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EQUATIONS FOR THE NON LINEAR EVOLUTION
OF THE RESISTIVE TEARING MODES IN TOROIDAL PLASMAS

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Following the tokamak ordering, we simplify the resistive MHD equations in toroidal geometry. We obtain a closed system of non linear equations for two scalar potentials of the magnetic and velocity fields and for plasma density and temperature. If we expand these equations in the inverse of aspect ratio they are exact to the two first orders. Our formalism should correctly describe the mode coupling by curvature effects /1/ and the toroidal displacement of magnetic surfaces /2/. It provides a natural extension of the well known cylindrical model /3/ and is now being solved on computer.

I - INTRODUCTION

The first systematic model of the non linear evolution of resistive modes has been achieved in cylindrical geometry and long wavelength approximation /3/. In that model, the plasma motion is incompressible and transverse ($\nabla \cdot \mathbf{v} = \nabla_{\perp}^2 \psi$, \mathbf{v} being the velocity, \mathbf{e}_z the cylindrical axis and ψ the velocity potential). The magnetic field perturbations $\delta \mathbf{B}$ are also transverse : $\delta \mathbf{B} = \nabla_{\perp} \psi \times \mathbf{e}_z$.

That model can be considered as a first approximation for toroidal plasmas following the tokamak ordering (safety factor q and poloidal β_p of order unity, inverse aspect ratio $\epsilon = a R_0^{-1}$ going to zero ; a : characteristic transverse plasma scale, R_0 : main torus radius).

For finite aspect ratio, new physical effects do appear. Primarily, there are first order corrections in the plasma equilibrium and in the structure of the perturbations (displacement, coupling). That corrections are included in our model, as explained below.

Secondly, there are second order corrections in the limits of stability and growth rates. These corrections involve parallel components of the velocity and magnetic field. For tearing modes, we have to consider, in addition to the geometrical parameter ϵ the temporal parameter S ($S = \tau_R / \tau_{HM}$, with $\tau_R = a^2 \eta_0^{-1}$ and $\tau_{HM}^2 = \rho_0 R_0^2 B_0^{-2}$, η_0 : average resistivity, ρ_0 : average plasma inertia, B_0 : vacuum toroidal field).

Let us say first some words about the second order corrections. Uncoupling of perpendicular and parallel motions and magnetic fields does not follow obviously the assumption of a poloidal β of order unity ($\beta \sim \epsilon^2 B_0^2$).

The parallel components of the perturbed velocity and magnetic field are related to u and ψ by curvature effects, geometrically of order ϵ^2 and proportional to β_p . Their effect upon the linear stability of the resistive modes is also dependant of the parameter, S .

From the linear analysis in plane geometry /4/ the condition for stability of the tearing mode was found to be $\Delta' < 0$ with standard notations. That condition is already modified if one includes the finite velocity associated with plasma resistive diffusion /5/ and reads $\Delta' < \Delta'_m$ with m finite and positive. But the most important effect is the curvature effect which has been successively computed in cylindrical /6/ and toroidal geometry /7/.

The linearized stability condition for the tearing mode reads :

$$\Delta' \lesssim S^{1/3} \epsilon^{5/12} \beta_p^{5/6} \left(\frac{d \ln q}{d \ln r} \right)^{-3/3} \quad (1,1)$$

That condition shows that it is not possible to make an expansion in aspect ratio valid for all physical parameters. Nevertheless that correlation has itself two limits of validity : it has been deduced from the linearized equations of the MHD fluid model. We know already that diamagnetic effects decrease the stabilizing effect of the good average curvature /1/. But we also know that in a non linear diffusion regime, temperature and density gradients will be flattened inside magnetic islands and inertia will become negligible /8/ at least for large S values.

Two scalar dependant models do not allow a correct description of the physics which leads to condition (1,1) but they have already provided a lot of very important results for the non linear evolution of tearing modes. Following the same approach we build a model which remains two scalar dependant (i.e. which involves two scalar potentials one for the fluid velocity, the other for the magnetic field) is mathematically coherent and has the new feature to include important toroidal effects, which appear already by expression of toroidal equations to the first order in ϵ .

Let us recall that, starting with toroidal equations, to zero order in ϵ , we recover the cylindrical approximation ; to first order we find two new effects : if the poloidal cross sections of the cylindrical magnetic surfaces are nested concentric circles, they become

displaced circles /2/ by a combined effect of the Laplace force and the pressure gradient (the displacement is of order ϵa). The second effect is the toroidal mode coupling /1/ : if we take two helical modes $\exp i(m_{1,2}\theta + n_{1,2}\psi)$ in straight geometry and if $n_1 = n_2$, $m_1 = m_2 \pm 1$ the coupling appears at order ϵ . It is easily larger than non linear effects, which are of order $(\delta B/B)$ or $(\delta B/B)^2$. Our system of equations includes these two effects. It has an exact energy balance. With a natural generalisation of the two cylindrical potentials u and ψ we obtain a simple but powerful extension of the known cylindrical model /3/.

II - BASIC CONSIDERATIONS

We consider a conducting fluid which is governed by the following system of equations :

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \underline{j} \times \underline{B} \quad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad (2.2)$$

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j} \quad (2.3)$$

$$\rho \left[\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T + (\gamma-1) T \nabla \cdot \underline{v} \right] = (\gamma-1) \left[\nabla \cdot \underline{k} \nabla T + \gamma \underline{j}^2 + \Omega \right] \quad (2.4)$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (2.5)$$

$$\nabla \times \underline{B} = \underline{j} \quad (2.6)$$

$$\nabla \cdot \underline{B} = 0 \quad (2.7)$$

Let \underline{E} be the electric field, \underline{B} the magnetic field, \underline{j} the current density, \underline{v} the fluid velocity, ρ the mass density, T the temperature, p the pressure ($p = \rho T$), γ the ratio of specific heats, η the resistivity with its temperature dependance, \underline{k} the tensor of thermal conductivity ; Ω is an additional heating mechanism (neutral injection...). The resistive ohm's law (2.3) is easily generalized to include the Hall effect which provides diamagnetic frequencies.

The plasma is surrounded by a rigid perfectly conducting wall. At such a boundary the appropriate conditions are :

$$\underline{n} \cdot \underline{B} = 0 \quad , \quad \underline{n} \cdot \underline{v} = 0 \quad (2.8)$$

The temperature T is given, uniform at each time ; $\underline{n} \times \underline{E}$ is also given : the tangential component of \underline{E} is toroidal and $R\mathbf{E}_\theta$ is uniform.

It can be shown that the system of equations above possess a simple equation for the time evolution of energy :

$$\frac{d}{dt} \int_{\Omega} d\tau \left\{ \frac{|B|^2}{2} + \frac{\rho |v|^2}{2} + \frac{pT}{\gamma-1} \right\} = \int_{\Omega} d\tau Q + \int_{\Gamma} dS n_i [B \times E + K_1 \nabla_{\perp} T] \quad (2.9)$$

Where the integration upon $d\tau$ is extended over the plasma domain and upon dS at its boundary.

It is convenient to look for a toroidal representation of the magnetic field which is close to the equilibrium case. Let us first introduce the vacuum toroidal field :

$$\underline{B}_c = B_c R_0 \nabla \phi \quad (2.10)$$

Then one can obtain the following exact and unique decomposition of the total field :

$$\underline{B} = \underline{B}_c + \nabla \chi \times \underline{A} = \underline{B}_c + \underline{B}_1 + \underline{B}_2 + \underline{B}_3 \quad (2.11)$$

$$\underline{A} = \psi \nabla \phi + R^2 \nabla \chi \times \nabla \phi \quad (2.12)$$

$$\underline{B}_1 = \nabla \psi \times \nabla \phi = \nabla \chi \times \underline{A} \quad (2.13)$$

$$\underline{B}_2 = K \nabla \phi = (\nabla \times \underline{A}_m)_\phi \quad (2.14)$$

$$\underline{B}_3 = \nabla \frac{\partial \chi}{\partial \phi} \quad (2.15)$$

$$K = -R^2 \Delta \chi \quad (2.16)$$

With a given K , χ is completely determined by the boundary condition $\chi = 0$. The gauge for the vector potential has been taken as $\nabla_{\perp} \cdot \left(\frac{\underline{A}}{R^2} \right) = 0$.

Explicit formulas for the toroidal and poloidal components of the current density follow easily :

$$\underline{J}_\phi = -\nabla \phi \times \nabla \psi \quad (2.17)$$

$$\underline{J}_m = \frac{1}{R^2} \nabla_m \frac{\partial \psi}{\partial \phi} + \nabla K \times \nabla \phi \quad (2.18)$$

where the toroidal operator L has the standard definition :

$$L\psi = R^2 \nabla \cdot \frac{1}{R^2} \nabla_m \psi \quad (2.19)$$

To progress towards the solution of the system of non linear MHD equations we introduce now the expected ordering of the physical quantities with respect with ϵ . With the Tokamak ordering the equilibrium magnetic field is such that :

$$|B_1| \sim \epsilon B_0 \quad |B_2| \sim \epsilon^2 B_0 \quad |B_3| = 0$$

If we assume the azimuthal gradients of the flux functions to remain smaller or of order ϵ as compared with the poloidal ones, we expect a similar ordering in the non linear evolution of the resistive modes (because $B_3 \sim \epsilon B_2$).

$$|B_1| \sim \epsilon B_c \quad |B_2| \sim \epsilon^2 B_c \quad |B_3| \sim \epsilon^3 B_c$$

From the field ordering, we deduce easily the current ordering and the properties of the Laplace force and subsequently of the velocities. Neglecting B_3 , one obtains the following expression of the two Laplace force components, which are exact to the two first orders in ϵ :

$$R^2 (\underline{J} \times \underline{B})_m \simeq -L\psi \nabla_m \psi + R_0 B_0 \left[\frac{\partial S_m}{\partial \varphi} \Big|_m - \nabla_m K \right] \quad (2.20)$$

$$R^2 (\underline{J} \times \underline{B})_\varphi \simeq \nabla_m K \times \nabla_m \psi - \frac{1}{2} \frac{\partial}{\partial \rho} (\nabla_m \psi)^2 \nabla \phi \quad (2.21)$$

A priori one could expect $(\underline{J} \times \underline{B})_m$ of order $\epsilon B_0^2/a$ and $(\underline{J} \times \underline{B})_\varphi$ of order $\epsilon^3 B_0^2/a$. In fact, in a quasi equilibrium situation the azimuthal pressure gradient is ϵ smaller than the poloidal gradient. The current is mainly force free and $(\underline{J} \times \underline{B})_m$ is of order $\epsilon^2 B_0^2/a$ in the Tokamak ordering.

When the fluid momentum and the pressure gradient "equally" balance the Laplace force the previous analysis implies that :

$$|\underline{V}_\phi| \sim \epsilon |\underline{V}_m|$$

of course both azimuthal and poloidal components of the plasma inertia may become negligible in a non linear diffusive regime. In such a case our ordering may be meaning-less but (inertia being negligible) that choice has no consequence.

III - DERIVATION OF A SIMPLIFIED SET OF TOROIDAL EQUATIONS

In this section we establish the resistive MHD equations for the Tokamak ordering. We start with ohm's law (2.3) which may also be written :

$$\frac{\partial \underline{A}}{\partial t} + \eta \underline{j} = \underline{v} \times \underline{B} + \underline{\nabla} u \quad (3.1)$$

where U has the obvious meaning of a scalar electrostatic potential. We take the projection of equation (3.1) in the magnetic field direction and obtain as an approximation exact to the two first orders in ϵ :

$$\frac{\partial \psi}{\partial t} - \eta \perp \psi = \frac{R^2}{R_0} \frac{\underline{B} \cdot \underline{\nabla} U}{B_0} \quad (3.2)$$

The poloidal component of $\underline{\nabla} U$ is ϵ^{-1} its parallel component when $\underline{A}_m \sim \epsilon \underline{A}_\varphi$ and $\underline{j}_m \sim \epsilon \underline{j}_\varphi$. Then, to the two first orders in ϵ , the poloidal component of (3.1) is equivalent to :

$$[\underline{v} \times \underline{B} + \underline{\nabla} u] \times \underline{\nabla} \psi \simeq 0$$

Taking into account $\underline{v}_\varphi \sim \epsilon \underline{v}_m$ we obtain :

$$v_m = \frac{R}{R_0} \frac{\underline{\nabla} U \times \underline{e}_\varphi}{B_0} \quad (3.3)$$

which is equivalent to :

$$\underline{\nabla} \cdot \frac{\underline{v}_m}{R^2} = 0$$

We have already pointed that condition (1.1) should be verified at least if we follow the linearized behaviour of the resistive modes. Equation (3.3) is obviously in agreement with our understanding of the quasielectrostatic approximation ($\beta p \rightarrow 0$). It is now convenient to set $\rho = \rho(R/R_0)^2$ which simplify the equation of continuity :

$$\frac{\partial \tilde{\rho}}{\partial t} + v_m \cdot \underline{\nabla} \tilde{\rho} = 0 \quad (3.4)$$

In the momentum equations, the poloidal component of the Laplace force (2.20) is of order $\frac{\epsilon^2 B_0}{a}$ and the v_m K contribution cannot be neglected. If we take the component : $\nabla_{\perp} \nabla_{\perp} X$ (momentum equations), we eliminate K to the zero cylindrical order but not to the first order. To obtain a closed set of equations, valid to the first two orders in ϵ , we have to take the component : $\nabla \phi \cdot \nabla X [R^2 \text{ momentum equations}]$ and obtain :

$$\frac{\partial \tilde{W}}{\partial t} + \tilde{V} \cdot (\tilde{v}_m \tilde{W}) = \tilde{V} \cdot \left[(-L\psi) \frac{\tilde{B}}{R^2} - \left(\frac{R^2}{R_0^2} \tilde{\nabla} p - \frac{\tilde{p}}{2} \tilde{\nabla} V^2 \right) \times \tilde{\nabla} \phi \right] \quad (3.5)$$

where it is sufficient to keep $\tilde{B} = B_0 + B_1$. \tilde{W} is defined by :

$$\tilde{W} = \tilde{\nabla} \cdot (\tilde{p} \tilde{v} \times \tilde{\nabla} \phi) = - \frac{1}{R_0 B_0} \tilde{\nabla} \cdot (\tilde{p} \tilde{\nabla}_m u) \quad (3.6)$$

We now set $\tilde{W} = -W \frac{R^2}{R_0^2}$, combine (3.4) and (3.5) and obtain a more convenient form of equation (3.5) :

$$\frac{\partial \tilde{W}}{\partial t} + \tilde{v}_m \cdot \tilde{\nabla} \tilde{W} = \left(\frac{R}{R_0} \right)^2 \tilde{\nabla} \cdot \left\{ \frac{\tilde{B}}{R_0^2} L\psi + \left(\frac{R^2}{R_0^2} \tilde{\nabla} p - \frac{\tilde{p}}{2} \tilde{\nabla} V^2 \right) \times \tilde{\nabla} \phi \right\} \quad (3.7)$$

with $\tilde{B} = B_0 \frac{R}{R_0} \nabla \phi - \nabla \phi \times \nabla \psi$

Our set of equations (3.2), (3.3), (3.4), (3.6) and (3.7) is now closed. It is nevertheless illuminating to indicate how we would compute the toroidal components K and V_{ϕ} . The equation for V_{ϕ} is simply the ϕ component of the momentum equations :

$$\frac{\partial}{\partial t} V_{\phi} + v_m \nabla V_{\phi} = \frac{1}{R} \left\{ \tilde{B}_m \nabla K - \frac{\partial}{\partial t} \left(\tilde{p} + \frac{B_m^2}{2} \right) \right\} \quad (3.8)$$

With our assumptions, the equation for K is not the corresponding component of the ohm's law but follows the momentum equation. To be coherent with equation (3.3), we take the component : $\tilde{\nabla} \cdot \tilde{\rho}^{-1}$ [momentum equation] to eliminate the inertia. In the linearized standard resistive theory [6] a similar equation maintains the pressure balance in the transverse direction to the magnetic field. The equation for the evolution of T is merely equation (2.4) with the substitution $\tilde{v} \rightarrow v_m$ and $\tilde{j} \rightarrow \tilde{j}_{\phi}$.

Let us now complete the model with an explicit formulation of the boundary conditions. $\mathbf{n} \cdot \nabla u = 0$ is equivalent to $u_\Gamma = C(\theta)$ where Γ is the boundary of a poloidal cross section ($\theta = \theta_0$). By the change of variable $u = \int C(\theta) d\theta + u$ compatible with (3.3), we may choose $u_\Gamma = 0$. $\mathbf{n} \cdot \mathbf{E} = 0$ is also equivalent to $\psi_\Gamma = D(\theta)$; but our choice of u implies now $D(\theta) = - \int_0^\tau (R E \theta)_\Gamma dt$ as follows from the definition of E_θ . It seems convenient to set $\tilde{\psi} = \psi + \int_0^\tau (R E \theta)_\Gamma dt$ which leaves $\tilde{\psi}_\Gamma = 0$. We recall that $(R E \theta)_\Gamma$ is given and constant on the boundary. The equation for $\tilde{\psi}$ is easily found with (3.2) and (3.3) to be

$$\frac{\partial \tilde{\psi}}{\partial t} + v_m \cdot \nabla \tilde{\psi} - \eta \Delta \tilde{\psi} = \frac{\partial u}{\partial \phi} + (R E \theta)_\Gamma \quad (3.9)$$

Adding equation (3.2) multiplied by $\nabla \cdot \frac{1}{R^2} \nabla m \psi$, equation (3.5) multiplied by u and equation (2.4) divided by $(\gamma - 1)$ we establish the equation for the time evolution of the total energy :

$$\frac{d}{dt} \int_{\Omega} d\tau \left[\frac{\tilde{\psi} |\nabla_L u|^2}{2 B_0^2} + \frac{|\nabla \psi|^2}{2 R^2} + \frac{p}{\gamma - 1} \right] = \int_{\Omega} d\tau Q + \int_{\Gamma} dS n \cdot \left[-E_\phi \frac{\partial \psi}{\partial n} + \chi_L \frac{\partial T}{\partial n} \right]$$

IV - NORMALIZED SYSTEM OF EQUATIONS

In this last chapter, we normalize the set of equations and indicate our future line of work.

One essential difficulty for studying the non linear evolution of the resistive modes on computers is due to the very different transverse scalings of the boundary layer and of the outside solution. We choose the simple procedure to normalize the plasma transverse dimensions to its radius ; for a numerical solution it will be necessary to increase the number of points in the grid in the vicinity of the boundary layers.

We normalize the time to τ_R the lengths to the plasma radius a , the flux ψ to $a^2 B_0$, the potential u and $(RE\phi)_r$ to $\eta_0 B_0$, the velocities to a/τ_R , \tilde{w} to $\tau_R R_0/\rho_0$ the density ρ to ρ_0 , the temperature T to T_0 and the resistivity η to η_0 . B_0 , η_0 , ρ_0 , T_0 , are the values on the magnetic axis at time $t = 0$. We had already introduced the factor S between the resistive and hydromagnetic characteristic times. Another important dimensionless parameter is obviously the poloidal $\beta_p = \frac{\rho_0}{\epsilon^2 B_0^2}$. Our final set of normalized equations reads :

$$\tilde{v} = \frac{R}{R_0} \nabla u \times e_\phi \quad (4.1)$$

$$\frac{\partial \psi}{\partial t} + \tilde{v} \cdot \nabla \psi = \eta L \psi + \frac{\partial u}{\partial \phi} + e \quad (4.2)$$

with
$$e = (RE\phi)_r / \eta_0 B_0$$

$$\frac{\partial \tilde{p}}{\partial t} + \tilde{v} \cdot \nabla \tilde{p} = 0 \quad (4.3)$$

with
$$\tilde{p} = \left(\frac{R}{R_0}\right)^2 p$$

$$\left(\frac{R}{R_0}\right)^2 \left[\frac{\partial \tilde{w}}{\partial t} + \tilde{v} \cdot \nabla \tilde{w} \right] = S^2 \nabla \cdot \left\{ \tilde{B} L \psi + \beta_p \frac{R}{R_0} \tilde{v} \times e_\phi \right\} \quad (4.4)$$

$$\tilde{w} = \left(\frac{R}{R_0}\right)^2 \nabla \cdot \left\{ \tilde{p} \tilde{v}_m U \right\} \quad \left| \quad - \nabla \cdot \left\{ \tilde{p} \frac{R_0}{R} \left(\frac{\nabla V^2}{2} \times e_\phi \right) \right\} \right. \quad (4.5)$$

$$\tilde{B} = \frac{R_0}{R} \left[\frac{R_0}{a} \frac{\partial \psi}{\partial \phi} + \nabla \psi \times e_\phi \right] \quad (4.6)$$

$$\tilde{\rho} \beta_T \left[\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T + (\gamma-1) T \nabla \cdot \underline{v} \right] = \quad (4.7)$$

$$(\gamma-1) \left\{ R^2 \lambda_{||} \nabla \cdot (\chi_e \nabla_e T) + R^2 \lambda_{\perp} \nabla \cdot (\chi_e \nabla_{\perp} T) + \eta (L \cdot \underline{v})^2 + \frac{R^2 \lambda_{\perp} \tau_e}{a^2} \right\}$$

in (4.7) $\lambda_{||}$ and λ_{\perp} are dimensionless parameters which give a measure of thermal conductivity, respectively

$$\frac{\tau_e T_0}{(a B_0)^2} \chi_e \quad \lambda_{||}, \lambda_{\perp}$$

Our equations include the main macroscopic features of the toroidal resistive modes and provide a significant step forwards with respect to previous models /3/. They give a simultaneous description of the three possible mechanisms for the mode coupling :

- the non linear mode coupling,
- the toroidal mode coupling,
- the poloidal mode coupling due to the shaping of the magnetic surfaces which has to be included in the JET's case.

We will study the influence of these coupling upon the development, saturation and overlapping of magnetic islands. One important goal remains to get a better theoretical understanding of the so called internal and main disruption.

Our model provides the basis of a tri dimensional code. That code will be completed in a next future.

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