repulsion' overtakes the 'attraction' at

$$r \approx 2m + Q^2/m \qquad (9)$$

while the outer horizon is at

$$r_+ \approx 2m - Q^2/2m \quad (10)$$

Thus the change-over of tidal 'forces' from 'attractive' to 'repulsive' occurs long before the event horizon is reached! This eliminates any doubt as to the coordinate system being responsible for the 'change-over' effect. For the extreme Reissner-Nordstrom black hole, $Q = m$ the 'change-over' occurs at $r \approx 10m/3$ (which is way out from the event horizon) and the 'force' goes as $3m^4 \ell/r^6$ for small r. In the case $Q > m$ (discussed in detail later) we start getting the 'repulsive effect' even further out and the force goes as $3Q^4 \ell/r^6$ for small r.

The 'attractive tidal force' reaches a maximum when r first satisfies the equation

$$r^3 - (8m/3 + 2Q^2/m)r^2 + (5Q^2/3)r - 3Q^4/m = 0. \quad (11)$$

which for $Q \ll m$, yields

$$A^1_{max} \approx (1 - 123Q^2/64m^2)27\ell/1024m^2 \quad (12)$$

at a distance of

$$r \approx 8m/3 + 11Q^2/8m \quad (13)$$

It can be seen from Eqs. (12) and (13) that the charge (Q) causes a reduction of the maximum 'force', and pushes the maximum further out. For Q = m we find that $A^{1}_{max} \approx 5.5 \times 10^{-3} \ell/m^2$ at $r \approx 4.72m$. Two points are now immeidately obvious: 1) that the freely falling observer sees the 'change-over' effect a long way from the event horizon; and 2) that the maximum acceleration (for the case Q = m) is a factor of over 200 less than the Newtonian tidal force at the event horizon!

4. The Tidal 'Forces' in the Kerr-Newmann Geometry

We are now in a better position to discuss Eq. (6) with a $\neq 0$. For a general orbit the 'force' lies somewhere between the rotating and non-rotating cases depending on the local value of θ. It is exactly the non-rotating one for an approach along the axis of rotation. Therefore, it is entirely adequate to consider the 'force' in the plane of rotation to get an idea of its general behavior.

Let us consider the uncharged case first ($Q = 0$). Here

$$A^1 = \frac{2m\ell}{r}\left[1 - \frac{2m}{r} + \frac{a^2}{r^2}\left(\frac{3}{2} + \frac{m}{2r} + \frac{2m^2}{r^2}\right)\right] \qquad (14)$$

It is clear that the effect here is not to enhance the 'repulsion' but to reduce it. However, there is nothing particularly interesting about the change in tidal 'forces' due to rotation in this case.

Looking at the case when $Q \neq 0$, we see that the spinning charge can give rise to a greater enhancement of the repulsive effect than the non-spinning case for sufficiently large values of Q. It is not particularly instructive to do a detailed analysis of the structure of the tidal 'forces' in the general case, but we see that there is a 'repulsive effect' going as $2a^2Q^6\ell/r^{10}$ for sufficiently small r. The rotating case has not been checked, inside the event horizon, with Eddington-Finklestein coordinates, but it is expected that the results obtained here would be valid since they are found to be valid for the non-rotating (charged) case.

5. 'Forces' as Seen in Terms of Geodesic

Having seen that a 'force' can be operationally defined for a freely falling observer, we would now like to consider something more like the Newtonian central force which gives rise to the tidal forces we have been considering, and see whether it also shows a structure similar to that shown by the tidal force as we proceed in towards the point particle. Since the tidal force is, in some sense, 'the gradient of a central force', one might expect a similar structure for the central force also. Thus, roughly speaking, an 'integration' of the tidal force discussed earlier would be expected to give something similar to the 'central force' we want to consider.

A better definition of the central force is obtained by considering the radial geodesic equation for the particular situation under discussion. The expression for $\ddot{r}$ is the 'force per unit mass' at the point. The geodesic equations in the plane of rotation are

$$\left.\begin{aligned}
&\ddot{t} - r^{-2}C[(EA_{,1}+BB_{,1})\dot{t} + (EB_{,1}-BE_{,1})\dot{\phi}]\dot{r} = 0 \\
&\ddot{\phi} - r^{-2}C[(BA_{,1}-AB_{,1})\dot{t} + (BB_{,1}+AE_{,1})\dot{\phi}]\dot{r} = 0 \\
&\ddot{r} + \tfrac{1}{2}[(C^{-1}A_{,1}-AC^{-1}_{,1})\dot{t}^2 + 2(C^{-1}B_{,1}-BC^{-1}_{,1})\dot{t}\dot{\phi} \\
&\qquad\qquad -(C^{-1}E_{,1}-EC^{-1}_{,1})\dot{\phi}^2 + C^{-1}_{,1}] = 0
\end{aligned}\right\} \qquad (15)$$

where A, B, C, D, E are given in Eqs. (5) (see Appendix C). The first two equations can be integrated to give

$$\left.\begin{aligned} \dot{t} &= C(E + B\Phi)r^{-2} \\ \dot{\phi} &= C(B - A\Phi)r^{-2} \end{aligned}\right\} \qquad (16)$$

where Φ is the angular momentum per unit mass (of the test particle experiencing the force). For a 'radially incoming' test particle ($\Phi = 0$) we have $\dot{t} = CE/r^2$, $\dot{\phi} = CB/r^2$. As shown in Appendix C the 'geodesic force' (per unit mass) acting inwards takes the simple form

$$\frac{m}{r^2}\left(1 + \frac{a^2}{r^2}\right) - \frac{Q^2}{r^3}\left(1 + \frac{2a^2}{r^2}\right) \qquad (17)$$

Let us consider the expression (17) for particular cases so as to understand the significance of the various terms appearing in it. First we take $Q = 0 = a$. In this case the 'force' reduces to m/r^2, the usual Newtonian force with no modifications. It does not show any change corresponding to the structure of the tidal force for the Schwarzschild black hole. This fact shows that the two definitions of force, by geodesics, and by geodesic deviation, are not equivalent.

The case of the charged black hole is altogether different. In the absence of rotation ($a = 0$), the expression (17) becomes $(m/r^2 - Q^2/r^3)$, which has no Newtonian analogue. Thus apart from the gravitational attraction of the point mass, there is, what we shall call, a Reissner-Nordstrom repulsion! As explained in section 3, the repulsion away from the black hole is due to an outward attraction by the electric energy in the field around the test particle. Since it goes as Q^2/r^3 and not Q^2/r^2 it will not 'cancel out' with the attraction from the other side and hence a net pull outwards remains. Clearly, the Reissner-Nordstrom 'electro-gravitic' repulsion, which is a new feature of relativity theory, overtakes the gravitational attraction at $r = Q^2/m$, outside the inner horizon. Thus, for an extreme Reissner-Nordstrom black hole, the change-over occurs at the event horizon!

For an uncharged, rotating black hole, expression (17) becomes $(m/r)(1 + a^2/r^2)$, clearly enhancing the attractive force due to the mass. This result could have been expected by noticing that rotation affects the geodesics by a dragging of inertial frames.

The complete expression can be easily understood now. The rotation merely enhances the attraction (or repulsion) due to the mass (or the charge). The fact that the expression (17) remains finite at the event horizon indicates that it is valid even inside the event horizon.

6. The Question of Naked Singularities in Relativity

So far we have been largely ignoring the case of $Q > m$ or $a > m$ so as to avoid naked singularities. Naked singularities are normally assumed to be non-physical as it is supposed that they would enable signals from the singularities (which are not describable by our present physics) to reach us. In particular, one could 'see' a particle falling into the singularity and achieving an infinite energy density. It is unfortunate that from this point of view, any of the so-called elementary particles must be unphysical if they are to be regarded as point particles. This fact is not given much importance because their Schwarzschild radius is much less (~20 orders of magnitude less) than the Planck length $(\hbar G/c^3)^{1/2} \sim 10^{-33}$cms. and hence their gravitational effects are negligible compared with the vacuum fluctuations.

None of these arguments are entirely convincing, and the 'cosmic censorship hypothesis'(4) that all singularities are 'clothed' by event horizons may be nothing more than a pious hope that we need not be bothered about the singularities themselves. We discard the cosmic censorship hypothesis, and show that the supposed problems associated with naked singularities are absent in the Kerr-Newmann geometry, and may not arise in general. Hence elementary particles, like electrons, can safely be regarded as point particles. The important point to remember is that there

<u>are</u> electro gravitic forces (enhanced by spin) which must be considered.

This above point is of particular relevance if Einstein's world view[5] (of regarding particles as space-time singularities which interact through the curvature of space-time) is to be taken seriously. One of us (A.Q) has considered the quantum theoretic implications of this world view[6]. A more explicit connection between the fundamentals of relativity and the quantum theory is being considered. For this world view to be internally consistent, it is necessary that the elementary particles be describable as singularities.

Let us, now, consider an electron as a spinning, charged, gravitating point particle. Its Schwarzschild radius is $\sim 1.35 \times 10^{-55}$ cms., its charge ($QG^{1/2}/c^2$) is $\sim 1.38 \times 10^{-34}$ cms., while its angular momentum per unit mass (a/c) is half its Compton wavelength $\sim 1.93 \times 10^{-11}$ cms. It is clar that $a \gg Q \gg m$ in this case. Thus, whereas the mass of the electron can not be considered gravitationally, it is not valid to ignore the curvature of space-time due to the electron's charge and spin.

The force due to a Kerr-Newmann singularity with negligible mass is $-(Q^2/r^2)(1 + 2a^2/r^2)$, which is entirely repulsive. For $r \gg a$, the force $\approx -Q^2/r^3$, which is indeed very small for an electron. However, for $r \ll a$, the force is appr mately $(-2a^2Q^2/r^5) \approx 1.49 \times 10^{-89}/r^5$ which rapidly increa s decreases. Assuming that the electro-gravitic repuls orce between an electron and a positron is simply twice the above value (which may not be a very good approximation), it exceeds the Coulomb attraction at $r \sim 4.65 \times 10^{-26}$ cms. For a proton being probed by an electron this effect occurs at $r \sim 3.12 \times 10^{-23}$ cms., as the more massive proton has a shorter Compton wavelength. Thus, for shorter distances, there is a net repulsion between the oppositely charged particles! It should be noted that this effect is of relevance in the grand unification scheme of strong, weak and electromagnetic forces, as the interaction distances talked of there(7) are $\sim 10^{-28} - 10^{-26}$ cms. General relativistic effects can not be ignored there!

We see now that, in terms of the orbit (or geodesic) of the test particle, there would be a repulsion from a naked singularity. Therefore we could still not see a particle falling into a singularity. Also, nothing could emerge out of the singularity due to the infinite force at the singularity. (Note that the force could not be repulsive as that would break up the singularity). Thus, even without censorship a Kerr-Newmann singularity is effectively clothed -- a sort of censorship without censorship! We conjecture that there is no need for censorship of <u>any</u> 'naked singularities' as they are repulsive instead of being attractive -- at least when one gets sufficiently close to them.

It appears that the elementary constituents of matter (whatever they may be) can be consistently regarded as space-time singularities. The more massive they are (up to a point), the closer can they be approached by a probe particle (as $a = \hbar/2mc$), the distance of appraoch going as $m^{-2/3}$ in our model. It would be interesting to see if this expectation fits with observation.

7. Summary and Discussion

We saw that for a freely falling observ , forces' acting on h can be operationally defined by using the device depicted in Fig. 1. This force is usually not considered in relatively theory, as the size of the device is generally much less than its distance from the gravitational source. However, the tidal force displays a surprising amount of structure, giving a deeper insight into the working of relativity the y, and leading us to consider forces in terms of radi acceleration of a test particle. The geodesic force, thus obtained, also displays structure, which makes 'naked singularities' physically acceptable.

The tidal forces could have been expected to display the structure they showed. An object falling radially in towards a non-rotating singularity can not go on increasing in length, but must start getting compressed at some stage. However, if there is any rotation the object can 'wrap itself round the singularity' even if it is falling 'radially inwards' ($\dot{\phi} \neq 0$ even when $\dot{\ } = 0$), except if its approach is along the axis of rotation.

The geodesic forces bring out an important feature of general relativity, viz. th electro-gravitic forces and the enhancement of forces by spin. These forces do not have a counterpart in classical physics. It is clear that a fully relativistic quantum field theory must contain a coupling between charged and neutral matter to deal with interaction distances ~10^{-25}cms. It is intriguing to ponder on a possible connection between the approach taken here and a grand unification scheme of all interactions.

To sum up, there are four major points made in this paper: 1) the usefulness of the concept of 'force' in relativity; 2) the absolute necessity of considering general relativity in a unification scheme; 3) the possibility of censorship without censorship as a means of maintaining the feasability of naked singularities; and 4) that relativity theory is a theory of <u>acceleration</u> and not merely a theory of gravitation.

Acknowledgments

Asghar Qadir expresses his gratitude to Professor J.A. Wheeler for providing financial support at the Centre for Theoretical Physics, University of Texas at Austin; and to the Council for International Exchange of Scholars under grant number 78- 169- A. The work of Prashant M. Valanju was supported by the U.S. Department of Energy under Contract No. EY-76-S-05-3992.

Appendix A

To derive the geodesic equation for θ, we evaluate all the non-zero Christoffel symbols $\left\{ {2 \atop a\ b} \right\}$. Now the components of the inverse metric tensor, g^{ab}, for the Kerr-Newmann metric, given in Eqs. (1) and (2), are found to be

$$g^{00}=CEr^{-2};g^{03}=BCr^{-2};g^{11}=-C^{-1};g^{22}=-D^{-1};g^{33}=-ACr^{-2} \qquad \text{(A1)}$$

where $C^{-1}r^2$ is the value of the determinant $(AE+B^2)$. Hence the required Christoffel symbols are

$$\left.\begin{array}{llll}
\left\{ {2 \atop 0\ 0} \right\} = -\frac{1}{2}D^{-1}A_{,2} & ; & \left\{ {2 \atop 0\ 3} \right\} = -\frac{1}{2}D^{-1}B_{,2} & ; \\
\left\{ {2 \atop 1\ 1} \right\} = -\frac{1}{2}D^{-1}C_{,2} & ; & \left\{ {2 \atop 1\ 2} \right\} = \frac{1}{2}D^{-1}D_{,1} & ; \\
\left\{ {2 \atop 2\ 2} \right\} = \frac{1}{2}D^{-1}D_{,2} & ; & \left\{ {2 \atop 3\ 3} \right\} = \frac{1}{2}D^{-1}E_{,2} & ;
\end{array}\right\} \qquad \text{(A2)}$$

Thus the θ geodesic equation

$$\ddot{\theta} + \left\{ {2 \atop a\ b} \right\} \dot{x}^a \dot{x}^b = 0 \qquad \text{(A3)}$$

becomes

$$\ddot{\theta} + D^{-1}\left(-\frac{1}{2}A_{,2}\dot{t}^2 - B_{,2}\dot{t}\,\dot{\phi} - \frac{1}{2}\ {}_{,2}\dot{r}^2 + D_{,1}\dot{r}\dot{\theta} + \frac{1}{2}D_{,2}\dot{\theta}^2 - \frac{1}{2}E_{,2}\dot{\phi}^2\right) = 0 \qquad \text{(A4)}$$

Now, taking the initial value of θ as $\pi/2$, we note that the derivatives of A, B, C, D and E with respect to θ vanish initially, as they all depend on $\mathrm{Sin}^2\theta$ or $\mathrm{Cos}^2\theta$. Thus Eq. (A4) becomes

$$\ddot{\theta} + D^{-1}D_{,1}\dot{r}\dot{\theta} = 0 \tag{A5}$$

Now, if the initial value of $\dot{\theta}$ is taken to be zero, i.e. the motion is started in the plane of rotation, $\ddot{\theta}$ is initially zero. Thus it must always remain zero, i.e. the motion, once started in the plane of rotation, always remains in the plane of rotation.

Appendix B

To evaluate A^1, the only non-zero component of our acceleration vector, we need to evaluate the component of the Riemann-Christoffel tensor

$$R^1{}_{010} = \left\{ {1 \atop 0\;0} \right\}_{,1} - \left\{ {1 \atop 0\;1} \right\}_{,0} + \left\{ {1 \atop a\;1} \right\}\left\{ {a \atop 0\;0} \right\} - \left\{ {1 \atop a\;0} \right\}\left\{ {a \atop 0\;1} \right\} \qquad \text{(B1)}$$

The Christoffel symbols appearing in Eq. (B1) are

$$\left.\begin{aligned}
&\left\{ {1 \atop 0\;0} \right\} = \tfrac{1}{2} C^{-1} A_{,1} \;; &&\left\{ {1 \atop 0\;1} \right\} = 0 \;; \\
&\left\{ {1 \atop 0\;3} \right\} = \tfrac{1}{2} C^{-1} B_{,1} \;; &&\left\{ {1 \atop 1\;1} \right\} = \tfrac{1}{2} C^{-1} C_{,1} \;; \\
&\left\{ {1 \atop 1\;2} \right\} = 0 \;; &&\left\{ {1 \atop 1\;3} \right\} = 0 \;; \\
&\left\{ {0 \atop 0\;1} \right\} = \tfrac{1}{2} C(EA_{,1} + BB_{,1}) r^{-2} \;; &&\left\{ {3 \atop 0\;1} \right\} = \tfrac{1}{2} C(BA_{,1} - AB_{,1}) r^{-2}
\end{aligned}\right\} \qquad \text{(B2)}$$

Thus Eq. (B1) reduces to

$$R^1{}_{010} = \tfrac{1}{2} C^{-1} A_{,11} - \tfrac{1}{4} C^{-2} A_{,1} C_{,1}$$

$$- \tfrac{1}{4}\left[\left(EA_{,1} + BB_{,1}\right) A_{,1} + \left(BA_{,1} - AB_{,1}\right) B_{,1}\right] r^{-2} \qquad \text{(B3)}$$

Now, inserting the values of B, C and E, given in Eq. (5), into Eq. (B3) we obtain

$$R^{1}{}_{010} = \frac{1}{2}\left[\left(A + a^2/r^2\right)A_{,11} - a^2/r^3 A_{,1}\right] \tag{B4}$$

Since

$$\left.\begin{aligned} A_{,1} &= 2\left(m/r^2 - Q^2/r^3\right) \\ A_{,11} &= 2\left(-2m/r^3 + 3Q^2/r^4\right) \end{aligned}\right\} \tag{B5}$$

Eq. (B4) further reduces to

$$\begin{aligned} R^{1}{}_{010} = &-\left(1 - 2m/r + Q^2/r^2 + a^2/r^2\right)\left(2m/r^3 - 3Q^2/r^4\right) \\ &-\left(a^2/r^3\right)\left(m/r^2 - Q^2/r^3\right) \end{aligned} \tag{B6}$$

On expanding this expression for R^{1}_{010} and inserting it into Eq. (4), we obtain Eq. (6).

Appendix C

To obtain the t, ϕ, r geodesic equations for the Kerr-Newmann geometry we need the following (non-zero) Christoffel symbols, not given in Eqs. (A2) and (B2):

$$\left.\begin{aligned} \left\{ {0 \atop 1\ 3} \right\} &= \tfrac{1}{2} C (EB_{,1} - BE_{,1})\, r^{-2}; \quad \left\{ {1 \atop 2\ 2} \right\} = -\tfrac{1}{2} C^{-1} D_{,1} \\ \left\{ {1 \atop 3\ 3} \right\} &= -\tfrac{1}{2} C^{-1} E_{,1}; \quad \left\{ {3 \atop 1\ 3} \right\} = \tfrac{1}{2} C (BB_{,1} + AE_{,1})\, r^{-2} \end{aligned}\right\} \quad ($$

Inserting the Eqs. (A2), (B2) and (C1) into the geodesic equations

$$\ddot{x}^a + \left\{ {a \atop b\ c} \right\} \dot{x}^b \dot{x}^c = 0 \qquad (a = 0,3) \tag{C2}$$

we obtain the first two of Eqs. (15). Now, from symmetry considerations there are two invariants of the motion

$$p_0 = \varepsilon \qquad p_3 = \tilde{\Phi} \tag{C3}$$

being the energy and angular momentum (of the test particle) 'as seen from infinity.' Thus, the covariant velocity components are

$$\left.\begin{aligned} \dot{x}_0 &= p_0/\varepsilon = 1 \\ \dot{x}_3 &= \dot{\tilde{\Phi}} = \Phi \quad \text{(say)} \end{aligned}\right\} \tag{C4}$$

Thus the required, contravariant velocity components are

$$\left.\begin{aligned}\dot{t} &= \dot{x}^0 = C^{-1}(E + B\Phi)r^{-2} \\ \dot{\phi} &= \dot{x}^3 = C^{-1}(B - A\Phi)r^{-2}\end{aligned}\right\} \tag{C5}$$

With a little bit of algebra these equations can be seen to satisfy the first two of Eqs. (15).

The radial geodesic equation is

$$\ddot{r} + \frac{1}{2}C^{-1}A_{,1}\dot{t}^2 + C^{-1}B_{,1}\dot{t}\dot{\phi} + \frac{1}{2}C^{-1}C_{,1}\dot{r}^2 - \frac{1}{2}C^{-1}E_{,1} = 0 \tag{C6}$$

The terms $\dot{t}$ and $\dot{\phi}$ have already been evaluated in terms of the metric components, and $\dot{r}^2$ is obtained by rewriting the metric equation, Eq. (1), as

$$\dot{r}^2 = (A\dot{t}^2 + 2B\dot{t}\dot{\phi} - E\dot{\phi}^2 - 1)C^{-1} \tag{C7}$$

(Remember that $\dot{\theta} = 0$ in the case under consideration.) Inserting Eqs. (C5) and (C7) into Eq. (C6) we obtain the last of Eqs. (15). We now replace B, C and E in Eq. (15) by their values in terms of A, as given in Eq. (5), to obtain

$$\ddot{r} + \frac{1}{2}\Big\{\Big[(a^2/r^2)(A_{,1}+2A/r)(r^2+2a^2-a^2A)^2 +$$
$$+\ 2a^2\left\{(a^2/r^2)(1-A)-(1+a^2/r^2)A_{,1}\right\}(1-A)(r^2+2a^2-a^2A) + \qquad \text{(C8)}$$
$$+\left\{a^2(2+a^2/r^2)A_{,1}-2r(1-a^4/r^4)A-(4a^2/r)(1+a^2/r^2)\right\}(1-A)^2\Big] \times$$
$$\times(a^2+r^2A)^{-2} + (A_{,1}-2a^2/r^3)\Big\} = 0$$

After some tedious algebra, this extremely complicated looking equation reduces to the beautifully simple one

$$\ddot{r} - (m/r^2)(1 + a^2/r^2) - (Q^2/r^3)(1 + 2a^2/r^2) = 0 \qquad \text{(C9)}$$

giving us expression (17) for the force defined in terms of the geodesic.

References and Footnotes

1. See, for example, Gravitation by C.W. Misner, K.S. Thorne and J.A. Wheeler (W.H. Freeman 1973).

2. A. Qadir and J.A. Wheeler: "Amalgamation of the black hole singularity with the big crunch singularity." (To appear in a volume in honor of Jules Gehenian, being edited by R. Debever and M. Demeur.)

3. R. Penrose: Confrontation of Cosmological Theories with Observational Data, Ed. M.S. Longair (D. Riedel 1974).

4. See, for example, R. Penrose's article in Physics and Contemporary Needs Vol. 1, edited by Riazuddin (Plenum Publishers 1977).

5. A. Einstein: Albert Einstein - Philosopher Scientist, P.A. Schlipp (ed.), (The Library of Living Philsophers, Tudor Publishing Company 1951).

6. A. Qadir: "A particle interpretation of the quantum formalism" (unpublished).

7. See, for example, S. Weinberg: Physics Today 30, 42(April 1977).

8. The factor $(1-2m/r)$ appearing in Eq. (7) is absent in the usual works on relativity theory (see, for example, pp. 860-861 of reference 1). This is because the experimental instrument is usually taken to be represented by a vector in the tangent plane to the curved space-time, rather than lying in the curved space-time itself. Thus the length ℓ is normally divided by the above factor when calculating the acceleration vector. We contend that the measuring device lies in the curved space-time and not in the tangent plane to it, and hence the above mentioned factor should remain. (We are indebted to Dr. R. Znajek for an illuminating discussion on this point.)

Figure Caption

Fig. 1 A spring of length ℓ connects two masses. The spring ends in a needle which rotates around on a dial which shows compression (-ve), stretching (+ve) or no change (0).

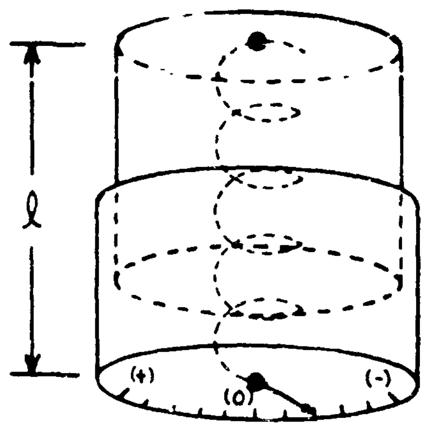

FIG. 1