VFR 8000281

LYCEN-7950
Juillet 1979

ON THE SU_2 UNIT TENSOR*

Maurice KIBLER+
Department of Applied Mathematics, University of Waterloo,
Waterloo, Ontario, N2I 3GI, Canada

Geneviève GRENET
Institut de Physique Nucléaire (et IN2P3),
Université Claude Bernard Lyon-1 - 69622 Villeurbanne Cedex, France

This paper deals with the $SU_2 \supset G^*$ unit tensor operators $t_{k\mu\alpha}$. In the case where the spinor point group G^* coincides with U_1, then $t_{k\mu\alpha}$ reduces (up to a constant) to the (Wigner-Racah-Schwinger) tensor operator $t_{kq\alpha}$, an operator which produces an angular momentum state $|j+\alpha, m+q>$ when acting the state $|jm>$. We first investigate those general properties of $t_{k\mu\alpha}$ which are independent of their realization. We then turn our attention to realizations of $t_{k\mu\alpha}$ in terms of two pairs of boson creation and annihilation operators. This leads us to look at the Schwinger calculus (found to be connected to the de Sitter algebra $so_{3,2}$) relative to one angular momentum or two coupled angular momenta. As a by-product, we give a procedure for producing recursion relationships between $SU_2 \supset U_1$ Wigner coefficients. Expressions for $t_{kq\alpha}$, which cover the cases k integer and half-an-odd-integer, are derived in terms of boson operators. When k is integer, the latter expressions can be rewritten in the enveloping algebra of so_3 or $so_{3,2}$ according to as $\alpha = 0$ or $\alpha \neq 0$. Finally, we study in two appendices some of the properties of (i) the Wigner and Racah operators for an arbitrary compact group and (ii) the $SU_2 \supset G^*$ coupling coefficients.

1. INTRODUCTION

The concept of a unit tensor is not specific to the (angular momentum) group SU_2.[1-4] As a matter of fact, a unit tensor can be defined for any (compact) topological group J as a set of irreducible unit tensor operators acting on the (pre-Hilbert) representation space $\mathcal{E}$ of J (cf. Appendix 1). The latter operators are frequently referred to as Wigner operators because their matrix elements between vectors of $\mathcal{E}$ transforming irreducibly under J are nothing but Wigner (or Clebsch-Gordan) coefficients of J (see also Ref. 1 and the introductory notes in Ref. 2). Therefore, they constitute basic ingredients for the Wigner-Racah algebra of J (see Refs. 1 and 3 for $J = U_n$.) In the case where J is a compact Lie group, the unit tensor operators comprize the unit adjoint tensor operators which play a key rôle in the solution of the inner multiplicity problem for the Wigner coefficients of J (see Ref. 3 for $J = U_n$ and Ref. 4 for $J = O_n$).

We shall be mainly concerned in this paper with unit tensor operators for $J = SU_2$ in a $SU_2 \supset U_1$ basis and more generally in a $SU_2 \supset G^*$ basis, where G^* stands for (according to molecular physics notation) the spinor group of a molecular or crystallographic point symmetry group G.

We define the $SU_2 \supset U_1$ unit tensor operator $t_{kq\alpha}$ by its $j'm'$ - jm matrix elements through

$$< j'm' | t_{kq\alpha} | jm > = \delta(j', j+\alpha)\,(-1)^{2k}\,(2j'+1)^{-1/2} < jkmq | j'm' > . \quad (1)$$

It clearly appears that the action of $t_{kq\alpha}$ on the state vector $|jm>$ leads to a state vector characterized by $j' = j+\alpha$ and $m' = m+q$.

Operators proportional (as far as matrix elements are concerned) to $t_{kq\alpha}$ enter various fields of Physics and Chemistry. The operator $t_{kq\alpha}$ resembles the operator $t(kq\alpha)$ introduced by Schwinger[5] in his famous treatment of angular momentum. In fact, we have

$$t(kq\alpha) = (-1)^{k+\alpha} \left[\frac{(2j+\alpha+k+1)!}{(2j+\alpha-k)!} \right]^{1/2} t_{kq\alpha} . \qquad (2)$$

The case $\alpha = 0$ deserves special consideration because it corresponds to the Racah[6] unit tensor operator u^k_q. Indeed, our normalization for $t_{kq\alpha}$ is chosen in a way that

$$u^k_q = t_{kqo} , \qquad (3)$$

as far as (shell model $\ell'm' - \ell m$) matrix elements are concerned. This choice justifies itself by the importance of the Racah unit tensor u^k for nuclear, atomic, molecular, and solid state spectroscopy. For instance, let us mention the growing interest of u^k for crystal- and ligand-field theory[7] and for the coupling between conduction electrons and moments of 3d and 4f ions in metals.[8] Another useful normalization is carried out by the Judd[9] tensor v^k :

$$v^k_q = (2k+1)^{1/2} t_{kqo} . \qquad (4)$$

Furthermore, we have the following correspondences :

$$O^o_k \sim t_{koo} ,$$

$$O^q_k(c) \sim \frac{1}{\sqrt{2}} [t_{k-qo} + (-1)^q t_{kqo}], \qquad q \neq 0 ,$$

$$O^q_k(s) \sim \frac{i}{\sqrt{2}} [t_{k-qo} - (-1)^q t_{kqo}], \qquad q \neq 0,$$

$$\tilde{O}_{kq} (\equiv R_{kq}) = \frac{1}{2^k} \left[\frac{(2j+k+1)!}{(2j-k)!} \right]^{1/2} t_{kqo} ,$$

$$T_{kq} = k! \left[\frac{(2j+k+1)!}{2^k (2k)! \, (2j-k)!} \right]^{1/2} t_{kqo} , \qquad (5)$$

where O, $\tilde{O}$ ($\equiv R$), and T denotes the operators defined by Stevens,[10] Buckmaster (Lindgård),[11] and Buckmaster et al.,[12] respectively. These latter operators are of central importance in the theory of magnetic resonance (electron paramagnetic resonance as well as electronic and nuclear double resonance) and related phenomena.[10-13] Note that when k is integer, all those of the preceding operators that are proportional to $t_{kq\alpha}$ transform under rotation like the spherical harmonic Y_{kq} while $O_k^q(c)$ and $O_k^q(s)$ transform like the so-called tesseral harmonics Z_{kq}^c and Z_{kq}^s, respectively.

There exist numerous realizations for the operators of type $t_{kq\alpha}$, which turn out to be useful in physical applications. We shall consider in turn the particular cases $\alpha = 0$ and α = integer, and the general case α = integer or half-integer.

The case $\alpha = 0$:

First, t_{kqo} can be realized in terms of (two) Bose operators. This yields (generally infinite) expansions that are of interest in the theory of magnetism (especially for spin wave calculations) and in nuclear physics (especially for a description of collective motions in nuclei). The most known ways to obtain such Bose operator expansions are probably through the use of the Holstein-Primakoff transformation and the Dyson and Maleev transformation.[14] Along this line, let us also mention the recent MME method[15] from which a Bose expansion of any $SU_2 \supset U_1$ tensor operator can be obtained by matching pertinent matrix elements.

Second, t_{kqo} can be realized in the enveloping algebra of su_2. This yields realizations which are known as polarized harmonics in nuclear physics[16] and as (diagonal) operator equivalents in solid state physics.[10-13] (The word diagonal refers to $\alpha = 0$.) The polarized form to t_{kqo} can be obtained in principle from

$$t_{kqo} = \frac{2^k}{k!} \left[\frac{4\pi}{2k+1} \; \frac{(2j-k)!}{(2j+k+1)!} \right]^{1/2} (\vec{J} \cdot \overrightarrow{\text{grad}})^k \mathcal{Y}_{kq}(\vec{r}) \, , \qquad (6)$$

i.e., by polarization of the solid harmonic $\mathcal{Y}_{kq}(\vec{r})$. However, the obtention of the operator equivalents form of t_{kqo} is easier in many respects. The operator equivalents have proved to be extremely fruitful for the understanding of the magnetic and optical properties of d^N and f^N partly-filled shell ions plunged into crystalline materials. Therefore, it is perhaps worthwhile to briefly discuss the state-of-the-art in the operator equivalents and the operator equivalents method originally introduced by Stevens[10] and used in molecular and solid state physics since more than 25 years.[10-13]

In many fields we need calculate matrix elements of interactions involving (or transforming like) harmonic polynomials $r^k Y_{kq}(\theta, \varphi)$ and $r^{-k-1} Y_{kq}(\theta, \varphi)$ or more generally quantities of type $f(r) Y_{kq}(\theta, \varphi)$. The Stevens prescription to get the matrix elements of $f(r) Y_{kq}(\theta, \varphi)$ within a space $\epsilon(j)$ of given angular momentum j proceeds as follows : (i) Express $f(r) Y_{kq}(\theta, \varphi)$ in terms of x, y, z (or $x \pm iy, z$) and symmetrize the obtained expression. (ii) Make the substitution $u \rightarrow J_u$ with $u = x, y, z$ and transform the obtained expression by using the angular momentum commutation rules. This leads to a diagonal operator equivalent for $f(r) Y_{kq}(\theta, \varphi)$ the (easily obtainable) matrix elements of which are (by virtue of the Wigner-Eckart theorem for the chain $SU_2 \supset U_1$) proportional within $\epsilon(j)$ to those of $f(r) Y_{kq}(\theta, \varphi)$. (iii) Evaluate the proportionality constant by working out twice one single matrix element.

The case α = integer :

A prescription similar to the Stevens one exists when α is an integer different from zero and permits to get the matrix elements of $f(r)\ Y_{kq}(\theta, \varphi)$ connecting states of angular momenta j and $j+\alpha$.[17] This yields $f(r)\ Y_{kq}(\theta, \varphi)$ to be mimicked by an off-diagonal operator equivalent, a quantity defined in the enveloping algebra of a 10-dimensional Lie algebra, namely, the Schwinger algebra s (to be explicited below).

Although the two afore-mentioned prescriptions to derive realizations of $t_{kq\alpha}$ with k , and therefore α , integer in the enveloping algebra of either su_2 ($\alpha = 0$) or s ($\alpha \neq 0$) both lie on the $SU_2 \supset U_1$ Wigner-Eckart theorem, it is to be noted that they lead to calculations of the matrix elements of $f(r)\ Y_{kq}(\theta, \varphi)$ which are far more tedious than the ones involved with a direct application of this theorem. It is the opinion of the authors that the development and the use of the operator equivalents method partly prevented and obscured the penetration of the Wigner-Racah (angular momentum) calculus in EPR and ENDOR spectroscopies and, to a less extent, in crystal-field theory. However, as a nice counterpart, it is probably true that the operator equivalents method largely contributed to the development of the (static and dynamic) spin Hamiltonian formalism originally introduced by Abragam and Pryce (cf. Ref. 13) and which is so useful for phenomenological descriptions of microwave resonance data.

There are many other ways to obtain realizations of t_{kqo} and $t_{kq\alpha}$, with α being integer , in the enveloping algebras of su_2 and s , respectively.[12, 16, 18-22] Let us mention, among others, the algorithms based on : (i) Step-up procedures with shift (lowering or raising) operators.[12, 16] (ii) Polynomial methods as e. g. in the approaches starting from the Molien generating function[18] or from the construction of orthogonal polynomials of a discrete real variable.[19] (iii) Analyses of formulas for Clebsch-Gordan coefficients.[20,22] (iv) Finite-difference and commutator calculus.[19,21]

The case α = integer and half-integer :

In the general case, a boson realization of $t_{kq\alpha}$ can be obtained in principle from the Schwinger generating function for $t(kq\alpha)$.[5] An alternative realization can be excerpted from the boson representation of the $SU_2 \supset U_1$ Wigner-Eckart theorem recently proposed by Yamamura et al.[23] in their investigation of the Schwinger representation of the quantized rotator and its application to nuclear structure theory. Finally, the present authors have recently reported a preliminary account of another boson realization of $t_{kq\alpha}$.[22]

It is one of the aims of this work to discuss realizations of $t_{kq\alpha}$ which cover both the cases k (and therefore q and α) integer and half-an-odd-integer. We start in Sec. 2 with those properties of the operators $t_{kq\alpha}$ (and more generally of the G^* symmetry adapted operators $t_{ka\Gamma\gamma\alpha}$) which do not depend on their realizations. We then turn our attention to the problem of constructing realizations of $t_{kq\alpha}$ in terms of boson operators. For that purpose, we devote Sec. 3 to (some aspects of) the boson representation of angular momentum introduced by Jordan and fully developed by Schwinger.[5] This representation is by now used in physical fields as distant as : the theory of magnetism,[24] the collective model and the shell model of the nucleus,[25] and the elementary particle physics.[26] A Lie-like approach to s , relevant for Schwinger's angular momentum calculus, is developed in Sec. 3 . It is proved that the $O_{3,2}$ de Sitter algebra constitutes the most general frame for an investigation of Schwinger's calculus. The pseudo-orthogonal groups $O_{p,q}$ have received considerable attention in the last 15 years. In particular, the de Sitter group $O_{3,2}$ has been recently discussed in various contexts : rigid rotator,[27,28] hydrogen atom,[29] symmetric top,[30] and mathematical physics.[31,32] It turns out to be one of the maximal subgroups of the conformal group $SO_{4,2}$ (locally isomorphic to $SU_{2,2}$ and $Sp_{4;\mathbb{R}}$). In addition, contraction of the de Sitter group $O_{3,2}$ (or $O_{4,1}$) yields the Poincaré group. All the closed connected subgroups of $O_{3,2}$ have been recently exhibited by Patera et al.[31] and the reader is referred to their work for the material concerning $O_{3,2}$ relevant to the present paper. We show in Sec. 4 how, when applied to a composite system (the Schwinger algebra of which is described by the chain $SO_{3,2} \otimes SO_{3,2} \supset SO_{3,2}$), Schwinger's calculus allows one to systematically derive families of recursion relationships for the $SU_2 \supset U_1$ Clebsch-Gordan coefficients. In particular, recurrence relations between 3-jm symbols recently (re)derived[33] from considerations on hypergeometric functions or recoupling formulas are obtained through simple manipulations of ladder operators. Section 5 goes back to our first preoccupation : it gives some realizations of the unit tensor operator $t_{kq\alpha}$. Several general formulas for $t_{kq\alpha}$ valid in the cases k integer and half-an-odd-integer are obtained in the boson representation of angular momentum. For the special (nevertheless physically important) case where k is integer, the formulas can be rewritten in the enveloping algebra of $so_{3,2}$. Many examples are given throughout the paper and useful formulas related to (1) the Lie algebra it is possible to associate with the Wigner-Racah algebra of any compact group and (2) some properties of the coupling coefficients for the chain $SU_2 \supset G^*$ are relagated in Appendices 1 and 2, respectively.

2. GENERAL PROPERTIES OF THE SU_2 UNIT TENSOR

Notations : We start from the Hilbert space $\mathcal{E} = \bigoplus_j \varepsilon(j)$, where

$$\varepsilon(j) = \left\{ |jm\rangle \ : \ m \text{ ranging} \right\} \qquad (7)$$

is an irreducible subspace of $\mathcal{E}$ associated with the IRC (irreducible representations class) j of SU_2 and spanned by the eigenvectors $|jm\rangle$ of the square J^2 and the 3-component J_3 (or J_z) of a generalized angular momentum operator. By introducing the subduction $SU_2 \downarrow G^*$, the space $\varepsilon(j)$ decomposes as $\varepsilon(j) = \bigoplus_{a\Gamma} \varepsilon(ja\Gamma)$, where Γ stands for an IRC of G^*, Γ_o being the identity IRC, and a denotes a branching multiplicity label to be used when Γ occurs several times in j. Indeed, we have

$$\varepsilon(ja\Gamma) = \left\{ |ja\Gamma\gamma\rangle \ : \ \gamma \text{ ranging} \right\}, \qquad (8)$$

where the $SU_2 \supset G^*$ symmetry adapted vector

$$|ja\Gamma\gamma\rangle = \sum_m |jm\rangle \langle jm | ja\Gamma\gamma\rangle \qquad (9)$$

is obtained from the $SU_2 \supset U_1$ vectors $|jm\rangle$ with the help of a unitary transformation, the matrix elements of which are $\langle jm | ja\Gamma\gamma\rangle$. By applying the same transformation to the $SU_2 \supset U_1$ operators $t_{kq\alpha}$, we obtain $SU_2 \supset G^*$ symmetry adapted unit tensor operators

$$t_{ka\Gamma\gamma\alpha} = \sum_q t_{kq\alpha} \langle kq | ka\Gamma\gamma\rangle . \qquad (10)$$

As a word of comment, it should be noted that the transformation from the $\{m\}$ scheme to the $\{a\Gamma\gamma\}$ scheme is chosen in such a way that, for fixed Γ, all sets $\{|ja\Gamma\gamma\rangle \ : \ \gamma \text{ ranging}\}$ and $\{t_{ka\Gamma\gamma\alpha} \ : \ \gamma \text{ ranging}\}$ span the same (rather than equivalent) irreducible matrix representation. This standardization turns out to be a necessary requirement for the application of Racah's factorization lemma[34] to the chain $SU_2 \supset G^*$ (cf. Appendices 1 and 2). When applied simultaneously to vectors and operators, the transformation under consideration allows us to transform Eq. (1) in the following $SU_2 \supset G^*$ symmetry adapted form

$$< j_1 a_1 \Gamma_1 \gamma_1 \mid t_{ka\Gamma\gamma\alpha} \mid j_2 a_2 \Gamma_2 \gamma_2 > \; = \; \delta(j_1, j_2+\alpha) \; f\begin{pmatrix} j_1 & j_2 & k \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a\Gamma\gamma \end{pmatrix}, \quad (11)$$

where the f symbol is defined via (cf. Appendix 2)[35, 36]

$$f\begin{pmatrix} j_1 & j_2 & k \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a\Gamma\gamma \end{pmatrix} = \sum_{m_1 q m_2} (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix}$$

$$< j_1 m_1 \mid j_1 a_1 \Gamma_1 \gamma_1 >^* < kq \mid ka\Gamma\gamma > \; < j_2 m_2 \mid j_2 a_2 \Gamma_2 \gamma_2 > . \quad (12)$$

To end up with our notational preliminaries, we define

$$\mathrm{tr}_{\epsilon(j)} A = \sum_m < jm \mid A \mid jm > ,$$
$$\mathrm{tr}_{\epsilon(ja\Gamma)} A = \sum_\gamma < ja\Gamma\gamma \mid A \mid ja\Gamma\gamma > . \quad (13)$$

Further, it will prove convenient to use the abbreviated form μ for $a\Gamma\gamma$. Note that to pass from G^* arbitrary to $G^* = U_1$, it will be sufficient to apply the following correspondence rules

$$f\begin{pmatrix} j_1 & j_2 & k \\ \mu_1 & \mu_2 & \mu \end{pmatrix} \longleftrightarrow (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \quad \text{and} \quad \mu \longleftrightarrow m \text{ or } q . \quad (14)$$

Then, $\delta(\mu_2 \mu_1)$ will be understood as $\delta(a_2 a_1)\, \delta(\Gamma_2 \Gamma_1)\, \delta(\gamma_2 \gamma_1)$ or as $\delta(m_2\, m_1)$. Finally, all other symbols will have their usual significance.

Phase choices : It is straightforward to verify that

$$t_{k\mu\alpha}{}^{\dagger} = (-1)^{k+\alpha} \sum_{\mu'} \begin{pmatrix} k \\ \mu\ \mu' \end{pmatrix} t_{k\mu'-\alpha} \qquad (15)$$

and

$$K\, t_{k\mu\alpha}\, K^{-1} = \sum_{\mu'} \begin{pmatrix} k \\ \mu\ \mu' \end{pmatrix} t_{k\mu'\alpha} \ , \qquad (16)$$

so that

$$t_{k\mu\alpha}{}^{\dagger} = (-1)^{k+\alpha}\, K\, t_{k\mu-\alpha}\, K^{-1} , \qquad (17)$$

where K is the Wigner time-reversal operator and the Herring-Wigner metric tensor[37] is defined through (cf. Appendix 2)[36]

$$\begin{pmatrix} j \\ \mu\ \mu' \end{pmatrix} = \sum_{m} (-1)^{j+m} < jm \mid j\mu >^{*} < j\text{-}m \mid j\mu' >^{*} . \qquad (18)$$

Note that when $G^{*} = U_1$, Eqs. (15) and (16) reduce to

$$t_{kq\alpha}{}^{\dagger} = (-1)^{-q+\alpha}\, t_{k-q-\alpha} \qquad (19)$$

and

$$K\, t_{kq\alpha}\, K^{-1} = (-1)^{k+q}\, t_{k-q\alpha} \ , \qquad (20)$$

respectively, in agreement with usual phase choices (cf. Ref. 38) .

Orthogonality and completeness relations : The relations

$$\sum_{\mu} t_{k\mu\alpha}{}^{\dagger}\, t_{k\mu\alpha'} = \delta(\alpha'\alpha)\ \delta(j+\alpha, k, j)\ (2j+1)^{-1} , \qquad (21)$$

and

$$\sum_{\alpha} [\, 2(j-\alpha) + 1\,]\ t_{k\mu\alpha}\ t_{k\mu'\alpha}{}^{\dagger} = \delta(\mu'\mu) , \qquad (22)$$

are valid when acting on $\epsilon(j)$. They follow from the orthogonality property of the f coefficients (cf. Appendix 2). It is to be observed that Eqs. (21) and (22) parallel the orthogonality relation (2.14) and the completeness relation (2.15), respectively, given in Ref. 3 for the unit tensor operators of U_n.

$\epsilon(j)$-orthogonality property : The $t_{k\mu\alpha}$'s are $\epsilon(j)$-mutually-orthogonal in the sense that

$$\mathrm{tr}_{\epsilon(j)}\; t_{k_1\mu_1\alpha_1}^{\dagger}\; t_{k_2\mu_2\alpha_2} = \delta(j+\alpha_1, j, k_1)\; \delta(k_2 k_1)\; \delta(\mu_2\mu_1)\; \delta(\alpha_2\alpha_1)\; (2k_1+1)^{-1}. \quad (23)$$

The proof of Eq. (23) directly follows from the orthogonality property of the f coefficients. In the particular case $\alpha_1 = \alpha_2 = 0$, Eq. (23) shows that the $(2j+1)^2$ operators u_μ^k, with μ ranging and $k = 0(1)\,2j$, acting on $\epsilon(j)$ span a pre-Hilbert space $\mathcal{G}$ with respect to the Hilbert-Schmidt scalar product $(A.B) = \mathrm{tr}_{\epsilon(j)}\, A^{\dagger} B$. The vectors $u_{\mu_1}^{k_1}$ and $u_{\mu_2}^{k_2}$ are thus mutually orthogonal on $\mathcal{G}$ (see also Ref. 19). Equation (23) generalizes the trace relation known for the $SU_2 \supset U_1$ diagonal unit tensor operators (cf. for example Ref. 39). This latter relation is very popular in the density matrix theory and its application to various fields as e.g. the optical pumping theory. It also specializes to the so-called barycenter rule which depicts the spectroscopic stability principle.

Fourier analysis : The coefficients $c(k\mu\alpha)$ of the development

$$H = \sum_{k\mu\alpha} c(k\mu\alpha)\; t_{k\mu\alpha} \quad (24)$$

of an arbitrary operator H acting on $\epsilon(j)$ are given by

$$c(k\mu\alpha)\; \delta(j+\alpha, j, k) = (2k+1)\; \mathrm{tr}_{\epsilon(j)}\; t_{k\mu\alpha}^{\dagger}\; H. \quad (25)$$

Eq. (25) is an immediate consequence of Eq. (23). As a concrete example, the $SU_2 \supset G^*$ Wigner projection-transfer operator (known when $G^* = U_1$ as the Hill-Wheeler integral in nuclear physics[40])

$$P_{j\mu\mu'} = \frac{2j+1}{|SU_2|} \int_{SU_2} dR \; D^j(R)_{\mu\mu'}{}^* \; P_R \tag{26}$$

can be developed as

$$P_{j\mu\mu'} = \sum_{k\,\mu''} (2k+1) \; f\begin{pmatrix} j & j & k \\ \mu & \mu' & \mu'' \end{pmatrix} \; u^k_{\mu''} \,. \tag{27}$$

As another example, the development

$$Y_{k\mu} = (-1)^{\ell} \, (2\ell+1)^{1/2} \left(\frac{2k+1}{4\pi}\right)^{1/2} \sum_{\alpha} (-1)^{\alpha} \left[2(\ell+\alpha)+1\right]^{1/2} \begin{pmatrix} \ell+\alpha & k & \ell \\ 0 & 0 & 0 \end{pmatrix} t_{k\mu\alpha} \tag{28}$$

holds as far as $\ell'\mu' - \ell\mu$ matrix elements are concerned.

$\varepsilon(ja\Gamma)$-orthogonality property : *The* $t_{ka\Gamma\gamma\alpha}$'s *are* $\varepsilon(ja\Gamma)$-*mutually-orthogonal in the sense that*

$$\begin{aligned} &\mathrm{tr}_{\varepsilon(ja\Gamma)} \, t_{k_1 a_1 \Gamma_1 \gamma_1 \alpha_1}{}^{\dagger} \; t_{k_2 a_2 \Gamma_2 \gamma_2 \alpha_2} \\ &\quad = \delta(\Gamma_2\Gamma_1) \; \delta(\gamma_2\gamma_1) \; \delta(\alpha_2\alpha_1) \; c(j a k_1 a_1 k_2 a_2 \Gamma\Gamma_1 \alpha_1), \end{aligned} \tag{29}$$

where the coefficient c does not depend on γ_2 and γ_1. The proof of Eq. (29) follows by applying Racah's lemma to the chain $SU_2 \supset G^*$ and by using the so-called orthogonality-completeness property for the Wigner coefficients of G^* (cf. Appendices 1 and 2).

Commutator relations : The commutator $([\,,\,] \equiv [\,,\,]_-)$ and anti-commutator $([\,,\,]_+)$ of two $SU_2 \supset G^*$ unit tensor operators are given by

$$\left[t_{k_1\mu_1\alpha_1} , t_{k_2\mu_2\alpha_2} \right]_{\pm} = \sum_{k_3\mu_3\alpha_3} \delta(\alpha_3, \alpha_1+\alpha_2)$$

$$(-1)^{2j+k_3+\alpha_3} (2k_3+1)\, f\begin{pmatrix} k_3 & k_2 & k_1 \\ \mu_3 & \mu_2 & \mu_1 \end{pmatrix}$$

$$\left[(-1)^{k_1+k_2+k_3+2\alpha_2} \begin{Bmatrix} k_1 & k_2 & k_3 \\ j & j+\alpha_3 & j+\alpha_2 \end{Bmatrix} \pm (-1)^{2\alpha_1} \begin{Bmatrix} k_1 & k_2 & k_3 \\ j+\alpha_3 & j & j+\alpha_1 \end{Bmatrix} \right] t_{k_3\mu_3\alpha_3} . \tag{30}$$

The proof of Eq. (30) proceeds along the same lines as the one described by Judd,[9] on the basis of angular momentum recoupling techniques, in the case of $G^* = U_1$ and $\alpha_1 = \alpha_2 = 0$. (See also Appendix 1 where the proof is outlined in the case of an arbitrary compact group.)

As a trivial by-product, Eq. (30) provides an expression of the product of the two operators $t_{k_1\mu_1\alpha_1}$ and $t_{k_2\mu_2\alpha_2}$ in terms of operators $t_{k_3\mu_3\alpha_3}$. Products of this type prove useful in various physical problems. In the case $\alpha_1 = \alpha_2 = 0$, they can be used in the calculation of time derivatives of tensor operators and in the evaluation of thermodynamic averages for systems described by a spin Hamiltonian (cf. Refs. 10-13 and more specifically the work by Lindgård in Ref. 11).

By way of illustration, let us consider the case $\Gamma_1 = \Gamma_2 = \Gamma_0$. Equation (30) then yields

$$\left[t_{k_1 a_1 \Gamma_0 \gamma_0 \alpha_1} , t_{k_2 a_2 \Gamma_0 \gamma_0 \alpha_2} \right]_{\pm} = \sum_{k_3 a_3 \alpha_3} \delta(\alpha_3, \alpha_1 + \alpha_2)$$

$$(-1)^{2j+k_3+\alpha_3} (2k_3+1) \; f \begin{pmatrix} k_3 & k_2 & k_1 \\ a_3 \Gamma_0 \gamma_0 & a_2 \Gamma_0 \gamma_0 & a_1 \Gamma_0 \gamma_0 \end{pmatrix}$$

$$\Bigg[(-1)^{k_1+k_2+k_3+2\alpha_2} \begin{Bmatrix} k_1 & k_2 & k_3 \\ j & j+\alpha_3 & j+\alpha_2 \end{Bmatrix}$$

$$\pm (-1)^{2\alpha_1} \begin{Bmatrix} k_1 & k_2 & k_3 \\ j+\alpha_3 & j & j+\alpha_1 \end{Bmatrix} \Bigg] \, t_{k_3 a_3 \Gamma_0 \gamma_0 \alpha_3} \, , \qquad (31)$$

which provides a development of the product of two G-invariant operators of type $t_{k a \Gamma_0 \gamma_0 \alpha}$ as a sum of G-invariant operators of the same type.

By specializing Eq. (30) to the case $\alpha_1 = \alpha_2 = 0$, it appears that, when acting on the space $\epsilon(j)$, the set $\left\{ u^k_\mu \; : \; \mu \text{ and } k \text{ ranging} \right\}$ can be used to generate some subalgebra of $g\ell_{2j+1\,;\,\mathbb{R}}$ with respect to the Lie product $[\;,\;]$. This result, of considerable importance for physical applications, was discussed by Racah[34] and Judd[9] in the case $G^* = U_1$. [The passage from $G^* = U_1$ to G^* arbitrary is achieved owing to the correspondence rules (14) and corresponds to a simple change of the structure constants in the Lie algebra of the group $GL_{2j+1\,;\,\mathbb{R}}$.[36]] As an illustration, su_2 and su_3 are generated by the sets $\left\{ u^1_\mu \; : \; \mu \text{ ranging} \right\}$ and $\left\{ u^1_\mu, \, u^2_{\mu'} \; : \; \mu \text{ and } \mu' \text{ ranging} \right\}$ acting on $\epsilon(1/2)$ and $\epsilon(1)$, respectively. More precisely, su_3 can be described in an $O_3 \supset G$ basis by

$$\left[u^1_\mu , u^1_{\mu'}\right] = \sum_{\mu''} f\begin{pmatrix} 1 & 1 & 1 \\ \mu'' & \mu' & \mu \end{pmatrix} u^1_{\mu''} ,$$

$$\left[u^1_\mu , u^2_{\mu'}\right] = \sqrt{5} \sum_{\mu''} f\begin{pmatrix} 2 & 2 & 1 \\ \mu'' & \mu' & \mu \end{pmatrix} u^2_{\mu''} , \qquad (32)$$

$$\left[u^2_\mu , u^2_{\mu'}\right] = \frac{-3}{\sqrt{5}} \sum_{\mu''} f\begin{pmatrix} 1 & 2 & 2 \\ \mu'' & \mu' & \mu \end{pmatrix} u^1_{\mu''} .$$

Similar expressions could be obtained for su_5 in an $O_3 \supset O$ basis, where O is the octahedral group, a basis of interest in those molecular problems where the chain $SU_5 \supset O_5 \supset O_3 \supset O$ is relevant as e.g. in the Jahn-Teller problem.[41]

3. THE SCHWINGER ALGEBRA

Let $a_\pm$ and $a_\pm^+$ be the operators defined on $\mathcal{E}$ by

$$a_\pm \,|\, jm> \;=\; (j \pm m)^{1/2} \,|\, j - 1/2\,,\; m \mp 1/2 > \qquad (33)$$

and

$$a_\pm^+ \,|\, jm> \;=\; (j \pm m + 1)^{1/2} \,|\, j + 1/2\,,\; m \pm 1/2 > \,, \qquad (34)$$

respectively. We easily deduce from Eqs. (33) and (34) the commutation relations

$$[a_+, a_+^+] = [a_-, a_-^+] = 1 \,,$$

$$[a_+, a_-] = [a_+, a_-^+] = [a_+^+, a_-] = [a_+^+, a_-^+] = 0 \,. \qquad (35)$$

The sets $\{a_+, a_+^+\}$ and $\{a_-, a_-^+\}$ are closed under hermitian conjugation :

$$a_\pm^+ = (a_\pm)^\dagger . \qquad (36)$$

They can be considered as two sets of boson operators corresponding to a two-dimensional isotropic harmonic oscillator (cf. Ref. 5) . These sets are interchanged under time-reversal operation :

$$K\, a_\pm\, K^{-1} = \mp\, a_\mp \,,$$

$$K\, a_\pm^+\, K^{-1} = \mp\, a_\mp^+ \,. \qquad (37)$$

By induction we get the useful formulas

$$< j'm' \,\big|\, (a_\pm)^p \,\big|\, jm > \;=\; \delta\left(j', j - \frac{p}{2}\right) \delta\left(m', m \mp \frac{p}{2}\right) \left[\frac{(j \pm m)\,!}{(j \pm m - p)\,!}\right]^{1/2} ,$$

$$< j'm' \,\big|\, (a_\pm^+)^p \,\big|\, jm > \;=\; \delta\left(j', j + \frac{p}{2}\right) \delta\left(m', m \pm \frac{p}{2}\right) \left[\frac{(j \pm m + p)\,!}{(j \pm m)\,!}\right]^{1/2} . \qquad (38)$$

From the four boson creation and annihilation operators $a_{\pm}$ and $a_{\pm}^{+}$, we can form ten linearly independent bilinear expressions. Let us put

$$
\begin{aligned}
k_{o}^{o} &= \frac{1}{2}\left(a_{+}^{+} a_{+} - a_{-}^{+} a_{-}\right), \quad k_{\pm}^{o} = a_{\mp} a_{\pm}^{+} , \\
k_{o}^{+} &= a_{+}^{+} a_{-}^{+} , \quad k_{\pm}^{+} = \mp a_{\pm}^{+} a_{\pm}^{+} , \\
k_{o}^{-} &= a_{+} a_{-} , \quad k_{\pm}^{-} = \pm a_{\mp} a_{\mp} , \\
J &= a_{+}^{+} a_{+} + a_{-}^{+} a_{-} + 1 .
\end{aligned} \tag{39}
$$

The notation k_{σ}^{ρ} follows the one by Atkins and Seymour[17] except for the operator J which is connected to their $\bar{j}$ by $J = 2\bar{j} + 1$. (J is the simplest operator to be introduced in order to close the set $\{k_{\sigma}^{\rho} : \rho \text{ and } \sigma \text{ ranging}\}$ under commutation.) Note that $-k_{+}^{+}$, k_{-}^{+}, k_{+}^{-}, and $-k_{-}^{-}$ identify to the operators $\hat{M}_{+}$, $\hat{N}_{+}$, $\hat{N}_{-}$, and $\hat{M}_{-}$, respectively, defined by Witschel.[42]

The operators $J_3 \equiv k_{o}^{o}$ and $J_{\pm} \equiv k_{\pm}^{o}$ satisfy the commutation relations

$$
\begin{aligned}
[J_3, J_{\pm}] &= \pm J_{\pm} , \\
[J_{+}, J_{-}] &= 2 J_3
\end{aligned} \tag{40}
$$

of the <u>spherical</u> angular momentum while $K_3 \equiv \frac{1}{2} J$ and $K_{\pm} \equiv k_{o}^{\pm}$ satisfy the commutation relations

$$
\begin{aligned}
[K_3, K_{\pm}] &= \pm K_{\pm} , \\
[K_{+}, K_{-}] &= -2 K_3
\end{aligned} \tag{41}
$$

of the <u>hyperbolic</u> angular momentum introduced by Schwinger.[5] The spherical and hyperbolic angular momentum operators can thus be regarded as the generators of the group SU_2 and $SU_{1,1}$ (two-to-one homomorphic to the group SO_3 and $SO_{2,1}$), respectively.

The properties of the k's and J can be readily deduced from those of the a's. By repeated application of Eqs. (33) and (34), we get (cf. Ref. 17)

$$
\begin{aligned}
k_{\pm}^{o} |jm> &= [(j\mp m)(j\pm m+1)]^{1/2} |j, m\pm 1> , \\
k_{o}^{o} |jm> &= m |jm> , \\
k_{o}^{+} |jm> &= [(j-m+1)(j+m+1)]^{1/2} |j+1, m> , \\
k_{o}^{-} |jm> &= [(j-m)(j+m)]^{1/2} |j-1, m> , \\
J |jm> &= (2j+1) |jm> , \\
k_{\pm}^{+} |jm> &= \mp [(j\pm m+1)(j\pm m+2)]^{1/2} |j+1, m\pm 1> , \\
k_{\pm}^{-} |jm> &= \pm [(j\mp m-1)(j\mp m)]^{1/2} |j-1, m\pm 1> . \qquad (42)
\end{aligned}
$$

(Observe there is a misprint in the expression for $k_{\mp}^{o}$ $|jm>$ given in Ref. 17.) The set $\{J, k_{\sigma}^{\rho} : \rho$ and σ ranging$\}$ is closed with respect to taking hermitian conjugate since

$$
\begin{aligned}
k_{-\sigma}^{-\rho} &= (k_{\sigma}^{\rho})^{\dagger} , \\
J &= J^{\dagger} .
\end{aligned} \qquad (43)
$$

In addition, time-reversal operation leads to

$$K\, k^{\rho}_{\sigma}\, K^{-1} = -k^{\rho}_{-\sigma} \,,$$
$$K\, J\, K^{-1} = J \,. \tag{44}$$

Finally, by finite induction we obtain the useful formulas

$$\langle j'm' \,|\, (k^{o}_{\pm})^{p} \,|\, jm \rangle = \delta(j', j)\ \delta(m', m \pm p) \left[\frac{(j \mp m)!\ (j \pm m + p)!}{(j \pm m)!\ (j \mp m - p)!} \right]^{1/2} ,$$

$$\langle j'm' \,|\, (k^{+}_{o})^{p} \,|\, jm \rangle = \delta(j', j+p)\ \delta(m', m) \left[\frac{(j - m + p)!\ (j + m + p)!}{(j - m)!\ (j + m)!} \right]^{1/2} ,$$

$$\langle j'm' \,|\, (k^{-}_{o})^{p} \,|\, jm \rangle = \delta(j', j-p)\ \delta(m', m) \left[\frac{(j - m)!\ (j + m)!}{(j - m - p)!\ (j + m - p)!} \right]^{1/2} ,$$

$$\langle j'm' \,|\, (k^{+}_{\pm})^{p} \,|\, jm \rangle = \delta(j', j+p)\ \delta(m', m \pm p)\ (\mp 1)^{p} \left[\frac{(j \pm m + 2p)!}{(j \pm m)!} \right]^{1/2} ,$$

$$\langle j'm' \,|\, (k^{-}_{\pm})^{p} \,|\, jm \rangle = \delta(j', j-p)\ \delta(m', m \pm p) \left[\frac{(j \mp m)!}{(j \mp m - 2p)!} \right]^{1/2} . \tag{45}$$

The set $\left\{ J,\ k^{\rho}_{\sigma} : \rho \text{ and } \sigma \text{ ranging} \right\}$ is closed under commutation and thus describes a Lie algebra we shall note s and refer to as the Schwinger algebra. The relevant Lie brackets are listed in Table 1. It is clear from

insert Table 1 around here

Table 1 that $(k^{+}_{o}, k^{+}_{+}, k^{+}_{-})$ and $(k^{-}_{o}, k^{-}_{+}, k^{-}_{-})$ behave as vectors under the generators J_3 and $J_{\pm}$ of SU_2, a fact which is at the root of the prescriptions used in Ref. 17 for obtaining diagonal and off-diagonal operator equivalents as polynomials in $\overline{j}$ and k^{ρ}_{σ}. Note that the second order invariant of s

$$I_2 = \frac{1}{6} J^2 - \frac{1}{6} K^2 - \frac{1}{24} \left(k^{+}_{+} k^{-}_{-} + k^{-}_{-} k^{+}_{+} + k^{+}_{-} k^{-}_{+} + k^{-}_{+} k^{+}_{-} \right) \tag{46}$$

TABLE 1. Lie brackets of the Schwinger algebra s . For instance $[k_0^-, k_0^+] = J$.

	k_0^0	k_+^0	k_-^0	k_0^+	k_+^+	k_-^+	k_0^-	k_+^-	k_-^-	J
k_0^0	0	k_+^0	$-k_-^0$	0	k_+^+	$-k_-^+$	0	k_+^-	$-k_-^-$	0
k_+^0	$-k_+^0$	0	$2k_0^0$	$-k_+^+$	0	$2k_0^+$	$-k_+^-$	0	$2k_0^-$	0
k_-^0	k_-^0	$-2k_0^0$	0	k_-^+	$-2k_0^+$	0	k_-^-	$-2k_0^-$	0	0
k_0^+	0	k_+^+	$-k_-^+$	0	0	0	$-J$	$-2k_+^0$	$2k_-^0$	$-2k_0^+$
k_+^+	$-k_+^+$	0	$2k_0^+$	0	0	0	$2k_+^0$	0	$-2(J+2k_0^0)$	$-2k_+^+$
k_-^+	k_-^+	$-2k_0^+$	0	0	0	0	$-2k_-^0$	$-2(J-2k_0^0)$	0	$-2k_-^+$
k_0^-	0	k_+^-	$-k_-^-$	J	$-2k_+^0$	$2k_-^0$	0	0	0	$2k_0^-$
k_+^-	$-k_+^-$	0	$2k_0^-$	$2k_+^0$	0	$2(J-2k_0^0)$	0	0	0	$2k_+^-$
k_-^-	k_-^-	$-2k_0^-$	0	$-2k_-^0$	$2(J+2k_0^0)$	0	0	0	0	$2k_-^-$
J	0	0	0	$2k_0^+$	$2k_+^+$	$2k_-^+$	$-2k_0^-$	$-2k_+^-$	$-2k_-^-$	0

assumes a simple form in terms of the Casimir operators J^2 and K^2 of su_2 and $su_{1,1}$, respectively.

It is not difficult to convert s to a semisimple noncompact Lie algebra. Let us introduce the antihermitian (on the space $\mathcal{E}$) operators

$$
\begin{aligned}
A' &= \frac{i}{2} (k_+^o + k_-^o) , \\
B' &= \frac{1}{2} (k_+^o - k_-^o) , \\
C' &= i\, k_o^o , \\
D' &= \frac{i}{2} J , \\
E' &= \frac{i}{2} (k_o^+ + k_o^-) , \qquad (47) \\
F' &= \frac{1}{2} (k_o^+ - k_o^-) , \\
G' &= \frac{1}{4} (-k_+^+ + k_-^+ - k_+^- + k_-^-) , \\
H' &= \frac{i}{4} (k_+^+ - k_-^+ - k_+^- + k_-^-) , \\
J' &= \frac{i}{4} (k_+^+ + k_-^+ + k_+^- + k_-^-) , \\
K' &= \frac{1}{4} (k_+^+ + k_-^+ - k_+^- - k_-^-) .
\end{aligned}
$$

It is then a straightforward but tedious piece of work to obtain the commutation relations reported in Table 2 . Comparison between Table 2 and the catalogue set

insert Table 2 around here

up by Patera et al.[31] for the de Sitter algebras of low dimensions shows that the algebra spanned by the set $\{A', B', \ldots, K'\}$ is isomorphic to $so_{3,2}$. The

TABLE 2. Lie brackets of the de Sitter algebra $so_{3,2}$. For instance [A', B'] = -C'.

	A'	B'	C'	D'	E'	F'	G'	H'	J'	K'
A'	O	-C'	B'	O	-G'	-H'	E'	F'	O	O
B'	C'	O	-A'	O	-J'	-K'	O	O	E'	F'
C'	-B'	A'	O	O	O	O	-J'	-K'	G'	H'
D'	O	O	O	O	-F'	E'	-H'	G'	-K'	J'
E'	G'	J'	O	F'	O	D'	A'	O	B'	O
F'	H'	K'	O	-E'	-D'	O	O	A'	O	B'
G'	-E'	O	J'	H'	-A'	O	O	D'	C'	O
H'	-F'	O	K'	-G'	O	-A'	-D'	O	O	C'
J'	O	-E'	-G'	K'	-B'	O	-C'	O	O	D'
K'	O	-F'	-H'	-J'	O	-B'	O	-C'	-D'	O

Schwinger algebra s thus turns out to be connected with the Lie algebra of the (non compact and semisimple) de Sitter group $SO_{3,2}$.

A two complex variables realization of s and $\{A', B', \ldots, K'\}$ in the Bargmann[43] space $\mathfrak{F}_2$ can be obtained with the help of

$$k^+_+ = -\xi^2 , \quad k^+_o = \xi\eta , \quad k^+_- = \eta^2 ,$$

$$k^o_+ = \xi \frac{\partial}{\partial\eta} , \quad k^o_o = \frac{1}{2}\left(\xi \frac{\partial}{\partial\xi} - \eta \frac{\partial}{\partial\eta}\right) , \quad k^o_- = \eta \frac{\partial}{\partial\xi} , \tag{48}$$

$$k^-_+ = \frac{\partial^2}{\partial\eta^2} , \quad k^-_o = \frac{\partial^2}{\partial\xi\,\partial\eta} , \quad k^-_- = -\frac{\partial^2}{\partial\xi^2} ,$$

$$J = \xi \frac{\partial}{\partial\xi} + \eta \frac{\partial}{\partial\eta} + 1 .$$

Note that I_2 writes in the preceding realization as

$$I_2 = -\frac{1}{12}\left[\left(\xi \frac{\partial}{\partial\xi} + \eta \frac{\partial}{\partial\eta}\right)^2 + 2\xi \frac{\partial}{\partial\xi} + 2\eta \frac{\partial}{\partial\eta} + \frac{7}{2}\right] . \tag{49}$$

By combining Eqs. (47) and (48), the algebra $\{A', B', \ldots, K'\}$ can be realized in $\mathfrak{F}_2$ by

$$A' = \frac{i}{2}\left(\xi \frac{\partial}{\partial\eta} + \eta \frac{\partial}{\partial\xi}\right) , \quad B' = \frac{1}{2}\left(\xi \frac{\partial}{\partial\eta} - \eta \frac{\partial}{\partial\xi}\right) ,$$

$$C' = \frac{i}{2}\left(\xi \frac{\partial}{\partial\xi} - \eta \frac{\partial}{\partial\eta}\right) , \quad D' = \frac{i}{2}\left(\xi \frac{\partial}{\partial\xi} + \eta \frac{\partial}{\partial\eta} + 1\right) ,$$

$$E' = \frac{i}{2}\left(\xi\eta + \frac{\partial^2}{\partial\xi\,\partial\eta}\right) , \quad F' = \frac{1}{2}\left(\xi\eta - \frac{\partial^2}{\partial\xi\,\partial\eta}\right) ,$$

$$G' = \frac{1}{4} \left(\xi^2 + \eta^2 - \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) ,$$

$$H' = \frac{i}{4} \left(-\xi^2 - \eta^2 - \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) ,$$

$$J' = \frac{i}{4} \left(-\xi^2 + \eta^2 - \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) ,$$

$$K' = \frac{1}{4} \left(-\xi^2 + \eta^2 + \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) . \tag{50}$$

Some other(physical) realizations of both $\{A', B', \ldots, K'\}$ and s could be obtained in terms of space and time coordinates by noticing the relevance of $SO_{3,2}$ as a spectrum generating group of the rigid rotator[27,28] and the symmetric top.[30]

4. BOSON-LIKE APPROACH TO RECURSION RELATIONS

We now go to the coupling of two angular momenta, the boson representations of which are described by the sets $\left\{a_{\pm}, a_{\pm}^{+}\right\}$ and $\left\{b_{\pm}, b_{\pm}^{+}\right\}$. The coupled-boson operators $\mathfrak{J}_3$, $\mathfrak{J}_{\pm}$, $\mathfrak{K}_3$, and $\mathfrak{K}_{\pm}$ introduced by Schwinger[5] can be rewritten as

$$
\begin{aligned}
\mathfrak{J}_3 &= \frac{1}{2}\left(a_+^+ a_+ + a_-^+ a_- - b_+^+ b_+ - b_-^+ b_-\right) , \\
\mathfrak{J}_+ &= a_+^+ b_+ + a_-^+ b_- , \\
\mathfrak{J}_- &= b_+^+ a_+ + b_-^+ a_- , \\
\mathfrak{K}_3 &= \frac{1}{2}\left(a_+^+ a_+ + a_-^+ a_- + b_+^+ b_+ + b_-^+ b_-\right) + 1 , \\
\mathfrak{K}_+ &= a_+^+ b_-^+ - a_-^+ b_+^+ , \\
\mathfrak{K}_- &= a_+ b_- - a_- b_+ .
\end{aligned} \tag{51}
$$

The operators $\mathfrak{J}_3$ and $\mathfrak{J}_{\pm}$ satisfy the SO_3 commutation relations (40) while $\mathfrak{K}_3$ and $\mathfrak{K}_{\pm}$ satisfy the $SO_{2,1}$ commutation relations (41) and, further, the set $\left\{\mathfrak{J}_3, \mathfrak{J}_{\pm}\right\}$ commutes with $\left\{\mathfrak{K}_3, \mathfrak{K}_{\pm}\right\}$.[5] Therefore, $\mathfrak{J}_3$, $\mathfrak{J}_{\pm}$, $\mathfrak{K}_3$, and $\mathfrak{K}_{\pm}$ can be regarded as the generators of a $SO_{3,1}$ Lie group. As a consequence of the SO_3 and $SO_{2,1}$ commutation relations, we can derive the following ladder relationships

$$
\begin{aligned}
\mathfrak{J}_3 \, |j_1 j_2 j m\rangle &= (j_1 - j_2) \, |j_1 j_2 j m\rangle , \\
\mathfrak{J}_+ \, |j_1 j_2 j m\rangle &= \left[(j - j_1 + j_2) \; (j + j_1 - j_2 + 1) \right]^{1/2} \, |j_1 + 1/2, \; j_2 - 1/2, \; j, m\rangle , \\
\mathfrak{J}_- \, |j_1 j_2 j m\rangle &= \left[(j + j_1 - j_2) \; (j - j_1 + j_2 + 1) \right]^{1/2} \, |j_1 - 1/2, \; j_2 + 1/2, \; j, m\rangle , \\
\mathfrak{K}_3 \, |j_1 j_2 j m\rangle &= (j_1 + j_2 + 1) \, |j_1 j_2 j m\rangle , \\
\mathfrak{K}_+ \, |j_1 j_2 j m\rangle &= \left[(j_1 + j_2 - j + 1) \; (j_1 + j_2 + j + 2) \right]^{1/2} \, |j_1 + 1/2, \; j_2 + 1/2, \; j, m\rangle , \\
\mathfrak{K}_- \, |j_1 j_2 j m\rangle &= \left[(j_1 + j_2 + j + 1) \; (j_1 + j_2 - j) \right]^{1/2} \, |j_1 - 1/2, \; j_2 - 1/2, \; j, m\rangle .
\end{aligned}
\tag{52}
$$

By applying the Schwinger operators $\mathfrak{J}_\pm$ on $|j_1 \mp 1/2, \; j_2 \pm 1/2, \; j, \; m\rangle$, we directly obtain by means of simple ladder techniques the three-term recursion relations

$$
\begin{aligned}
&\left[(j - j_1 + j_2 + 1) \, (j + j_1 - j_2) \right]^{1/2} \langle j_1 j_2 m_1 m_2 | j m\rangle \\
&\quad = \left[(j_1 + m_1) \, (j_2 + m_2 + 1) \right]^{1/2} \langle j_1 - 1/2, \; j_2 + 1/2, \; m_1 - 1/2, \; m_2 + 1/2 \; | \; j m\rangle \\
&\quad + \left[(j_1 - m_1) \, (j_2 - m_2 + 1) \right]^{1/2} \langle j_1 - 1/2, \; j_2 + 1/2, \; m_1 + 1/2, \; m_2 - 1/2 \; | \; j m\rangle , \\
&\left[(j + j_1 - j_2 + 1) \, (j - j_1 + j_2) \right]^{1/2} \langle j_1 j_2 m_1 m_2 | j m\rangle \\
&\quad = \left[(j_1 + m_1 + 1) \, (j_2 + m_2) \right]^{1/2} \langle j_1 + 1/2, \; j_2 - 1/2, \; m_1 + 1/2, \; m_2 - 1/2 \; | \; j m\rangle \\
&\quad + \left[(j_1 - m_1 + 1) \, (j_2 - m_2) \right]^{1/2} \langle j_1 + 1/2, \; j_2 - 1/2, \; m_1 - 1/2, \; m_2 + 1/2 \; | \; j m\rangle .
\end{aligned}
\tag{53}
$$

[What we have just done parallels a well-known procedure to get (trivial) recursion relations by taking convenient matrix elements of the composite shift operator $J_\pm (a, b) = J_\pm (a) + J_\pm (b) \equiv k^o_\pm + \ell^o_\pm$, where the k's and ℓ's are given by

Eqs. (39) in terms of the a's and b's, respectively.] In a similar way, the action of $\mathcal{K}_{\pm}$ on $|j_1 \mp 1/2, j_2 \mp 1/2, j, m>$ produces

$$\left[(j_1+j_2-j)(j_1+j_2+j+1)\right]^{1/2} <j_1 j_2 m_1 m_2 | jm>$$
$$= \left[(j_1+m_1)(j_2-m_2)\right]^{1/2} <j_1-1/2, j_2-1/2, m_1-1/2, m_2+1/2 | jm>$$
$$- \left[(j_1-m_1)(j_2+m_2)\right]^{1/2} <j_1-1/2, j_2-1/2, m_1+1/2, m_2-1/2 | jm> ,$$

(54)

$$\left[(j_1+j_2+j+2)(j_1+j_2-j+1)\right]^{1/2} <j_1 j_2 m_1 m_2 | jm>$$
$$= \left[(j_1+m_1+1)(j_2-m_2+1)\right]^{1/2} <j_1+1/2, j_2+1/2, m_1+1/2, m_2-1/2 | jm>$$
$$- \left[(j_1-m_1+1)(j_2+m_2+1)\right]^{1/2} <j_1+1/2, j_2+1/2, m_1-1/2, m_2+1/2 | jm> .$$

Equations (53) and (54) have been derived in term of the 3-jm symbol by Schulten and Gordon on the basis of recoupling relations and by Barut and Wilson with the aid of Rainville's contiguous relations for generalized hypergeometric functions.[33]

At this point, we forsee a general procedure to obtain recursion relations for $SU_2 \supset U_1$ Wigner coefficients (and possibly SU_2 Racah coefficients). The procedure may be described as follows : (i) construct polynomials in the $\mathcal{J}$, $\mathcal{K}$, k, and ℓ's , (ii) express them in the a and b's (and/or if possible k and ℓ's) , and (iii) take convenient matrix elements of the so-obtained expressions.

To illustrate the procedure, let us consider the $\mathcal{K}\mathcal{K}$, $\mathcal{K}\mathcal{J}$, and $\mathcal{J}\mathcal{J}$ bilinear products

$$\mathcal{K}_- \mathcal{K}_- = -k_+^- \ell_-^- - 2k_0^- \ell_0^- - k_-^- \ell_+^- ,$$

$$\mathcal{J}_- \mathcal{J}_- = k_+^- \ell_-^+ + 2k_o^- \ell_o^+ + k_-^- \ell_+^+ ,$$

$$\mathcal{J}_- \mathcal{K}_- = -k_+^- \ell_-^o - 2k_o^- \ell_o^o - k_-^- \ell_+^o ,$$

$$\mathcal{J}_+ \mathcal{J}_+ = k_+^+ \ell_-^- + 2k_o^+ \ell_o^- + k_-^+ \ell_+^- ,$$

$$\mathcal{K}_+ \mathcal{K}_+ = -k_+^+ \ell_-^+ - 2k_o^+ \ell_o^+ - k_-^+ \ell_+^+ ,$$

$$\mathcal{J}_+ \mathcal{K}_+ = -k_+^+ \ell_-^o - 2k_o^+ \ell_o^o - k_-^+ \ell_+^o ,$$

$$\mathcal{J}_+ \mathcal{K}_- = k_+^o \ell_-^- + 2k_o^o \ell_o^- + k_-^o \ell_+^- , \qquad (55)$$

$$\mathcal{J}_- \mathcal{K}_+ = k_+^o \ell_-^+ + 2k_o^o \ell_o^+ + k_-^o \ell_+^+$$

$$\frac{1}{4}(\mathcal{K}_+ \mathcal{K}_- + \mathcal{K}_- \mathcal{K}_+ - \mathcal{J}_+ \mathcal{J}_- - \mathcal{J}_- \mathcal{J}_+ - 2)$$

$$= -k_+^o \ell_-^o - 2k_o^o \ell_o^o - k_-^o \ell_+^o .$$

[The 9 preceding operators are clearly defined in the enveloping algebra of $s(k) \otimes s(\ell)$ and are indeed proportional to scalar products of the 6 SU_2 vectors $V^\rho = (V_o^\rho, V_+^\rho, V_-^\rho)$ with $V = k, \ell$ and $\rho = o, +, -$; see also Eq. (83) below.] It can be verified that the action of $\mathcal{J}_+ \mathcal{K}_+$ on $|j_1+1, j_2, j, m>$ yields the four-term recursion relation

$$[(j_1+j_2-j)(j_1+j_2+j+1)(j-j_1+j_2+1)(j+j_1-j_2)]^{1/2} < j_1 j_2 m_1 m_2 | jm>$$
$$= [(j_1+m_1)(j_1+m_1-1)(j_2-m_2)(j_2+m_2+1)]^{1/2} < j_1-1, j_2, m_1-1, m_2+1 | jm>$$
$$- 2m_2 [(j_1-m_1)(j_1+m_1)]^{1/2} < j_1-1, j_2, m_1, m_2 | jm> \qquad (56)$$
$$- [(j_1-m_1)(j_1-m_1-1)(j_2+m_2)(j_2-m_2+1)]^{1/2} < j_1-1, j_2, m_1+1, m_2-1 | jm> .$$

Similarly, the 8 remaining operators lead to

$$[(j_1+j_2+j+3)(j_1+j_2-j+2)(j_1+j_2+j+2)(j_1+j_2-j+3)]^{1/2}\langle j_1 j_2 m_1 m_2 | jm\rangle$$

$$= [(j_1-m_1+1)(j_1-m_1+2)(j_2+m_2+1)(j_2+m_2+2)]^{1/2}\langle j_1+1,\ j_2+1,\ m_1-1,\ m_2+1 | jm\rangle$$

$$-2[(j_1-m_1+1)(j_1+m_1+1)(j_2-m_2+1)(j_2+m_2+1)]^{1/2}\langle j_1+1,\ j_2+1,\ m_1,\ m_2 | jm\rangle$$

$$+[(j_1+m_1+1)(j_1+m_1+2)(j_2-m_2+1)(j_2-m_2+2)]^{1/2}\langle j_1+1,\ j_2+1,\ m_1+1,\ m_2-1 | jm\rangle,$$

$$[(j+j_1-j_2+2)(j-j_1+j_2-1)(j+j_1-j_2+1)(j-j_1+j_2)]^{1/2}\langle j_1 j_2 m_1 m_2 | jm\rangle$$

$$= [(j_1-m_1+1)(j_1-m_1+2)(j_2-m_2)(j_2-m_2-1)]^{1/2}\langle j_1+1,\ j_2-1,\ m_1-1,\ m_2+1 | jm\rangle$$

$$+2[(j_1-m_1+1)(j_1+m_1+1)(j_2-m_2)(j_2+m_2)]^{1/2}\langle j_1+1,\ j_2-1,\ m_1,\ m_2 | jm\rangle$$

$$+[(j_1+m_1+1)(j_1+m_1+2)(j_2+m_2)(j_2+m_2-1)]^{1/2}\langle j_1+1,\ j_2-1,\ m_1+1,\ m_2-1 | jm\rangle,$$

$$[(j_1+j_2+j+2)(j_1+j_2-j+1)(j+j_1-j_2+1)(j-j_1+j_2)]^{1/2}\langle j_1 j_2 m_1 m_2 | jm\rangle$$

$$= -[(j_1-m_1+1)(j_1-m_1+2)(j_2-m_2)(j_2+m_2+1)]^{1/2}\langle j_1+1, j_2,\ m_1-1,\ m_2+1 | jm\rangle$$

$$-2m_2[(j_1-m_1+1)(j_1+m_1+1)]^{1/2}\langle j_1+1,\ j_2,\ m_1,\ m_2 | jm\rangle$$

$$+[(j_1+m_1+1)(j_1+m_1+2)(j_2+m_2)(j_2-m_2+1)\langle j_1+1, j_2,\ m_1+1,\ m_2-1 | jm\rangle,$$

$$[(j-j_1+j_2+2)(j+j_1-j_2-1)(j-j_1+j_2+1)(j+j_1-j_2)]^{1/2}\langle j_1 j_2 m_1 m_2 | jm\rangle$$

$$= [(j_1+m_1)(j_1+m_1-1)(j_2+m_2+1)(j_2+m_2+2)]^{1/2}\langle j_1-1,\ j_2+1,\ m_1-1,\ m_2+1 | jm\rangle$$

$$+2[(j_1-m_1)(j_1+m_1)(j_2-m_2+1)(j_2+m_2+1)]^{1/2}\langle j_1-1,\ j_2+1,\ m_1,\ m_2 | jm\rangle$$

$$+[(j_1-m_1)(j_1-m_1-1)(j_2-m_2+1)(j_2+m_2+1)]^{1/2}\langle j_1-1, j_2+1, m_1+1, m_2-1 | jm\rangle,$$

(57)

$$[(j_1+j_2-j-1)(j_1+j_2+j)(j_1+j_2-j)(j_1+j_2+j+1)]^{1/2} \langle j_1 j_2 m_1 m_2 | jm \rangle$$

$$= [(j_1+m_1)(j_1+m_1-1)(j_2-m_2)(j_2-m_2-1)]^{1/2} \langle j_1-1, j_2-1, m_1-1, m_2+1 | jm \rangle$$

$$-2[(j_1-m_1)(j_1+m_1)(j_2-m_2)(j_2+m_2)]^{1/2} \langle j_1-1, j_2-1, m_1, m_2 | jm \rangle$$

$$+[(j_1-m_1)(j_1-m_1-1)(j_2+m_2)(j_2+m_2-1)]^{1/2} \langle j_1-1, j_2-1, m_1+1, m_2-1 | jm \rangle ,$$

$$[(j_1+j_2+j+2)(j_1+j_2-j+1)(j-j_1+j_2+1)(j+j_1-j_2)]^{1/2} \langle j_1 j_2 m_1 m_2 | jm \rangle$$

$$= -[(j_1+m_1)(j_1-m_1+1)(j_2+m_2+1)(j_2+m_2+2)]^{1/2} \langle j_1, j_2+1, m_1-1, m_2+1 | jm \rangle$$

$$+2m_1[(j_2-m_2+1)(j_2+m_2+1)]^{1/2} \langle j_1, j_2+1, m_1, m_2 | jm \rangle$$

$$+[(j_1-m_1)(j_1+m_1+1)(j_2-m_2+1)(j_2-m_2+2)]^{1/2} \langle j_1, j_2+1, m_1+1, m_2-1 | jm \rangle ,$$

$$[(j_1+j_2-j)(j+j_2+j+1)(j+j_1-j_2+1)(j-j_1+j_2)]^{1/2} \langle j_1 j_2 m_1 m_2 | jm \rangle$$

$$= [(j_1+m_1)(j_1-m_1+1)(j_2-m_2)(j_2-m_2-1)]^{1/2} \langle j_1, j_2-1, m_1-1, m_2+1 | jm \rangle$$

$$+2m_1[(j_2-m_2)(j_2+m_2)]^{1/2} \langle j_1, j_2-1, m_1, m_2 | jm \rangle$$

$$-[(j_1-m_1)(j_1+m_1+1)(j_2+m_2)(j_2+m_2-1)]^{1/2} \langle j_1, j_2-1, m_1+1, m_2-1 | jm \rangle ,$$

$$[j_1(j_1+1)+j_2(j_2+1)-j(j+1)] \langle j_1 j_2 m_1 m_2 | jm \rangle$$

$$= -[(j_1+m_1)(j_1-m_1+1)(j_2-m_2)(j_2+m_2+1)]^{1/2} \langle j_1, j_2, m_1-1, m_2+1 | jm \rangle$$

$$-2m_1 m_2 \langle j_1 j_2 m_1 m_2 | jm \rangle$$

$$-[(j_1-m_1)(j_1+m_1+1)(j_2+m_2)(j_2-m_2+1)]^{1/2} \langle j_1, j_2, m_1+1, m_2-1 | jm \rangle .$$

Needless to say that, all or part of the relations (56) and (57) are probably scattered in the literature. We note for instance that Eq. (56) appears in Ref.(38) [cf. Eq. (3. 7. 13)] as function of the 3-jm symbol.

As a more sophisticated illustration, we find that the operator

$$j_1 k_o^- \mathcal{J}_+ \mathcal{K}_+ + (j_1+1)\, k_o^+ \mathcal{J}_- \mathcal{K}_-$$

$$= j_1 (j_1+1)(2j_1+1) \left[k_o^o + \ell_o^o - \frac{1}{4} k_o^o \left(\mathcal{K}_+ \mathcal{K}_- + \mathcal{K}_- \mathcal{K}_+ - \mathcal{J}_+ \mathcal{J}_- - \mathcal{J}_- \mathcal{J}_+ \right) \right] \tag{58}$$

produces, after some rearrangement, the relation

$$\left[(m_2-m_1) - m \frac{(j_1-j_2)(j_1+j_2+1)}{j(j+1)} \right] < j_1 j_2 m_1 m_2 \mid jm >$$

$$= \frac{[(j+m+1)(j-m+1)(j+j_1+j_2+2)(j_1+j_2-j)(j+j_2-j_1+1)(j+j_1-j_2+1)]^{1/2}}{(j+1)[(2j+1)(2j+3)]^{1/2}} < j_1 j_2 m_1 m_2 \mid j+1, m >$$

$$+ \frac{[(j+m)(j-m)(j+j_1+j_2+1)(j_1+j_2-j+1)(j+j_2-j_1)(j+j_1-j_2)]^{1/2}}{j[(2j-1)(2j+1)]^{1/2}} < j_1 j_2 m_1 m_2 \mid j-1, m > \tag{59}$$

given by Racah[6] in the Condon and Shortley notation and recently rederived by Barut and Wilson and also by Akyeampong and Rashid.[33]

5. BOSON REALIZATIONS OF THE SU_2 UNIT TENSOR

We are now in a position to develop $t_{kq\alpha}$ as a polynomial in the four boson operators $a_\pm$ and $a^+_\pm$.

5.1. Basic Realization

The Majumdar-realization : A boson development of $t_{kq\alpha}$, valid as far as $t_{kq\alpha}$ acts on $\epsilon(j)$, is provided by

insert Eqs. (60) - (67) here

Outline of proof : The $j'm'-jm$ matrix element of $t_{kq\alpha}$ as given by Eq. (60) is easily set up by using Eq. (38). The so-obtained relation can be transformed to yield

$$\langle j'm' \left| t_{kq\alpha} \right| jm \rangle = \delta(j', j+\alpha)\ \delta(m', m+q)$$

$$(-1)^{k-q} \left[\frac{(j-m)!}{(j+m)!} \ \frac{(j+\alpha+m+q)!}{(j+\alpha-m-q)!} \right]^{1/2}$$

$$\left[\frac{(k+\alpha)!\,(k-q)!}{(k-\alpha)!\,(2j+\alpha-k)!\,(2j+\alpha+k+1)!\,(k+q)!} \right]^{1/2}$$

$$\sum_z (-1)^z \frac{(2j+2\alpha-z)!\ (k-\alpha+z)!\,(j+m)!}{z!\ (k+\alpha-z)!\ (z-q-\alpha)!\ (j+\alpha+m+q-z)!} \ . \qquad (68)$$

Comparison between Eq. (68) and the Majumdar hypergeometric formula[44] specialized to $\langle j\,k\,m\,q \,|\, j'm' \rangle$ leads to recognize Eq. (1) in Eq. (68) so that the proof of Eq. (60) is completed. Equations (61)-(67) are then deducible from Eq. (60) owing

$$t_{kq\alpha} = \left[\frac{(k+\alpha)!\,(k-q)!}{(k-\alpha)!\,(2j+\alpha-k)!\,(2j+\alpha+k+1)!\,(k+q)!}\right]^{1/2} (-1)^{k-q}\,(a_+^+)^{q+\alpha}\,(a_-)^{q-\alpha}$$

$$\left\{(-1)^{q+\alpha}\,\frac{(2j+\alpha-q)!\,(k+q)!}{(q+\alpha)!\,(k-q)!} + [1-\delta(k\ q)] \sum_{z=q+\alpha+1}^{k+\alpha} (-1)^z\,\frac{(2j+2\alpha-z)!\,(k-\alpha+z)!}{z!\,(k+\alpha-z)!\,(z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+\alpha+J_z+q-z+t)\right\}, \quad q \geqslant \alpha \geqslant 0, \tag{60}$$

$$t_{k-q\alpha} = \left[\frac{(k+\alpha)!\,(k-q)!}{(k-\alpha)!\,(2j+\alpha-k)!\,(2j+\alpha+k+1)!\,(k+q)!}\right]^{1/2} (-1)^{2k+q+\alpha}\,(a_-^+)^{q+\alpha}\,(a_+)^{q-\alpha}$$

$$\left\{(-1)^{q+\alpha}\,\frac{(2j+\alpha-q)!\,(k+q)!}{(q+\alpha)!\,(k-q)!} + [1-\delta(k\ q)] \sum_{z=q+\alpha+1}^{k+\alpha} (-1)^z\,\frac{(2j+2\alpha-z)!\,(k-\alpha+z)!}{z!\,(k+\alpha-z)!\,(z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+\alpha-J_z+q-z+t)\right\}, \quad q \geqslant \alpha \geqslant 0, \tag{61}$$

$$t_{kq-\alpha} = \left[\frac{(k+\alpha)!\,(k-q)!}{(k-\alpha)!\,(2j-\alpha-k)!\,(2j-\alpha+k+1)!\,(k+q)!}\right]^{1/2} (-1)^{2k}\,(a_-)^{q+\alpha}\,(a_+^+)^{q-\alpha}$$

$$\left\{(-1)^{q+\alpha}\,\frac{(2j-\alpha-q)!\,(k+q)!}{(q+\alpha)!\,(k-q)!} + [1-\delta(k\ q)] \sum_{z=q+\alpha+1}^{k+\alpha} (-1)^z\,\frac{(2j-z)!\,(k-\alpha+z)!}{z!\,(k+\alpha-z)!\,(z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j-J_z-z+t)\right\}, \quad q \geqslant \alpha \geqslant 0, \tag{62}$$

$$t_{k-q-\alpha} = \left[\frac{(k+\alpha)!\,(k-q)!}{(k-\alpha)!\,(2j-\alpha-k)!\,(2j-\alpha+k+1)!\,(k+q)!}\right]^{1/2} (-1)^{k-\alpha}\,(a_+)^{q+\alpha}\,(a_-^+)^{q-\alpha}$$

$$\left\{(-1)^{q+\alpha}\,\frac{(2j-\alpha-q)!\,(k+q)!}{(q+\alpha)!\,(k-q)!} + [1-\delta(k\ q)] \sum_{z=q+\alpha+1}^{k+\alpha} (-1)^z\,\frac{(2j-z)!\,(k-\alpha+z)!}{z!\,(k+\alpha-z)!\,(z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+J_z-z+t)\right\}, \quad q \geqslant \alpha \geqslant 0, \tag{63}$$

$$t_{kq\alpha} = \left[\frac{(k+q)!\ (k-\alpha)!}{(k-q)!\ (2j+\alpha-k)!\ (2j+\alpha+k+1)!\ (k+\alpha)!}\right]^{1/2} (-1)^{k-q}\ (a_+^+)^{\alpha+q}\ (a_-^+)^{\alpha-q}$$

$$\left\{(-1)^{q+\alpha}\frac{(2j)!\ (k+\alpha)!}{(q+\alpha)!\ (k-\alpha)!} + [1-\delta(k\ \alpha)] \sum_{z=q+\alpha+1}^{k+q} (-1)^z \frac{(2j+\alpha+q-z)!\ (k-q+z)!}{z!\ (k+q-z)!\ (z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+\alpha+J_z+q-z+t)\right\}, \quad \alpha \geqslant q \geqslant 0, \tag{64}$$

$$t_{k-q\alpha} = \left[\frac{(k+q)!\ (k-\alpha)!}{(k-q)!\ (2j+\alpha-k)!\ (2j+\alpha+k+1)!\ (k+\alpha)!}\right]^{1/2} \cdot (-1)^{2k+q+\alpha}\ (a_-^+)^{\alpha+q}\ (a_+^+)^{\alpha-q}$$

$$\left\{(-1)^{q+\alpha}\frac{(2j)!\ (k+\alpha)!}{(q+\alpha)!\ (k-\alpha)!} + [1-\delta(k\ \alpha)] \sum_{z=q+\alpha+1}^{k+q} (-1)^z \frac{(2j+\alpha+q-z)!\,(k-q+z)!}{z!\ (k+q-z)!\ (z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+\alpha-J_z+q-z+t)\right\}, \quad \alpha \geqslant q \geqslant 0, \tag{65}$$

$$t_{kq-\alpha} = \left[\frac{(k+q)!\ (k-\alpha)!}{(k-q)!\ (2j-\alpha-k)!\ (2j-\alpha+k+1)!\ (k+\alpha)!}\right]^{1/2} (-1)^{2k}\ (a_-)^{\alpha+q}\ (a_+)^{\alpha-q}$$

$$\left\{(-1)^{q+\alpha}\frac{(2j-2\alpha)!\,(k+\alpha)!}{(q+\alpha)!\ (k-\alpha)!} + [1-\delta(k\ \alpha)] \sum_{z=q+\alpha+1}^{k+q} (-1)^z \frac{(2j-\alpha+q-z)!\ (k-q+z)!}{z!\,(k+q-z)!\ (z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j-J_z-z+t)\right\}, \quad \alpha \geqslant q \geqslant 0, \tag{66}$$

$$t_{k-q-\alpha} = \left[\frac{(k+q)!\ (k-\alpha)!}{(k-q)!\ (2j-\alpha-k)!\ (2j-\alpha+k+1)!\,(k+\alpha)!}\right]^{1/2} (-1)^{k-\alpha}\ (a_+)^{\alpha+q}\ (a_-)^{\alpha-q}$$

$$\left\{(-1)^{q+\alpha}\frac{(2j-2\alpha)!\ (k+\alpha)!}{(q+\alpha)!\ (k-\alpha)!} + [1-\delta(k\ \alpha)] \sum_{z=q+\alpha+1}^{k+q} (-1)^z \frac{(2j-\alpha+q-z)!\,(k-q+z)!}{z!\,(k+q-z)!\ (z-q-\alpha)!} \prod_{t=1}^{z-q-\alpha} (j+J_z-z+t)\right\}, \quad \alpha \geqslant q \geqslant 0. \tag{67}$$

to appropriate combinations of the following symmetry relations

$$< j_1 j_2 m_1 m_2 \mid jm >$$
$$= (-1)^{j_1+j_2-j} < j_1 j_2 \, -m_1 \, -m_2 \mid j\,-m >$$
$$= (-1)^{j_2+m_2} \left(\frac{2j+1}{2j_1+1}\right)^{1/2} < j j_2 \, -m \, m_2 \mid j_1 \, -m_1 > , \qquad (69)$$

and

$$< j_1 j_2 m_1 m_2 \mid j, m_1+m_2 > = (-1)^{j-j_1-m_2} \left(\frac{2j+1}{j+j_1+m_2+1}\right)^{1/2}$$
$$< \tfrac{1}{2}(j_1+j-m_2),\ j_2,\ -\tfrac{1}{2}(j-j_1-m_2)+m_1,\ j-j_1$$
$$\Big| \tfrac{1}{2}(j_1+j+m_2),\ \tfrac{1}{2}(j-j_1+m_2)+m_1 > , \qquad (70)$$

which describe ordinary[16, 38] and Regge[45] symmetry properties of the $SU_2 \supset U_1$ Wigner coefficients, respectively.

Numerical examples : As a first (simple) example, we have

$$t_{321} = \left[\frac{4!\ 1!}{2!\ (2j-2)!\ (2j+5)!\ 5!}\right]^{1/2} (-1)^5 \ (a_+^+)^3 \ a_-$$
$$\left[(-1)^3 \ \frac{(2j-1)!\ \ 5!}{3!\ 1!} + (-1)^4 \ \frac{(2j-2)!\ 6!}{4!\ 0!\ 1!} \ (j+J_z) \right]$$
$$= k_+^+ \ J_+ \left[\frac{10\,(2j-2)!}{(2j+5)!}\right]^{1/2} (3J_z - j + 2) , \qquad (71)$$

in accordance with Ref. 17 . As a second (more complicated) example, we get

$$t_{631} = \left[\frac{7!\,3!}{5!\,(2j-5)!\,(2j+8)!\,9!} \right]^{1/2} (-1)^9 \, (a_+^+)^4 \, (a_-)^2$$

$$\left[(-1)^4 \frac{(2j-2)!\,9!}{4!\,3!} + (-1)^5 \frac{(2j-3)!\,10!}{5!\,2!\,1!} (j+J_z) \right.$$

$$\left. + (-1)^6 \frac{(2j-4)!\,11!}{6!\,1!\,2!} (j+J_z-1)(j+J_z) + (-1)^7 \frac{(2j-5)!\,12!}{7!\,0!\,3!} (j+J_z-2)(j+J_z-1)(j+J_z) \right], \quad (72)$$

which can be rearranged as

$$t_{631} = - \left[\frac{9!\,(2j-5)!}{5!\,(2j+8)!\,7!\,3!} \right]^{1/2} (a_+^+)^4 \, (a_-)^2$$

$$[(2j-4)(2j-3)(2j-2) \times 210 - (2j-4)(2j-3)(j+J_z) \times 1260 + (2j-4)(j+J_z-1)(j+J_z) \times 2310$$

$$- (j+J_z-2)(j+J_z-1)(j+J_z) \times 1320]$$

$$= -6 \left[\frac{10\,(2j-5)!}{(2j+8)!} \right]^{1/2} k_+^+ \, J_+^2$$

$$(22 J_z^3 - 11 j J_z^2 - 4 j^2 J_z + j^3 + 88 J_z^2 - 41 j J_z - 3 j^2 + 142 J_z - 40 j + 84), \quad (73)$$

in disagreement with Ref. 17 . The formulas (60) - (67) for $t_{kq\alpha}$ also allow us to handle the case when k is half-an-odd-integer in contradistinction to the known prescriptions[10, 17] for constructing operator equivalents. For instance, we obtain

$$t_{17/2\ 11/2\ 7/2} = -10 \left[\frac{182\,(2j-5)!}{5\,(2j+13)!} \right]^{1/2} (a_+^+)^9 \, (a_-)^2$$

$$(34 J_z^3 + 12 j^2 J_z - 42 j J_z^2 + 186 J_z^2 + 33 j^2 - 177 j J_z + 374 J_z - 198 j + 264). \quad (74)$$

The operator construction : As a particular case, Eq. (64) shows that

$$t_{jmj} = (-1)^{2j} (2j+1)^{-1/2} [(j+m)!\,(j-m)!]^{-1/2} (a_+^+)^{j+m} (a_-^+)^{j-m}, \quad m \geqslant 0, \quad (75)$$

is valid when acting on the boson vacuum space $\epsilon(o)$. On the other hand, Eq. (1) yields

$$|jm> \;=\; (-1)^{-2j}\,(2j+1)^{1/2}\; t_{jmj}\; |o\,o> . \qquad (76)$$

Combination of Eqs. (75) and (76) gives

$$|jm> \;=\; [\,(j+m)!\,(j-m)!\,]^{-1/2}\;(a_+^+)^{j+m}\,(a_-^+)^{j-m}\;|o\,o> , \qquad (77)$$

in agreement with Eq. (1. 13) of Ref. 5 describing the boson-like operator construction of the angular momentum eigenvectors.

The case $k = 1/2$: Eqs. (60)-(67) show that the four basic boson operators $a_\pm$ and $a_\pm^+$ are connected to the unit tensor operators $t_{1/2\; q\; \alpha}$ acting on $\epsilon(j)$ through

$$t_{1/2\;\pm 1/2\;\;1/2} \;=\; -\,[\,2\,(j+1)\,(2j+1)\,]^{-1/2}\; a_\pm^+ \;,$$

$$t_{1/2\;\pm 1/2\;-1/2} \;=\; -\,[\,2j\,(2j+1)\,]^{-1/2}\; a_\mp \;. \qquad (78)$$

Let $W^k(j,\alpha)$ be the SU_2 tensor defined by its components

$$W_q^k(j,\alpha) \;=\; (-1)^{-2k}\;[\,2\,(j+\alpha)+1\,]^{1/2}\; t_{kq\alpha} \qquad (79)$$

acting on $\epsilon(j)$. By making use of Eqs. (79) and (78), it can be verified that the Schwinger operators defined by Eq. (51) can be nicely rewritten in terms of zeroth-order tensor products :

$$\mathcal{J}_3 \;=\; -\,\frac{1}{\sqrt{2}}\Big[\,(2j_1+1)\,\big\{W^{1/2}(j_1,1/2)\;W^{1/2}(j_1,-1/2)\big\}_o^o \;-\;(2j_2+1)\,\big\{W^{1/2}(j_2,1/2)\;W^{1/2}(j_2,-1/2)\big\}_o^o\Big],$$

$$\mathcal{J}_\pm = \mp \left[2(2j_1+1)(2j_2+1)\right]^{1/2} \left\{ w^{1/2}(j_1, \pm 1/2)\ w^{1/2}(j_2, \mp 1/2) \right\}^o_o ,$$

$$\mathcal{K}_3 = -\frac{1}{\sqrt{2}} \Big[(2j_1+1) \left\{ w^{1/2}(j_1, 1/2)\ w^{1/2}(j_1, -1/2) \right\}^o_o + (2j_2+1) \left\{ w^{1/2}(j_2, 1/2)\ w^{1/2}(j_2, -1/2) \right\}^o_o \Big] + 1 ,$$

$$\mathcal{K}_\pm = -\left[2(2j_1+1)(2j_2+1)\right]^{1/2} \left\{ w^{1/2}(j_1, \pm 1/2)\ w^{1/2}(j_2, \pm 1/2) \right\}^o_o . \qquad (80)$$

The case k integer : As a particular case, when k is integer, Eqs. (60)-(67) can be rewritten in terms of the k's by remarking that

$$\begin{aligned}
(a_+^+)^{q+\alpha} (a_-)^{q-\alpha} &= (-1)^\alpha (k_+^+)^\alpha (k_+^o)^{q-\alpha} , \\
(a_-^+)^{q+\alpha} (a_+)^{q-\alpha} &= (k_-^+)^\alpha (k_-^o)^{q-\alpha} , \\
(a_-)^{q+\alpha} (a_+^+)^{q-\alpha} &= (k_+^-)^\alpha (k_+^o)^{q-\alpha} , \\
(a_+)^{q+\alpha} (a_-^+)^{q-\alpha} &= (-1)^\alpha (k_-^-)^\alpha (k_-^o)^{q-\alpha} , \\
(a_+^+)^{\alpha+q} (a_-^+)^{\alpha-q} &= (-1)^q (k_+^+)^q (k_o^+)^{\alpha-q} , \\
(a_-^+)^{\alpha+q} (a_+^+)^{\alpha-q} &= (k_-^+)^q (k_o^+)^{\alpha-q} , \\
(a_-)^{\alpha+q} (a_+)^{\alpha-q} &= (k_+^-)^q (k_o^-)^{\alpha-q} , \\
(a_+)^{\alpha+q} (a_-)^{\alpha-q} &= (-1)^q (k_-^-)^q (k_o^-)^{\alpha-q}
\end{aligned} \qquad (81)$$

hold once k , and therefore q and α , are integers.

The case k = 1 : Eqs. (60)-(67) show that the nine basic operators k_σ^ρ are connected to the unit tensor operators $t_{1q\alpha}$ acting on $\varepsilon(j)$ through

$$t_{100} = [j(j+1)(2j+1)]^{-1/2}\, k_o^o \,,$$

$$t_{1\pm 10} = \mp [2j(j+1)(2j+1)]^{-1/2}\, k_\pm^o \,,$$

$$t_{101} = [(j+1)(2j+1)(2j+3)]^{-1/2}\, k_o^+ \,,$$

$$t_{1\pm 11} = \mp [(2j+1)(2j+2)(2j+3)]^{-1/2}\, k_\pm^+ \,,$$

$$t_{10-1} = -[j(2j-1)(2j+1)]^{-1/2}\, k_o^- \,,$$

$$t_{1\pm 1-1} = \pm [2j(2j-1)(2j+1)]^{-1/2}\, k_\pm^- \,. \qquad (82)$$

By using Eqs. (79) and (82), the operators defined by Eq. (55) can be rewritten as

$$\mathcal{K}_- \mathcal{K}_- = 2\,[3j_1(2j_1+1)j_2(2j_2+1)]^{1/2} \left\{ W^1(j_1,-1)\, W^1(j_2,-1) \right\}_o^o \,,$$

$$\mathcal{J}_- \mathcal{J}_- = 2\,[3j_1(2j_1+1)(j_2+1)(2j_2+1)]^{1/2} \left\{ W^1(j_1,-1)\, W^1(j_2,1) \right\}_o^o \,,$$

$$\mathcal{J}_- \mathcal{K}_- = -2\,[3j_1(2j_1+1)j_2(j_2+1)]^{1/2} \left\{ W^1(j_1,-1)\, W^1(j_2,o) \right\}_o^o \,,$$

$$\mathcal{J}_+ \mathcal{J}_+ = 2\,[3(j_1+1)(2j_1+1)j_2(2j_2+1)]^{1/2} \left\{ W^1(j_1,1)\, W^1(j_2,-1) \right\}_o^o \,,$$

$$\mathcal{K}_+ \mathcal{K}_+ = 2\,[3(j_1+1)(2j_1+1)(j_2+1)(2j_2+1)]^{1/2} \left\{ W^1(j_1,1)\, W^1(j_2,1) \right\}_o^o \,,$$

$$\mathcal{J}_+ \mathcal{K}_+ = 2\,[3(j_1+1)(2j_1+1)j_2(j_2+1)]^{1/2} \left\{ W^1(j_1,1)\, W^1(j_2,o) \right\}_o^o \,,$$

$$\mathcal{J}_+ \mathcal{K}_- = 2\,[\,3j_1(j_1+1)\,j_2\,(2j_2+1)\,]^{1/2}\left\{W^1(j_1, o)\;W^1(j_2,-1)\right\}^o_o ,$$

$$\mathcal{J}_- \mathcal{K}_+ = -2\,[\,3j_1(j_1+1)(j_2+1)(2j_2+1)\,]^{1/2}\left\{W^1(j_1, o)\;W^1(j_2, 1)\right\}^o_o ,$$

$$\frac{1}{4}\,(\mathcal{K}_+\mathcal{K}_- + \mathcal{K}_-\mathcal{K}_+ - \mathcal{J}_+\mathcal{J}_- - \mathcal{J}_-\mathcal{J}_+ - 2)$$

$$= 2\,[3j_1(j_1+1)\,j_2\,(j_2+1)^{1/2}\left\{W^1(j_1, o)\;W^1(j_2, o)\right\}^o_o . \qquad (83)$$

The diagonal unit tensor : As a limiting case, when $\alpha = 0$, Eqs. (60)-(67) reduce to

$$u^k_{\pm q} = \left[\frac{(k-q)!}{(k+q)!\,(2j-k)!\,(2j+k+1)!}\right]^{1/2}(-1)^{(k\pm k+2q)/2}\,J_\pm^{\,q}\left\{(-1)^q\,\frac{(2j-q)!\,(k+q)!}{q!\,(k-q)!}\right.$$

$$\left.+\,[1-\delta(k\;q)]\sum_{z=q+1}^{k}(-1)^z\,\frac{(2j-z)!\,(k+z)!}{z!\,(k-z)!\,(z-q)!}\prod_{t=1}^{z-q}(j\mp J_z+q-z+t)\right\},\quad q\geqslant 0 . \qquad (84)$$

Specialization of Eq. (84) leads to the well-known result

$$u^k_{\pm k} = (-1)^{(k\pm k)/2}\,\frac{1}{k!}\left[\frac{(2k)!\,(2j-k)!}{(2j+k+1)!}\right]^{1/2} J_\pm^{\,k} . \qquad (85)$$

We note that Eqs. (84) and (85) are in agreement with the lemma[19] according to which

$$\left\{ J_\pm^{\,r}\;J_z^{\,s} :\; s = 0(1)\,(2j-r)\;;\; r = 0\,(1)\,2j \right\}$$

constitutes a basis of $\mathcal{T}$.

As a by-product, it is to be noted that Eq. (84) supplies a trick for obtaining a closed form expression of any harmonic polynomial $r^k Y_{kq}$ in Cartesian coordinates. It is sufficient in Eq. (84) : (i) to ignore all terms of order lower than k which are of quantum-mechanical origin, (ii) to make the replacements $J_\pm \rightarrow x \pm i y$, $J_z \rightarrow z$, and $J^2 \rightarrow r^2 = x^2 + y^2 + z^2$, and (iii) to multiply the so-obtained expression by the factor

$$\frac{1}{2^k} \left(\frac{2k+1}{4\pi} \right)^{1/2} \left[\frac{(2j+k+1)!}{(2j-k)!} \right]^{1/2} .$$

Such a trick (cf. also Ref. 16) constitutes the reciprocal part of the Stevens prescription described in Sec. 1. As an illustration, Eq. (84) yields

$$u_4^7 = 4 \sqrt{462} \left[\frac{(2j-7)!}{(2j+8)!} \right]^{1/2} J_+^{\ 4} (13 J_z^3 + 78 J_z^2 + 179 J_z - 3 J_z J^2 - 6 J^2 + 150) . \quad (86)$$

and application of the trick leads to

$$r^7 Y_{74} = \frac{3}{64} \sqrt{\frac{770}{\pi}} (x + i y)^4 (13 z^3 - 3 z r^2) . \quad (87)$$

A trick corresponding to the reciprocal part of the Atkins and Seymour[17] prescription could also be stated. Nevertheless, its interest would be quasi-null except for checking purposes.

The algebra su_3 : Eqs. (60)-(67) show that the set $\left\{ u_q^k : q \text{ ranging and } k = 0(1)2 \right\}$ acting on $\varepsilon(1)$ can be realized by

$$u_o^o = \frac{1}{\sqrt{3}} ,$$

$$u_o^1 = \frac{1}{\sqrt{6}} J_z ,$$

$$u_{\pm 1}^1 = \mp \frac{1}{2\sqrt{3}} J_\pm ,$$

$$u^2_o = \frac{1}{\sqrt{30}} (3J_z^2 - 2) ,$$

$$u^2_{\pm 1} = \frac{1}{\sqrt{20}} J_\pm (-1 \mp 2J_z) ,$$

$$u^2_{\pm 2} = \frac{1}{\sqrt{20}} J_\pm^2 . \tag{88}$$

It can be verified that Eq. (88) satisfies the commutation relations (32) specialized to the case $G^* = U_1$.

5.2. Other Realizations

We now turn to list some alternative boson realizations of $t_{kq\alpha}$.

The van der Waerden-realization :

$$t_{kq\alpha} = (-1)^{k+\alpha} \left[\frac{(k+q)!\,(k-q)!\,(k+\alpha)!\,(k-\alpha)!\,(2j+\alpha-k)!}{(2j+\alpha+k+1)!} \right]^{1/2}$$

$$\sum_z (-1)^z \frac{(a_-^+)^{k-q-z}\,(a_-)^{k-\alpha-z}\,(a_+^+)^{q+\alpha+z}\,(a_+)^z}{(k-q-z)!\,(k-\alpha-z)!\,(q+\alpha+z)!\,z!} . \tag{89}$$

The Zukauskas-Mauza-realization :

$$t_{kq\alpha} = (-1)^{q+\alpha} \left[\frac{(k+q)!\,(k-q)!\,(k+\alpha)!\,(k-\alpha)!\,(2j+\alpha-k)!}{(2j+\alpha+k+1)!} \right]^{1/2}$$

$$\sum_z (-1)^z \frac{(a_+)^{k-q-z}\,(a_+^+)^{k+\alpha-z}\,(a_-)^{q-\alpha+z}\,(a_-^+)^z}{(k-q-z)!\,(k+\alpha-z)!\,(q-\alpha+z)!\,z!} . \tag{90}$$

The Wigner-realization :

$$t_{kq\alpha} = (-1)^{k-q} \left[\frac{(k+\alpha)!\,(k-\alpha)!\,(2j+\alpha-k)!}{(k+q)!\,(k-q)!\,(2j+\alpha+k+1)!} \right]^{1/2}$$

$$\sum_z (-1)^z \frac{(a_+^+)^z\,(a_+)^{k-q}\,(a_+^+)^{k+\alpha-z}\,(a_-^+)^{k+\alpha-z}\,(a_-)^{k+q}\,(a_-^+)^z}{(k+\alpha-z)!\,z!} \,. \quad (91)$$

The Racah-realization :

$$t_{kq\alpha} = (-1)^{2k} \left[\frac{(k+q)!\,(k-q)!\,(2j+\alpha-k)!}{(k+\alpha)!\,(k-\alpha)!\,(2j+\alpha+k+1)!} \right]^{1/2}$$

$$\sum_z (-1)^z \frac{(a_+^+)^{k+q-z}\,(a_+)^{k-\alpha}\,(a_+^+)^z\,(a_-)^{k+q-z}\,(a_-^+)^{k+\alpha}\,(a_-)^z}{(k+q-z)!\,z!} \,. \quad (92)$$

Outline of proof : It is sufficient to verify that the $j'm'-jm$ matrix element of $t_{kq\alpha}$ as given by Eqs. (89)-(92) verifies Eq. (1) with the $SU_2 \supset U_1$ Clebsch-Gordan coefficient responding to the formulas of van der Waerden,[46] Zukauskas and Mauza,[47] Wigner,[48] and Racah.[6]

In all honesty, it is to be mentioned that the van der Waerden realization (89) turns out to be a rewriting in our terminology of the boson representation of the $SU_2 \supset U_1$ Wigner-Eckart theorem recently derived by Yamamura et al.[23]

5.3. General Comments

At this point, a brief comparison between the four latter realizations and the Majumdar-realization is in order. First, in the case when k is integer, inspection of Eqs. (60)-(67) and (89)-(92) shows that only the Majumdar-realization of $t_{kq\alpha}$ can be easily rewritten in terms of the operators k_σ^ρ defined by Eq. (39). Second, this realization allows one to get, with merely simple algebraic manipulations, diagonal and off-diagonal operator equivalents in a form very close to the one underlying the already published tables.[10-13, 17, 19] Such a form could be also obtainable in principle from the other realizations. However, this would require longwinded manipulations of commutation relations. Third, Eqs. (60)-(67) are particularly appropriate to hand and machine calculations. For instance, the determination of the J_z polynomial part of the Majumdar-realization of $t_{kq\alpha}$ amounts in last analysis to solving a Cramer system. In this connection, Eqs. (60)-(67) have been coded on a computer to produce tables of diagonal ($\alpha = 0$) and off-diagonal ($\alpha \neq 0$) operator equivalents $t_{ka\Gamma\gamma\alpha}$ adapted to the cubical, tetragonal, and trigonal groups for use in molecular and solid state physics.[49]

To close Sec. 5, it is perhaps worth stressing that the operator $W^k_q(j, \alpha)$ defined by Eq. (79) is actually a $SU_2 \supset U_1$ Wigner operator in the sense that (cf. Appendix 1)

$$\langle j'm' | W^k_q(j, \alpha) | jm \rangle = \delta(j', j+\alpha) \langle jkmq | j'm' \rangle . \qquad (93)$$

In the same vein, a SU_2 Racah operator $R(j_1\alpha_1 j_2\alpha_2 jk)$ may be defined via (cf. Appendix 1)

$$\langle j_1' j_2' j' m' | R(j_1\alpha_1 j_2\alpha_2 jk) | j_1 j_2 j m \rangle$$

$$= \delta(j_1', j_1+\alpha_1)\ \delta(j_2', j_2+\alpha_2)\ \delta(j'j)\ \delta(m'm)\ W(j_1, j_2, j_1+\alpha_1, j_2+\alpha_2 ; jk) . \qquad (94)$$

It is immediate to prove that (cf. Appendix 1)

$$R(j_1\alpha_1 j_2\alpha_2 jk) = (-1)^{j_1+\alpha_1+j_2-j-k} [2(j_1+\alpha_1)+1]^{-1/2} [2(j_2+\alpha_2)+1]^{-1/2}$$

$$(2k+1)^{1/2} \left\{ W^k(j_1, \alpha_1)\ W^k(j_2, \alpha_2) \right\}^o_o \qquad (95)$$

verifies indeed Eq. (94). Boson realizations of $R(j_1\alpha_1 j_2\alpha_2 jk)$ can be obtained by combining Eqs. (79) and (95) with the various boson realizations of $t_{kq\alpha}$ given in this section. In this respect, the Schwinger operators $\mathcal{J}_\pm$ and $\mathcal{K}_\pm$ are of type $R(j_1\alpha_1\, j_2\alpha_2\, j\, 1/2)$ while the operators $\mathcal{J}_\rho \mathcal{J}_\sigma$, $\mathcal{J}_\rho \mathcal{K}_\sigma$, and $\mathcal{K}_\rho \mathcal{K}_\sigma$ are of type $R(j_1\alpha_1\, j_2\alpha_2\, j\, 1)$.

6. CONCLUDING REMARKS

We started from the general properties of the unit tensor operators $t_{k\mu\alpha}$ adapted to a chain $SU_2 \supset G^*$ which can be obtained without specifying the realization and ended with boson realizations of the operators $t_{kq\alpha}$ adapted to the chain $SU_2 \supset U_1$.

Among the $t_{k\mu\alpha}$'s, we have paid a special attention to the set $\left\{t_{1q\alpha} : q \text{ and } \alpha \text{ ranging}\right\}$. Such a nine-dimensional set is easily completed with a tenth operator to produce the ten-dimensional Schwinger algebra. We have converted this algebra to the Lie algebra of the de Sitter group $SO_{3,2}$. It is to be noted that the Lie group $SO_{3,2}$ has already been exhibited in connection with Schwinger's calculus. Indeed, Györghyi and Kövesi-Domokos[27] have proved that $O_{3,2}$ enters the spherical rotor problem as a dynamical group : the $O_{3,2}$ group can be realized in terms of Dirac brackets of the Cartesian coordinates and the canonically conjugate momenta for the spherical rotor. It is also to be noted that $SO_{3,2}$ has been exhibited in an apparently distant field, namely, the Coulomb problem. In fact, Englefield[29] has obtained an $O_{3,2}$ Lie algebra in terms of ladder operators acting on the hydrogen-like eigenfunctions and therefore similar to the operators k_σ^ϱ although they are constructed without any relation to the Jordan-Schwinger representation of orbital angular momentum. The de Sitter group $SO_{3,2}$ thus appears as the most general structure of relevance for the Schwinger two-dimensional oscillator approach to the angular momentum theory.

The coupled-boson representation describing the coupling of two angular momenta then appears to be depicted by $SO_{3,2} \otimes SO_{3,2}$. We have shown how the simple manipulation of ladder operators of type : (i) coupled-boson operators $\mathfrak{J}$ and $\mathfrak{K}$ (related to $SO_{3,1}$) and (ii) bilinear expressions in $\mathfrak{J}$ and $\mathfrak{K}$ (related to the enveloping algebra of $SO_{3,2} \otimes SO_{3,2}$) allow to obtain, without any intermediate step, basic families of recursion relations for $SU_2 \supset U_1$ Wigner coefficients. We conjecture that any recursion relation for the $SU_2 \supset U_1$ Wigner coefficients (and possibly the SU_2 Racah coefficients) could be reached on the same pattern through a detailed study of the chain $SO_{3,2} \otimes SO_{3,2} \supset SO_{3,2} \supset SO_{3,1}$. We are enclined to think our procedure to get recursion relations shares some ressemblances (although we have been unable to correctly connect the two procedures) with the one of Shaw [33] based on the use of a Lorentz group $\mathcal{L}(C_2)$. The real reason for this is to be found in the fact that both procedures are very close, in a certain way, to the Bargmann [43] treatement of the three-dimensional rotation group.

The various boson realizations of the $SU_2 \supset U_1$ unit tensor operators $t_{kq\alpha}$ settle an important result of this study. We limited ourself to five realizations which mimick five basic formulas for the $SU_2 \supset U_1$ Wigner coefficient. Although we conjecture it is possible to get a boson realization of $t_{kq\alpha}$ from any factorial expression of the $SU_2 \supset U_1$ Wigner coefficient, actually, the Majumdar-realization seems to be the most efficient for producing polarized harmonics or operator equivalents in a form suitable for subsequent application. Of course, it should be possible to pass from one realization to another by using either the commutation relations of the four basic boson operators or symmetries of the $SU_2 \supset U_1$ Wigner coefficient. For example, the Zukauskas-Mauza- and the Wigner-realizations are linked by an ordinary symmetry while the two Majumdar-realizations (60) and (65) are connected by a Regge symmetry. This suggests that the 72 ordinary and Regge symmetries of the 3-jm symbol could be handled by looking at the symmetries of the $t_{kq\alpha}$'s.

The properties of the $t_{k\mu\alpha}$'s can be classified in two parts. Some of them clearly show that the $t_{k\mu\alpha}$'s serve as basis for expanding physical interactions. Some others emphasize the interest of the $t_{k\mu\alpha}$'s for Lie group theoretical analyses of the spectrum of physical interactions. We close by mentioning some avenues of future outlook.

Chains of groups : In the case k integer, Eqs. (60)-(67) can be considered as a solution to the problem of constructing $so_3 \supset so_2$ operators in the enveloping algebra of $so_{3,2}$. A related problem, of interest for producing recursion relations, would be the state labelling problem associated with the chain $SO_{3,2} \otimes SO_{3,2} \supset SO_{3,2} \supset SO_{3,1}$. The two mentioned problems require for the part $SO_{3,2} \supset SO_3 \otimes SO_2 \supset SO_3 \supset SO_2$ the consideration of a set of four commuting operators taken from the enveloping algebra of $so_{3,2}$ since, in addition to the two (second and fourth order) invariants of $SO_{3,2}$, two extra labelling operators are needed to completely label the $SO_{3,2} \supset SO_3 \supset SO_2$ states. The method of the generating function recently developed by the Montréal group[50] could be a good starting point for tackling these problems.

Half-integer rank tensor : It is generally more or less stated in physics textbooks that only integer rank irreducible tensors are suitable for physical applications. We ascertain, however, that calculation of intensities in x-ray and ultraviolet photoelectron spectroscopy[51] as well as in the recently developed bremsstrahlung isochromat spectroscopy[52] for transition-metal and rare-earth ions in crystalline matrices requires the introduction of half-an-odd-integer irreducible tensors. We hope that part of the material reported here shall be useful in those fields where half-integer rank tensors are of interest.

Theory of complex spectra : Specialization to the case $G^* = U_1$ and inversion of Eq. (27) yields

$$\delta(j, k, j,)u^k_q = \sum_{mm'} (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} P^j_{mm'} . \qquad (96)$$

Equation (96) curiously bears the same form that the intrashell operators introduced by Harter and Patterson[53] in their alternative (Gel'fand) basis for the theory of complex spectra. This can be easily understood by realizing that the set

$\left\{P^{j}_{\mu\mu'} : \mu \text{ and } \mu' \text{ ranging}\right\}$ furnishes a basis of the Lie algebra of the unitary group U_{2j+1}. We fell it would be worthwile to explore these matters in more detail.

Projection operator : The projection operator[40] $P_{jm} = P^{j}_{mm}$ [cf. Eq. (26)] can be realized in terms of boson operators by introducing the boson realizations of u^{k}_{q} obtainable from Sec. 5 into Eq. (27). This leads to expressions for P_{jm} that can be proved to be equivalent to the von Neumann (product) form and the Löwdin (sum and product) forms of P_{jm}. We are presently examining the respective merits, with special reference to computer coding, of the various possible realizations of P_{jm}.

Lie superalgebra : In the recent years, several works have been devoted to mathematical studies of low dimensional Z_2-graded Lie algebras [cf. Ref. (54)]. The concept of Z_2-graded Lie algebra is particularly appropriate for describing fermion-bose supersymmetry in various contexts as e. g. in the one of space-time symmetries for particle physics. We note the (infinite) set $\left\{t_{k\mu\alpha} : \alpha, \mu, \text{ and } k \text{ ranging}\right\}$ is closed under anticommutation and commutation so that the $t_{k\mu\alpha}$'s could be used for generating Lie superalgebras.

Factorization method : It has been already noticed by Witschel[42] that there exists a connection between the Schrödinger-Infeld-Hull factorization method[55] and the Schwinger boson calculus. A Lie-like aspect of this connection arises in this paper. In fact, the Schwinger algebra admits several subalgebras of type $\mathcal{G}(a, b)$, where $\mathcal{G}(a, b)$ denotes the complex four-dimensional Lie algebra discussed by Miller[55] in relation with the factorization method. In the same vein, let us mention that the realizations of $t_{kq\alpha}$ given here enable to obtain useful expressions for the O_3 shift operators recently investigated by Hughes and Yadegar[56] and which turn out to be relevant to the factorization method and to the classification and analysis of representations of various Lie groups.

Symplecton calculus : The Jordan-Schwinger boson representation of angular momentum has been generalized by Biedenharn and Louck to give the so-called symplecton representation.[57] Such a generalization leads to replace the boson calculus by a symplecton calculus. It would be (at least curious and perhaps) interesting, especially in view of the rotational SU_3 nuclear model which motivated the introduction of the symplecton representation, to translate some of the results of this paper in the language of symplecton operators.

ACKNOWLEDGMENTS

One of us (M. K.) would like to thank J. Paldus and J. Čížek for their kind and stimulating hospitality at the Quantum Theory Group of the University of Waterloo in the fall of 1978, during which time part of this work was completed. He would also like to thank J. Patera and J. Harnad (Université de Montréal), R. T. Sharp (McGill University), and K. B. Wolf (Universidad Nacional Autonoma de Mexico) for helpful conversations which provided the starting point of the $so_{3,2}$ analysis of the Schwinger algebra. The authors are grateful to J. Sau (Université Lyon-I) for calling our attention on Refs. (23) and (25).

APPENDIX I : ON THE WIGNER-RACAH ALGEBRA OF A COMPACT GROUP

Notations : Let J be a compact topological (finite or continuous) group. We use j or k to denote an IRC of J. Then, j_o stands for the identity IRC of J. Let $\varepsilon(j) = \{ |jm> : m \text{ ranging} \}$ be an irreducible subspace, associated with j, of the representation space $\mathcal{E}$ of J. We note $[j]$ the dimension of $\varepsilon(j)$. A definite choice of basis is assumed for each space $\varepsilon(j)$, thus defining the $\{m\}$ scheme, and the various J irreducible tensorial sets of vectors and operators of $\mathcal{E}$ are required to transform according to this scheme. Further, $< j_1 j_2 m_1 m_2 | bjm>$ is a Wigner coefficient of J compatible with the $\{m\}$ scheme, b being a Kronecker multiplicity label to be used when j occurs several times in $j_1 \otimes j_2$. Finally, $\Delta(j_1 | j_1 \otimes j_2) = 0$ or 1 according to the frequency of j in $j_1 \otimes j_2$ is equal to 0 or different from 0. The notations for a subgroup G of J can be deduced from the preceding ones thanks to the correspondences : $j \longrightarrow \Gamma$, $m \longrightarrow \gamma$, and $b \longrightarrow \beta$. The subduction $J \downarrow G$ amounts in many respects to replace m by $\mu \equiv a\Gamma\gamma$, where a is a branching multiplicity label, thus defining the $\{a\Gamma\gamma\}$ scheme. The Wigner coefficients of J in the $\{a\Gamma\gamma\}$ scheme are referred to as $J \supset G$ Wigner coefficients.

The orthogonality-completeness property

: The $J \supset G$ Wigner coefficients satisfy a well-known unitary property. In view of this property and the completeness property for the representation matrices of J in the $\{a\Gamma\gamma\}$ scheme, we can derive the following orthogonality-completeness relation[36]

$$\sum_{j_2 \mu_2 b} [j_2] < j_1 j_2 \mu_1 \mu_2 | bj\mu > < j_1 j_2 \mu_1' \mu_2 | bj\mu' >^* = \delta(\mu_1' \mu_1)\, \delta(\mu' \mu)\, [j]$$

and its dual relation

$$\sum_{\mu_1 \mu} \langle j_1 j_2 \mu_1 \mu_2 | bj\mu \rangle^* \langle j_1 j_2' \mu_1 \mu_2' | b'j\mu \rangle$$

$$= \Delta(j|j_1 \otimes j_2)\ \delta(j_2' j_2)\ \delta(\mu_2' \mu_2)\ \delta(b'b)\ [j_2]^{-1}\ [j].$$

Similar relations hold when replacing $\Sigma_{j_2 \mu_2 b}$ and $\Sigma_{\mu_1 \mu}$ by $\Sigma_{j_1 \mu_1 b}$ and $\Sigma_{\mu_2 \mu}$, respectively.

The Wigner operator : A unit tensor operator $\hat{t}_{j_2 \mu_2 bj}$ may be defined through

$$\langle j'\mu | \hat{t}_{j_2 \mu_2 bj} | j_1 \mu_1 \rangle$$

$$= \delta(j'j)\ e^{i\varphi(bjj_1j_2)}\ [j]^{-1/2} \langle j_1 j_2 \mu_1 \mu_2 | bj\mu \rangle^*,$$

where φ is a J-dependent phase factor. We call the operator

$$\hat{W}^{j_2}_{\mu_2}(bj) = e^{-i\varphi(bjj_1j_2)}\ [j]^{1/2}\ t_{j_2 \mu_2 bj}$$

a Wigner operator since its $j\mu - j_1\mu_1$ matrix element is nothing but than $\langle j_1 j_2 \mu_1 \mu_2 | bj\mu \rangle^*$. Application to $\hat{W}^{j_2}_{a_2 \Gamma_2 \gamma_2}(bj)$ of the Wigner-Eckart theorem for G in the $\{a\Gamma\gamma\}$ scheme directly leads to Racah's lemma :

$$\langle j_1 j_2 a_1 \Gamma_1 \gamma_1\ a_2 \Gamma_2 \gamma_2 | bja\Gamma\gamma \rangle$$

$$= \sum_{\beta} (j_1 a_1 \Gamma_1 + j_2 a_2 \Gamma_2 | bja\beta\Gamma)\ \langle \Gamma_1 \Gamma_2 \gamma_1 \gamma_2 | \beta\Gamma\gamma \rangle,$$

where $(j_1 a_1 \Gamma_1 + j_2 a_2 \Gamma_2 | bja\beta\Gamma)$ is , in modern parlance, a $J \supset G$ isoscalar factor. This connection between the Racah lemma and the Wigner-Eckart theorem takes it origin in the fact that they are both more or less direct corollaries of Schur's lemma.

The Racah operator : As a consequence of the great orthogonality theorem (again a corollary of Schur's lemma !) for the representation matrices of J in the $\{a\Gamma\gamma\}$ scheme, the recoupling coefficients

$$< j_1\,(j_2j_3)\,b_{23}\,j_{23}\,b'\,j'\,\mu' \mid (j_1j_2)\,b_{12}\,j_{12}\,b_3\,b\,j\,\mu >$$

$$= \sum_{\substack{\mu_1\mu_2\mu_3\\ \mu_{12}\mu_{23}}} <j_1j_2\mu_1\mu_2 \mid b_{12}j_{12}\mu_{12}> \; < j_{12}j_3\mu_{12}\mu_3 \mid bj\mu >$$

$$< j_2j_3\mu_2\mu_3 \mid b_{23}j_{23}\mu_{23}>^{*} <j_1\,j_{23}\mu_1\mu_{23} \mid b'\,j'\,\mu' >^{*}$$

enjoy the property

$$< j_1\,(j_2j_3)\,b_{23}\,j_{23}\,b'\,j'\,\mu' \mid (j_1j_2)\,b_{12}\,j_{12}\,j_3\,b\,j\,\mu >$$

$$= \delta(j'j)\;\delta(\mu'\mu)\;< j_1\,(j_2j_3)\,b_{23}\,j_{23}\,b'\,j \mid (j_1j_2)\,b_{12}\,j_{12}\,j_3\,b\,j>$$

where

$$<j_1\,(j_2\,j_3)\,b_{23}\,j_{23}\,b'\,j \mid (j_1j_2)\,b_{12}\,j_{12}\,j_3\,b\,j>$$

$$= [j]^{-1}\sum_{\mu} <j_1\,(j_2j_3)\,b_{23}\,j_{23}\,b'\,j\,\mu \mid (j_1j_2)\,b_{12}\,j_{12}\,j_3\,b\,j\,\mu >$$

is a J Racah coefficient.

A Racah operator $\hat{R}\;(j_1j_1'j_2j_2'jkbb'b_1'b_2')$ may be defined through

$$< (j_1''j_2'')\, b'j'\mu' \;\Big|\; \hat{R}(j_1j_1'j_2j_2'jkbb'b_1'b_2') \;\Big|\; (j_1j_2)\, bj\mu >$$

$$= \delta(j_1''j_1')\ \delta(j_2''j_2')\ \delta(j'j)\ \delta(\mu'\mu)$$

$$[j]^{-1/2}\,[k]^{-1/2} < j_1\ (j_2j_2')\ b'\,kb_2'j_1' \;\big|\; (j_1j_2)\ bj\ j_2'b_1'j_1' > .$$

The Racah operator $\hat{R}\,(j_1j_1'j_2j_2'jkbb'b_1'b_2')$ is connected to the Wigner operators $\hat{W}^k\ (b_1'j_1')$ and $\hat{W}^k\ (b_2'j_2')$ acting on two distinct spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ by the tensor product relation

$$\hat{R}\ (j_1j_1'j_2j_2'jkbb'b_1'b_2')$$

$$\sim \left\{\hat{W}^k\ (b_1'j_1')\ \hat{W}^k\ (b_2'j_2')\right\}^{j_o}$$

in the case where k *is orthogonal or symplectic.*

Commutation relations : The commutator acting on $\mathcal{E}(j)$ of two Wigner operators is given by[36]

$$\left[\hat{W}^{k_1}_{\mu_1}\ (b_1j_1),\ \hat{W}^{k_2}_{\mu_2}\ (b_2j_2)\right]_{\pm}$$

$$= \sum_{bk_3\mu_3b_3j_3} \hat{W}^{k_3}_{\mu_3}\ (b_3j_3)$$

$$\Big[\ \delta(j_3j_1) < j\,(k_2k_1)\ bk_3\ b_3j_3 \;\big|\; (jk_2)\ b_2j_2k_1b_1j_1 > \ < k_2k_1\mu_2\mu_1 \big| b\,k_3\,\mu_3 >$$

$$\mp\, \delta(j_3j_2) < j\,(k_1k_2)\ b\,k_3\,b_3j_3 \;\big|\; (jk_1)\ b_1j_1\,k_2\,b_2j_2 > < k_1k_2\mu_1\mu_2 \;\big|\; bk_3\mu_3 > \Big]^* .$$

The proof is based on recoupling techniques. From the definition of the Racah coefficient as function of the Wigner coefficient, we may derive, by making use of the unitarity and orthogonality-completeness properties of the Wigner coefficients, a relation that corresponds to the matrix elements of the desired commutation relation, thus completing the proof.

The latter relation generalizes the ones obtained in the diagonal case $(j_1 = j_2 = j_3 = j)$ for : (i) the chain $SO_3 \supset SO_2$ in connection with atomic and nuclear spectroscopy[9, 34] and (ii) a finite group in connection with the Jahn-Teller effect.[41]

Lie algebra and Wigner-Racah algebra : In the diagonal case $(j_1 = j_2 = j_3 = j)$, it can be seen that the general commutation relation of $\hat{W}^{k_1}_{\mu_1}(b_1 j)$ and $\hat{W}^{k_2}_{\mu_2}(b_2 j)$ defines the structure of the Lie group $GL_{[j]}; \mathbb{R}$ and of its subgroups. This result gives a precise meaning to the word algebra in what is generally referred to as Wigner-Racah algebra. The Lie algebra $\oplus_j \, g\ell_{[j]}; \mathbb{R}$ can thus be associated with the Wigner-Racah algebra of the compact group J. It should be observed that the basic ingredients, viz, (i) the Wigner operators of J, (ii) the (basis- or G-dependent) Wigner coefficients of J, and (iii) the (basis- or G-independent) Racah coefficients of J, all appear in the defining Lie law of the Wigner-Racah algebra of J.

We establish now a parallel between this appendix and the main body of the present paper. In the case $J = SU_2$, there is no multiplicity label b. Assuming the phase choice $\varphi(j_1 j_2 k) = 2k\pi$, the operator $\hat{t}_{k,\,q,\,j+\alpha}$ identifies to $t_{kq\alpha}$ as defined by Eq. (1). On the other hand, the operators $\hat{W}^k_q(j+\alpha)$ and $\hat{R}(j_1, j_1+\alpha_1, j_2, j_2+\alpha_2, j, k)$ identify to $W^k_q(j, \alpha)$ and $R(j_1 \alpha_1 j_2 \alpha_2 j k)$ as defined by Eqs. (79) and (95), respectively. Lastly, specialization of the general commutation relation to the chain $SU_2 \supset G^*$ leads to Eq.(30).

APPENDIX 2 : THE f SYMBOL

The f symbol (f for fractional) defined via Eq. (12) can be equivalently rewritten as

$$f\begin{pmatrix} j_1' & j_2 & k \\ \mu_1 & \mu_2 & \mu \end{pmatrix} = (-1)^{2k} \, (2j_1+1)^{-1/2} \, < j_2 k \mu_2 \mu \mid j_1 \mu_1 >^{*} ,$$

where

$$< j_2 k \mu_2 \mu \mid j_1 \mu_1 > = \sum_{m_2 q m_1}$$

$$< j_2 m_2 \mid j_2 \mu_2 >^{*} \, < k q \mid k \mu >^{*} < j_2 k m_2 q \mid j_1 m_1 > \, < j_1 m_1 \mid j_1 \mu_1 >$$

is a $SU_2 \supset G^*$ symmetry adapted Clebsch-Gordan coefficient.[36] It constitutes an extension of the f coefficients relative to the cubical, tetragonal, and trigonal double groups used by Racah, Low, and some of their students[35] in crystal-field analysis for iron group ions in solids. The main interest of the f symbol lies in the concise form

$$< \tau_1 j_1 \mu_1 \mid T^k_\mu \mid \tau_2 j_2 \mu_2 > = (\tau_1 j_1 \| T^k \| \tau_2 j_2) \quad f\begin{pmatrix} j_1 & j_2 & k \\ \mu_1 & \mu_2 & \mu \end{pmatrix}$$

the Wigner-Eckart theorem assumes when transcribed in a $SU_2 \supset G^*$ basis.

The $1\text{-}j\mu$ symbol defined through Eq. (18) derives from the f symbol since

$$f\begin{pmatrix} 0 & j & j' \\ \Gamma_o & \mu & \mu' \end{pmatrix} = \delta(j'j) \, (2j+1)^{-1/2} \begin{pmatrix} & j & \\ \mu & & \mu' \end{pmatrix}^{*} .$$

We may think of the f symbol as a third-rank tensor (second-rank covariant and first-rank contravariant). A third-rank covariant, and therefore more symmetrical, tensor arises with the introduction of the 3-jμ symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = \sum_{\mu_3'} \begin{pmatrix} j_3 \\ \mu_3 \, \mu_3' \end{pmatrix}^* f \begin{pmatrix} j_3 & j_2 & j_1 \\ \mu_3' & \mu_2 & \mu_1 \end{pmatrix}$$

which identifies to the Wigner 3-jm symbol when $G^* = U_1$.

By using the unitarity property of both the $SU_2 \supset U_1$ Wigner coefficients and the transformation from the $\{m\}$ scheme to the $\{\mu \equiv a\Gamma\gamma\}$ scheme, the f (or 3-jμ) symbol may be seen to satisfy orthogonality relations[36] similar to the ones of the 3-jm symbol. In addition, it may be also fractioned[36] according to the Racah lemma sketched in Appendix 1. Finally, the SU_2 recoupling coefficients can be redefined with the help of the f (or 3-jμ) symbol. In this regard, we have

$$\sum_{\mu_3} f \begin{pmatrix} j_1 & j_3 & k_2 \\ \mu_1 & \mu_3 & \mu_2' \end{pmatrix} f \begin{pmatrix} j_3 & j_2 & k_1 \\ \mu_3 & \mu_2 & \mu_1' \end{pmatrix}$$

$$= \sum_{k_3 \mu_3'} (-1)^{j_1 + j_2 + k_1 - k_2} (2k_3 + 1) \begin{Bmatrix} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} f \begin{pmatrix} k_3 & k_1 & k_2 \\ \mu_3' & \mu_1' & \mu_2' \end{pmatrix} f \begin{pmatrix} j_1 & j_2 & k_3 \\ \mu_1 & \mu_2 & \mu_3' \end{pmatrix} ,$$

a relation that is central in the derivation of Eq. (30).

* Work supported in part by the France-Canada exchange programme, the Quantum Theory Group of the University of Waterloo, and the Institut National de Physique Nucléaire et de Physique des Particules (IN2P3). This work constitutes a part of Chapter 1 of the State Doctorate Thesis in preparation by G. Grenet.

+ Permanent address : Institut de Physique Nucléaire, 43 Bd du 11 Novembre, 69621 Villeurbanne, France.

[1] L.C. Biedenharn, Phys. Lett. 3, 254 (1963) ; L.C. Biedenharn, J. Math. Phys. 4, 436 (1963) ; G.E. Baird and L.C. Biedenharn, J. Math. Phys. 4, 1449 (1963) ; ibid. 5, 1723, 1730 (1964).

[2] L. C. Biedenharn and H. van Dam, Quantum Theory of Angular Momentum (Academic Press, New York, 1965).

[3] J.D. Louck and L. C. Biedenharn, J. Math. Phys. 11, 2368 (1970). See also : L. C. Biedenharn, J.D. Louck, E. Chacón, and M. Ciftan, J. Math. Phys. 13, 1957 (1972) ; L. C. Biedenharn and J.D. Louck, J. Math. Phys. 13, 1985 (1972) ; J.D. Louck and L. C. Biedenharn, J. Math. Phys. 14, 1336 (1973).

[4] M.K.F. Wong, J. Math. Phys. 12, 1530 (1971).

[5] J. Schwinger, Report U.S. Atom. Energy Comm. NYO-3071, 1952, published in Ref. 2. See also : P. Jordan, Z. Physik 94, 531 (1935).

[6] G. Racah, Phys. Rev. 62, 438 (1942). See also : J.P. Elliott, Proc. Roy. Soc. (London) A 245, 128 (1958) ; G. Racah and J. Stein, Phys. Rev. 156, 58 (1967).

[7] G. Grenet and M. Kibler, J. Solid State Chem. 23, 1 (1978).

[8] L.L. Hirst, Adv. Phys. 27, 231 (1978).

[9] B. R. Judd, Operator Techniques in Atomic Spectroscopy (McGraw-Hill, New York, 1963).

[10] K. W. H. Stevens, Proc. Phys. Soc. 65, 209, 311 (1952). See also : R. J. Elliott and K. W. H. Stevens, Proc. Roy. Soc. A 218, 553 (1953) ; ibid. 219, 387 (1953) ; B. R. Judd, Proc. Roy. Soc. A 227, 552 (1955) ; ibid·Mol. Phys. 2, 407 (1959) ; R. J. Elliott, B. R. Judd, and W. A. Runciman, Proc. Roy. Soc. A 240, 509 (1957) ; J. M. Baker, B. Bleaney, and W. Hayes, Proc. Roy. Soc. A 247, 141 (1958) ; D. A. Jones, J. M. Baker, and D. F. D. Pope, Proc. Phys. Soc. 74, 249 (1959) ; M. T. Hutchings, Sol. State Phys. 16, 227 (1964) ; ibid. Proc. Phys. Soc. 85, 397 (1965).

[11] H. A. Buckmaster, Can. J. Phys. 40, 1670 (1962). See also : D. Smith and J. H. M. Thornley, Proc. Phys. Soc. 89, 779 (1966) ; R. J. Birgeneau, Can. J. Phys. 45, 3761 (1967) ; P.-A. Lindgård, J. Phys. C 8, 3401 (1975).

[12] H. A. Buckmaster, R. Chatterjee, and Y. H. Shing, Phys. Stat. Sol. (a) 13, 9 (1972).

[13] W. Low, Paramagnetic Resonance in Solids (Academic Press, New York, 1960) ; A. Abragam and B. Bleaney, Electron Paramagnetic Resonance of Transition Ions (Clarendon Press, Oxford, 1970).

[14] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940) ; F. J. Dyson, Phys. Rev. 102, 1217 (1956) ; S. V. Maleev, Zh. Eksperim. i Theor. Fiz. 33, 1010 (1957) ; ibid. Sov. Phys. - JETP 7, 1048 (1958) ; G. T. Trammell, J. Appl. Phys. 31, 362 S (1960) ; ibid. Phys. Rev. 131, 932 (1963) ; B. Grover, Phys. Rev. A 140, 1944 (1965).

[15] P.-A. Lindgård and O. Danielsen, J. Phys. C 7, 1523 (1974) ; P.-A. Lindgård and A. Kowalska, J. Phys. C 9, 2081 (1976) ; A. Kowalska and P.-A. Lindgård, Physica 86, 1111 (1977).

[16] M. E. Rose, Elementary Theory of Angular Momentum (Wiley, New York, 1957).

[17] P. W. Atkins and J. C. Dobson, Proc. Roy. Soc. A 321, 321 (1971) ; P. W. Atkins and P. A. Seymour, Molec. Phys. 25, 113 (1973). See also: W. Witschel, Molec. Phys. 26, 419 (1973).

[18] B. Meyer, Can. J. Math. 6, 135 (1954). See also : J. Patera and P. Winternitz, J. Chem. Phys. 65, 2725 (1976) ; L. Michel, in : Group Theoretical Methods in Physics, Eds. R. T. Sharp and B. Kolman (Academic Press, New York, 1977).

[19] P. L. Corio, J. Math. Phys. 9, 1067 (1968). See also : P. L. Corio, SIAM J. Appl. Math. 29, 661 (1975).

[20] J. M. Caola, Phys. Lett. 47 A, 357 (1974).

[21] A. K. Bose, Phys. Lett. 50 A, 425 (1975) ; A. K. Bose and R. A. B. Devine, Phys. Lett. 55 A, 267 (1975).

[22] G. Grenet and M. Kibler, Phys. Lett. 68 A, 147 (1978) ; ibid. 71 A, 323 (1979).

[23] M. Yamamura, T. Suzuki, and H. Ichihashi, Progr. Theor. Phys. (Kyoto) 60, 197 (1978). See also : H. Ichihashi and M. Yamamura, Progr. Theor. Phys. (Kyoto) 60, 753 (1978).

[24] D. C. Mattis, The Theory of Magnetism (Harper and Row, 1965).

[25] T. Kishimoto and T. Tamura, Nucl. Phys. A 192, 246 (1972) ; ibid. A 270, 317 (1976) ; O. Scholten, F. Iachello, and A. Arima, Ann. Phys. (N. Y.) 115, 325 (1978) ; T. Otsuka, A. Arima, and F. Iachello, Nucl. Phys. A 309, 1 (1978) ; O. Castaños, E. Chacón, A. Frank, and M. Moshinsky, J. Math. Phys. 20, 35 (1979).

[26] O. Hara and T. Gotō, Progr. Theor. Phys. (Kyoto) Suppl. No. 41, 56 (1968).

[27] G. Györgyi and S. Kövesi-Domokos, Report IPNO/TH123 (Paris Univ., 1968).

[28] R. L. Anderson, S. Kumei, and C. E. Wulfman, J. Math. Phys. 14, 1527 (1973).

[29] M. J. Englefield, Group Theory and the Coulomb Problem (Wiley-Interscience, New York, 1972). See also : W. Witschel, J. Phys. A : Math., Nucl. Gen. 7, 1847 (1974).

[30] A. Bohm and R. B. Teese, J. Math. Phys. 17, 94 (1976).

[31] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 18, 2259 (1977).

[32] M. Flato and C. Fronsdal, Lett. Math. Phys. 2, 421 (1978).

[33] K. Schulten and R. G. Gordon, J. Math. Phys. 16, 1961, 1971 (1975) ; A. O. Barut and R. Wilson, J. Math. Phys. 17, 900 (1976) ; D. A. Akyeampong and M. A. Rashid, Report IC/75/145 (Miramare - Trieste, 1975) ; R. Shaw, Preprint (Hull Univ., 1975).

[34] G. Racah, Phys. Rev. 76, 1352 (1949) ; Ergeb. Exakt. Naturw. 37, 28 (1965).

[35] G. Schönfeld, M. Sc. Thesis (Jerusalem Univ., 1960) ; M. Flato, M. Sc. Thesis (Jerusalem Univ., 1961) ; G. Racah and M. Flato, unpublished ; G. Rosengarten, M. Sc. Thesis (Jerusalem Univ., 1961) ; W. Low, in : Quantum Electronics, Ed. C. H. Townes (Columbia Univ. Press, New York, 1960) ; W. Low, in : Advances in Quantum Electronics, Ed. J. R. Singer (Columbia Univ. Press, New York, 1961) ; W. Low and G. Rosengarten, J. Molec. Spectrosc. 12, 319 (1964) ; M. Flato, J. Molec. Spectrosc. 17, 300 (1965) ; W. Low, in : Spectroscopic and Group Theoretical Methods in Physics, Eds. F. Bloch, S. G. Cohen, A. de-Shalit, S. Sambursky, and I. Talmi (North-Holland, Amsterdam, 1968).

[36] M. Kibler, J. Molec. Spectrosc. 26, 111 (1968) ; ibid. C. R. Acad. Sci. (Paris) B268, 1221 (1969) ; ibid. Int. J. Quantum Chem. 3, 795 (1969) ; ibid. Report LYCEN 6946 (Lyon Univ., 1969) ; ibid. Report LYCEN 7843 (Lyon Univ., 1978) ; ibid. in : Recent Advances in Group Theory and Their Application to Spectroscopy, Ed. J. C. Donini (Plenum, New York, 1979).

[37] E.P. Wigner, 1940 manuscript, published in Ref. 2 .

[38] A.R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton Univ. Press, New Jersey, 1960).

[39] J. Killingbeck, J. Phys. A 8, L 121 (1975).

[40] P.-O. Löwdin, Rev. Mod. Phys. 36, 966 (1964) ; J. Shapiro, J. Math. Phys. 6, 1680 (1965) ; Yu. F. Smirnov, in : Clustering Phenomena in Nuclei (IAEA, Vienna, 1969). See also : J.-L. Calais, Tech. Note No. 25 (Uppsala Univ., 1959) and Int. J. Quantum Chem. 2, 715 (1968).

[41] D.R. Pooler, J. Phys. A 11, 1045 (1978).

[42] W. Witschel, J. Phys. A 7, 1847 (1974).

[43] V. Bargmann, Rev. Mod. Phys. 34, 829 (1962), reprinted in Ref. 2 .

[44] S.D. Majumdar, Progr. Theor. Phys. (Kyoto) 20, 798 (1958).

[45] K. Venkatesh, J. Math. Phys. 19, 2060 (1978).

[46] B.L. van der Waerden, Die Gruppenteoretische Methode in der Quantenmechanik (Springer, Berlin, 1932).

[47] K. Zukauskas and E. Mauza, Vilniaus universiteto Mokslo darbai 7, 55 (1957). See also : A.P. Yutsis, I.B. Levinson, and V.V. Vanagas, Matematicheskii Apparat Teorii Momenta Kolichestva Dvizheniya (Vilnius, U.S.S.R., 1960). Translated by A. Sen and R.N. Sen : Mathematical Apparatus of the Theory of Angular Momentum (Israel Program for Scientific Translations, Jerusalem, 1962).

[48] E.P. Wigner, Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren (Vieweg, Braunschweig, 1931). Translated by J.J. Griffin : Group Theory and its Application to the Quantum Mechanics of Atomic Spectra (Academic Press, New York, 1959).

[49] G. Grenet and M. Kibler, unpublished.

50 J. Patera, R.T. Sharp, and P. Winternitz, J. Math. Phys. 19, 2362 (1978) ; P.E. Desmier and R.T. Sharp, J. Math. Phys. 20, 74 (1979) ; J. Patera and R.T. Sharp, in : Recent Advances in Group Theory and Their Application to Spectroscopy, Ed. J.C. Donini (Plenum, New York, 1979) ; ibid. in : Proceedings of the V II International Colloquium on Group Theoretical Methods in Physics (Springer, Berlin, 1979).

51 P.A. Cox, Struct. Bonding 24, 59 (1975) ; G. Grenet, Y. Jugnet, Tran Minh Duc, and M. Kibler, Chem. Phys. Lett. 62, 125 (1979).

52 J.K. Lang, Y. Baer, and P.A. Cox, Phys. Rev. Lett. 42, 74 (1979).

53 W.G. Harter, Phys. Rev. A 8, 2819 (1973) ; W.G. Harter and C.W. Patterson, Phys. Rev. A 13 , 1067 (1975) ; C.W. Patterson and W.G. Harter, Phys. Rev. A 15, 2372 (1977).

54 N.B. Backhouse, J. Phys. A 12, 21 (1979) ; M. Bednář and V.F. Šachl, J. Math. Phys. 20, 367 (1979) ; P.D. Jarvis and H.S. Green, Preprint THEP 78/9-7 (Southampton Univ., 1978).

55 E. Schrödinger, Proc. Roy. Irish Acad. A 46, 9 (1940) ; L. Infeld and T.E. Hull, Rev. Mod. Phys. 23, 21 (1951) ; W. Miller, Jr., Lie Theory and Special Functions (Academic, New York, 1968).

56 J.W.B. Hughes and J. Yadegar, J. Math. Phys. 19, 2068 (1978).

57 L.C. Biedenharn and J.D. Louck, Ann. Phys. (N.Y.) 63, 459 (1971) ; ibid. in : Group Theory and its Applications, Vol. II, Ed. E.M. Loebl (Academic, New York, 1971).