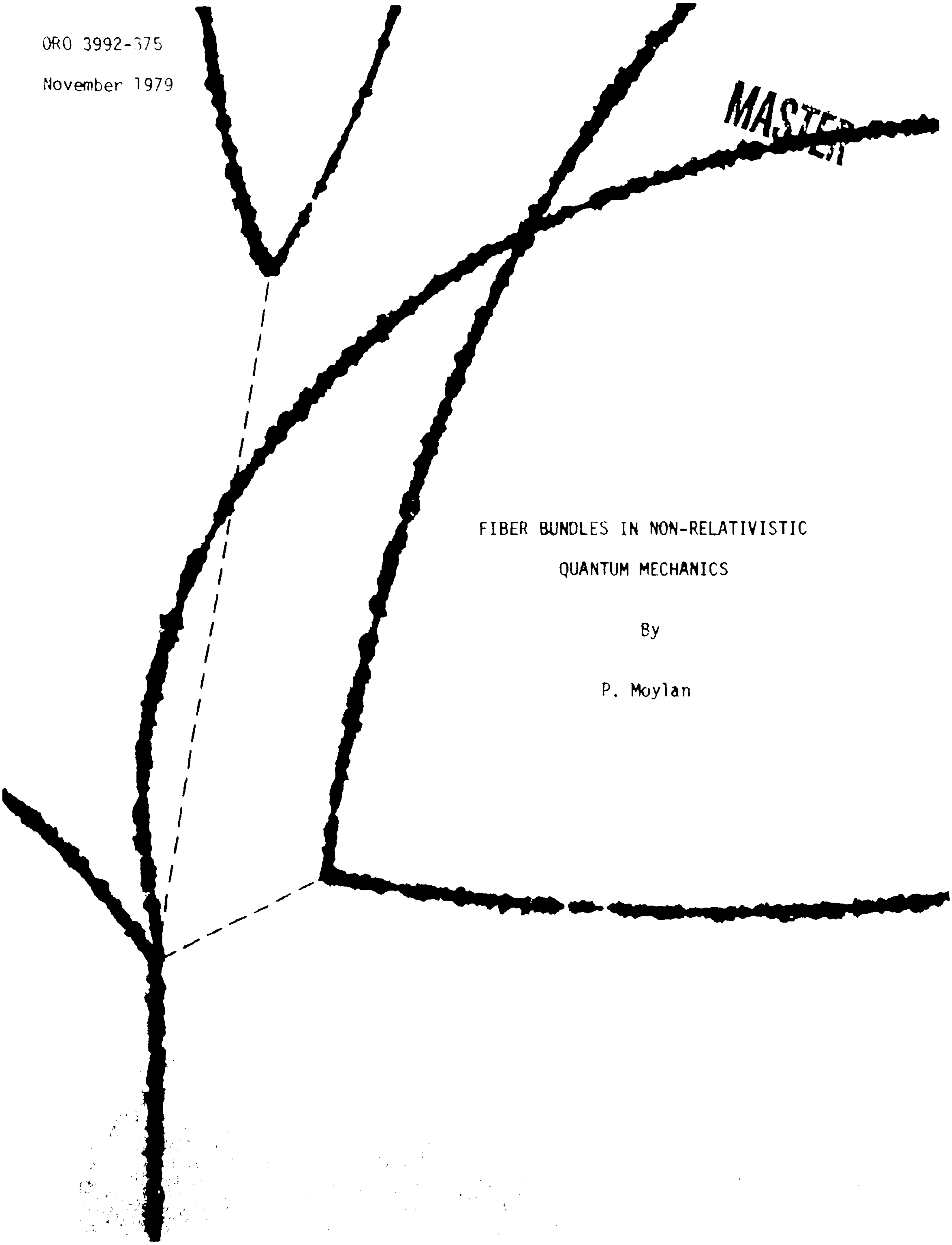
ORO 3992-375

November 1979



FIBER BUNDLES IN NON-RELATIVISTIC QUANTUM MECHANICS

By

P. Moylan

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Abstract

The problem of describing a quantum mechanical system with symmetry by a fiber bundle is considered. The quantization of a fiber bundle is introduced. Fiber bundles for the Kepler problem and the rotator are constructed. The fiber bundle concept provides a new model for a physical system: it provides us with a model for an elementary particle with extension having integral values of spin.

I. Introduction

Frequently in physics, the introduction of a new concept to solve new problems can be employed to shed light on old, familiar problems. Such is the case with the fiber bundle idea as the present paper shows. It has been known for quite some time[1)] that a soluble quantum mechanical problem admits of a symmetry group. Particular irreducible representation spaces of the symmetry group contain all states of a given energy and provide the solutions to the Schrodinger equation. For this reason the group is called the "group of degeneracy".[2)] Also, given a symmetry group, we can construct a larger group by adjoining usually noncompact generators. Such groups have been called spectrum generating or dynamical groups. The existence of a symmetry group or a dynamical group provides a means by which we can associate with the physical system a certain structure over the space we are interested in called a fiber bundle.

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1) S. Lie, "Über Gruppen von Transformation" in Gesamalte Abhandlungen, vol. V.
W. Miller, Symmetry and Separation of Variables, Addison-Wesley (1977).

2) A. O. Barut, Phys. Rev., 135, B839 (1964).

In the following two non-relativistic quantum mechanical systems are described in terms of fiber bundles. First we deal with the necessary notions of fiber bundle theory. Next in section III, we consider the quantization of an arbitrary fiber bundle. Then in section IV, we give fiber bundle descriptions to two important quantum mechanical systems: the Kepler problem and the rotator. In order to give the Kepler problem a fiber bundle description, we find it necessary to first discuss it in a way that goes back to Pauli and Fock.[3,4] Next we give a fiber bundle description of a mathematical model of the non-relativistic rotator, which was introduced by A. Böhm.[4] The analog of the fiber bundle ih Minkowski space is a DeSitter fiber bundle.[5]

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[3] M. Bander, C. Itzykson, Review of Modern Physics, <u>38</u>, p. 330-34?, (1960).

J. Schwinger, J. Math. Phys., <u>5</u>, 1606, (1964).

M. Engelfield, <u>Group Theory and the Coulomb Problem</u>, Wiley-Interscience, (1964).

R. Finkelstein, <u>Nonrelativistic Mechanics</u>, Benjamin, (1973).

M. Bander, C. Itzykson, Reviews of Modern Physics, <u>38</u>, p. 346-358, (1966).

[4] A. Böhm, Nuovo Cimento, <u>43</u>, p. 665-683, (1966).

[5] W. Drechler, Fortschritte der Physik, <u>23</u>, 607-617, (1975).

II. Construction of a Fiber Bundle

In this paper, we are going to construct some fiber bundles over three-dimensional Euclidean space. All fiber bundles over contractible Euclidean space are trivializable and hence can be reduced to tensor products, a simplifying fact used throughout this paper. First we describe the notion of a fiber bundle and some related concepts.[6)7)] A fiber bundle $E(B,F,\pi,G)$ over a manifold, B, is given by the following collection of objects

(i) a manifold, E, called the total space or fiber bundle;

(ii) a manifold, B, called the base space;

(iii) a manifold, F, called the fiber;

(iv) an onto mapping π from E to B called the projection. For $x \in B$; $\pi^{-1}(x) = F_x$ is called the fiber over x.

(v) a covering of B with neighborhoods $\{U_j\}$ such that $\pi^{-1}(U_j)$ is homeomorphic to the topological product $U_j \times F$ under a homeomorphism ϕ_{U_j} defined to

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6) W. Drechsler, M. E. Mayer, Fiber Bundle Techniques in Gauge Theories, Springer-Verlag, (1977).

7) S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience, (1963).

be such that for $p = (x,\xi_x)\varepsilon E$ with $x\varepsilon U_j$ and $\xi_x \varepsilon F_x$ we have $\phi_{U_j}(p) = (x,\bar{\phi}_{U_j}(\xi_x))$ with $\bar{\phi}_{U_j}$ denoting a (topological) homomorphism of F_x onto F.

(vi) a Lie group G of homomorphisms of F onto itself called the structural group of the bundle, such that for $x\varepsilon U_i \cap U_j$ the homomorphism $\phi_{U_i} \cap \phi_{U_j}^{-1}$ is an element of G which depends continuously on x.

If the structural group G of the bundle is the same manifold as F itself (for example, if G acts on itself by left translation) then we call the bundle $P(B,G,\pi,G,\phi)$ a principle fiber bundle over B and we write it as P(B,G). The fiber bundle, $E(B,F,\pi_E,G,\phi')$ associated with the principle fiber bundle P(B,G) and having fiber F is defined as follows: Let F be a manifold on which G acts effectively as a transformation group. If E is identified with the coset space $K = P \times F/G$ and π_E is the mapping of K onto B induced by the mapping π of P onto B, then we can construct a family of homomorphisms $\{\phi'_{U_i}\}$ of $U_i \times F$ onto $\pi_E^{-1}(U)$ using the ones from the fiber bundle P(B,G).

Now we define the important concept of soldering for a certain type of fiber bundle. The importance of soldering in physics is that it firmly links the fibers of those fiber bundles, for which it is possible to define a soldering, to the tangent spaces of the space. A fiber bundle

E(B,F,T,G,P) over B associated to the principle fiber bundle P(B,G) is called <u>soldered</u> to B if the following are satisfied:

(i) F is the homogeneous space G/G' where G' is the stability subgroup leaving a point, OεF, fixed.

(ii) dim F = dim B = n

(iii) The bundle E(B,F,G,P) admits a cross section which will be identified with B.

(iv) If T'(B) is the union of all the spaces of all the tangent vectors to F_x at xεB for all x and $T_*(B)$ is the tangent bundle over B, then we can identify T'(B) and $T_*(B)$ by an isomorphism.

For our purposes we need a modification of (iv). Let T*(B) be the fiber bundle over B with fiber, TR_n^*, the "momentum" space--the space of characters of the n-dimensional translation group--and as structural group the group of affine transformations. Then (iv) becomes replaced by:

(iv') If T'(B) is the union of all spaces of all tangent vectors to F_x at xεB for all x, then we can identify T'(B) and T*(B) by an isomorphism.

The fiber bundles considered here satisfy the conditions (i), (ii), (iii), and (iv').

III. Quantization of a Fiber Bundle

According to the axioms of quantum mechanics, a physical observable is represented by a linear operator which acts in a linear space (space of physical states).[8] Unless the observable is to represent a nonclassical degree of freedom such as isospin it must be an element of some extension algebra of the enveloping algebra[9] of the Heizenberg algebra or possibly a limit of such elements. In the fiber bundle case we have to decide what functions of P_1 and Q_1 to choose for the elements of the fiber over a point. In general there is no unique choice. A choice must depend upon other considerations--such as wanting to exploit some symmetry, which is the case for the rotator example constructed below.

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[8] A. Böhm, Quantum Mechanics, Springer-Verlag, (1979).

[9] J. Dixmier, Enveloping Algebras, North-Holland Publishing Co., (1978).

A. Böhm, Rigged Hilbert Space and Quantum Mechanics, p. 49, Springer-Verlag, (1978).

IV. A Fiber Bundle Description of Two Nonrelativistic Quantum Mechanical Systems

A. The Kepler Problem

We will describe the Kepler problem in terms of a fiber bundle. As is well known, the problem of two bodies moving in an inverse square central field of force is equivalent to determining the motion of the center of the mass and the motion of a fictitious particle of mass $\frac{m_1 m_2}{m_1+m_2}$ which is the relative motion of the two particles. In the center of a mass system, the Schrödinger equation for the two body problem becomes an equation for a fictitious particle moving in a potential $V(\vec{r}) = \frac{x}{r}$, for the case of an inverse square-law force. r is the relative separation of the two particles. The fiber bundle that we will construct will be a fiber bundle over the three dimensional space of the center of the mass. Local coordinates in the fiber will be related to the relative separation coordinates (r_x, r_y, r_z) via the three dimensional Fourier transform. The fiber will be a curved space and the equation of motion in the fiber[6)] for the particle will be the wave equation for a free particle moving in the curved space.

Before we proceed with the fiber bundle construction, however, we must make a mathematical digression to demonstrate the equivalence of the Schrödinger equation for a free particle moving in a $\frac{1}{r}$ potential and the wave equation for a free particle moving in a curved space. The Schrödinger equation for a free particle moving in a potential $V(\vec{r}) = \frac{\alpha}{r}$ is:

$$\left(\frac{\nabla^2}{2} - \frac{\alpha}{r}\right)\psi(\vec{r}) = E\psi(\vec{r}) \tag{1}$$

with units chosen so that $\hbar = c = 1$ and the reduced mass equal to unity. The wave equation for a free particle moving in a curved space is the eigenvalue equation of the Laplace operator on that space. The Laplace operator is defined for an arbitrary pseudo-Riemannian space to be:[10)]

$$\Delta = \frac{1}{g^{1/2}} \frac{\partial}{\partial x_\mu} \left(g^{1/2} g^{\mu\nu} \frac{\partial}{\partial x^\nu}\right) \tag{2}$$

where $g^{\mu\nu}$ is the contravariant metric tensor and g is the determinant of the metric tensor, $g_{\mu\nu}$. The particular

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[10)] S. Helgason, Differential Geometry and Symmetric Spaces, p. 387, Academic Press, New York, (1962).

curved space which we choose depends upon whether we are considering the bound states or scattering states. For the bound states, the space has constant positive curvature and can be embedded into four dimensional Euclidean space. It is the three sphere there, S_3, that is, the set of all points $\{(x_1,x_2,x_3,x_4) | x_1^2+x_2^2+x_3^2+x_4^2 = 1\}$ (see Figure (1)). For the scattering states, the space has constant negative curvature and can be embedded into four dimensional Minkowski space. There it is the two-sheeted unit hyperboloid, $T_3^+ \cup T_3^- = T_3$ (see Fig. (2)). For the zero energy case, the space has zero curvature and is isometric to Euclidean space. The symmetry group for the eigenvalue equation of the Laplace operator turns out to be SO(4) in the case of bound states, SO(3,1) in the case of scattering states and E(3) in the case of zero energy. Although our analysis applies to it, we will not study the limiting case of zero energy and will now demonstrate the equivalence for the bound and scattering states.

First we map from the configuration space, R^3, for the relative motion to the momentum space in the obvious way. We then define a new wave function $\phi(\vec{p})$ by the Fourier transform. It is:

$$\phi(\vec{p}) = \frac{1}{(2\pi)^{1/2}} \int \phi(\vec{r}) e^{-i\vec{p}\cdot\vec{r}} d^3r \tag{3}$$

The Schrödinger equation for the Kepler problem in momentum space thus reads:

$$\left(\frac{p^2}{2} - E\right)\phi(\vec{p}) = -\frac{\alpha}{2\pi^2} \int \frac{d^3q\,\phi(\vec{q})}{|\vec{q} - \vec{p}|^2} \tag{4}$$

Here we have used the fact that the Fourier transform of the convolution of two functions is the product of their Fourier transforms.[11)]

Now introduce steriographic projection. Two cases need to be considered:

Case I	let $p_o^2 = -2E$	if $E<0$		(5a)
Case II	let $p_o^2 = +2E$	if $E>0$		(5b)

Eqn. (4) reads for the two cases:

$$(p^2 + p_0^2)\phi(\vec{p}) = \frac{\alpha}{\pi^2} \int \frac{d^3p'\,\phi(\vec{p}\,')}{|\vec{p}\,' - \vec{p}|^2} \quad \text{if } E < 0 \tag{6a}$$

$$(p^2 + p_0^2)\phi(\vec{p}) = \frac{\alpha}{\pi^2} \int \frac{d^3p'\,\phi(\vec{p}\,')}{|\vec{p}\,' - \vec{p}|^2} \quad \text{if } E > 0 \tag{6b}$$

[11)] W. Rudin, Functional Analysis, p. 167, McGraw Hill, (1973).

For case I, replace p by $p/\frac{1}{2}p_o$ and embed the momentum space into a four dimensional Euclidean space, R^4, in such a way that the momentum space, T^*_3, is a hyperplane passing through the point (-1,0,0,0) (Fig. (1)). Then perform a steriographic projection onto the unit sphere, S_3, minus the north pole, in R^4 as shown in Fig. (1). The equations of steriographic projection read:

$$\left.\begin{aligned} \xi_i &= \frac{2p_0 p_i}{p^2 + p_0^2} \qquad (i = 1,2,3) \\ \xi_0 &= \frac{p^2 - p_0^2}{p^2 + p_0^2} \end{aligned}\right\} \tag{7a}$$

For case II, replace p by $p/\frac{1}{2}p_o$ and embed the momentum space into a four dimensional pseudo-Euclidean space with Minkowski metric having signature (+---), M_4, in such a way that the momentum space, T^*_3, is a space-like hypersurface passing through the point (-1,0,0,0) (Fig. (2)). Performing a steriographic projection onto the two-sheeted hyperboloid, T_3, in M_4, we obtain the following equations:

$$\left.\begin{aligned} \xi_i &= \frac{2p_0 p_i}{p_0^2 - p^2} \qquad (i = 1,2,3) \\ \xi_0 &= \frac{p_0^2 + p^2}{p_0^2 - p^2} \end{aligned}\right\} \tag{7b}$$

The region $p^2 < p_0^2$ is mapped onto the upper sheet and the region $p^2 > p_0^2$ is mapped onto the lower one. The measures on S_3 and T_3^+ are in terms of steriographic projection coordinates[3)]

$$d\Omega_{S_3} = \frac{(2p_0)^2}{(p_0^2 + p^2)^3}\, d^3p \tag{8a}$$

$$d\Omega_{T_3^+} = \frac{(2p_0)^2}{(p_0^2 - p^2)^3}\, d^3p \tag{8b}$$

We observe that

$$|\vec{\xi} - \vec{\xi}'|^2 = \frac{4p_0^2}{(p_0^2 + p^2)(p_0^2 + p'^2)}\, |\vec{p} - \vec{p}'|^2 \quad \text{for } \vec{\xi}, \vec{\xi}' \in S_3 \tag{9a}$$

$$|\vec{\xi} - \vec{\xi}'|^2 = \frac{4p_0^2}{(p_0^2 - p^2)(p_0^2 - p'^2)}\, |\vec{p} - \vec{p}'|^2 \quad \text{for } \vec{\xi}, \vec{\xi}' \in T_3^+ \text{ or } T_3^- \tag{9b}$$

Now we define the following functions in terms of steriographic projection coordinates on S_3 and T_3

$$\phi(\vec{\xi}) = (p_0)^{-1/2} \left[\frac{p_0^2 + p^2}{2p_0}\right] \phi(\vec{p}) \quad \text{for } \phi(\vec{p}) \text{ satisfying Eqn. (6a)} \tag{10a}$$

$$\phi(\vec{\xi}) = (p_0)^{-1/2} \left[\frac{p_0^2 - p^2}{2p_0}\right] \phi(\vec{p}) \quad \text{for } \phi(\vec{p}) \text{ satisfying Eqn. (6b)} \tag{10b}$$

Using Eqns. (8a), (8b), (9a), (9b), (10a) and (10b) we rewrite Eqns. (6a) and (6b) in terms of the ξ coordinates and obtain:

$$\phi(\vec{\xi}) = \frac{\alpha}{2p_0\pi^2}\int \frac{\phi(\vec{\xi}\,')}{|\vec{\xi} - \vec{\xi}\,'|^2}\, d\Omega'_{S_3} \qquad \vec{\xi},\vec{\xi}\,' \in S_3 \tag{11a}$$

$$\phi(\vec{\xi}) = \frac{\alpha\epsilon(\vec{\xi})}{2p_0\pi^2}\int \frac{\phi(\vec{\xi}\,')}{|\vec{\xi} - \vec{\xi}\,'|^2}\, d\Omega'_{T_3} \qquad \vec{\xi},\vec{\xi}\,' \in T_3 \tag{11b}$$

where

$$\epsilon(\vec{\xi}) = \begin{cases} +1 & \text{if } \vec{\xi} \in T_3^+ \\ -1 & \text{if } \vec{\xi} \in T_3^- \end{cases}$$. [12)]

For case 1 the mapping defined by Eqn. (10a) effects a one-to one and isometric mapping which is unitary from the Hilbert space, which is obtained by taking all linear combinations of bound state eigenfunctions of Eqn. (1) and their limits, to the Hilbert space of L^2 functions on the sphere, S_3. This is seen as follows: using Eqns. (10a) and (8a), we have

$$\int |\phi(\vec{\xi})|^2\, d\Omega_{S_3} = \int \frac{p_0^2 + p^2}{2p_0^2}\, |\phi(\vec{p})|^2\, d^3p \tag{12}$$

Using the virial theorem which states that

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12) Bander, Itzykson, p. 348.

$$E\int|\phi(\vec{p})|^2 d^3p = -\int\frac{p^2}{2}|\phi(\vec{p})|^2\, d^3p \qquad (13)$$

it follows

$$\int d\Omega_{S_3}|\phi(\xi)|^2 = \int d^3p|\phi(\vec{p})|^2 \qquad (14)$$

Therefore the mapping given by Eqn. (10a) preserves scalar products. We will state that the functions on the sphere as given by Eqn. (11a) are dense in $L_2(S_3)$ shortly. Thus the mapping can be extended on the one side to the Hilbert space $L_2(S_3)$ and on the other side to the Hilbert space of linear combinations (and their limits) of bound state eigenfunctions of the Hamiltonian (Eqn. (1)). It is unitary because of Eqn. (14). For case II, no Hilbert space model exists for the mapping defined by Eqn. (10b). The appropriate mathematical tool to describe this mapping is the Rigged Hilbert space.[13)]

So far we have shown the equivalence of determining solutions to the Schrodinger equation for a particle in a 1/r potential and the solutions of two integral equations in momentum space--Eqns. (11a) and (11b). Now we show that the solutions of these two integral equations are

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[13)] A. Bohm, Rigged Hilbert Space and Quantum Mechanics.

certain eigenfunctions of the Laplace operators on S_3 and T_3. For case I, the Riemann space, S_3, embedded in R_4, with coordinate system

$$(\xi_0,\xi_1,\xi_2,\xi_3) = (\cos\rho,\sin\rho\cos\theta,\sin\rho\sin\theta\cos\phi,\sin\rho\sin\theta\sin\phi) \tag{15a}$$

has line element[14)]

$$ds = d\rho^2 + \sin^2\rho d\theta^2 + \sin^2\rho\sin^2\theta d\phi^2 \tag{16a}$$

By using Eqn. (2) the Laplacian is found to be

$$\Delta_I = \frac{\partial^2}{\partial\rho^2} + \frac{1}{(\sin\rho)^2}\left\{\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} \tag{17a}$$

For case II, the Riemann space, T_3^+, embedded in M_4, with coordinate system and line element

$$(\xi_0,\xi_1,\xi_2,\xi_3) = (\cosh\rho,\sinh\rho\cos\theta,\sinh\rho\sin\theta\cos\phi,\sinh\rho\sin\theta\sin\phi) \tag{15b}$$

$$ds = d\rho^2 + \sinh^2\rho d\theta^2 + \sinh^2\rho\sin^2\theta d\phi^2 \tag{16b}$$

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14) Misner, Thorne and Wheeler, Gravitation, p. 723, Freeman, (1973).

has as Laplacian

$$\Delta_{II} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{(\sinh\rho)^2}\left\{\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} \tag{17b}$$

A complete set of eigenfunctions of Eqn. (17a) are the four-dimensional spherical harmonics $Y_{n\ell m}(\vec{\xi})$. For n an integer, the spherical harmonic $Y_{n\ell m}(\vec{\xi})$ is defined to be $1/r^{n-1}$ multiplied by a homogeneous polynomial, $y_{n\ell m}(\vec{r})$, of degree n-1 in $\vec{r} = |\vec{r}|\vec{\xi}$ which is required to satisfy

$$\Delta^4 y_{n\ell m}(\vec{r}) = \Delta^4 |\vec{r}|^{n-1} Y_{n\ell m}(\vec{\xi}) = 0 \qquad (\vec{r} \in R^4) \tag{18}$$

where Δ^4 is the Laplacian on R^4 and $\vec{\xi} \in S_3$. Here ℓ,m provide a three-dimensional classification of the number of linearly independent four-dimensional spherical harmonics of a given degree, n-1. The largest value of ℓ is the degree of the polynomial $r^{n-1}Y_{n\ell m}(\vec{\xi})$. Thus $-\ell \leq m \leq \ell$ and $0 \leq \ell \leq n - 1$ give a total of n^2 linearly independent spherical harmonics of degree n - 1. We see the spherical harmonics form a complete set of eigenfunctions as follows. The Euclidean Laplace operator in R^4 is in spherical polar coordinates

$$\Delta^4 = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{\Delta_I}{r^2} \tag{19}$$

where Λ_I is given by Eqn. (17a). Using Eqn. (18) we obtain

$$\Lambda_I Y_{n\ell m}(\vec{\xi}) = (n-1)(n+1)Y_{n\ell m}(\vec{\xi}) \tag{20}$$

which shows that they are eigenfunctions of Eqn. (17a). These functions satisfy the following three conditions:

$$\sum_{n=0}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} Y_{n\ell m}(\vec{\xi}_1) Y_{n\ell m}(\vec{\xi}_2)^* = \delta_{S_3}(\vec{\xi}_1,\vec{\xi}_2) \tag{21a}$$

(completeness condition)

($\delta_{S_3}(\xi_1,\xi_2)$ is the δ-function on the three sphere)

$$\int_{S_3} d\Omega_{S_3} Y_{n\ell m}(\vec{\xi}) Y^*_{n'\ell'm'}(\vec{\xi}) = \delta_{n'n}\delta_{\ell'\ell}\delta_{m'm} \tag{22a}$$

(orthogonality condition)

$$Y_{n\ell m}(\vec{\xi}) = \frac{n}{2\pi^2} \int \frac{d\Omega'_{S_3}}{|\vec{\xi} - \vec{\xi}'|^2} Y_{n\ell m}(\vec{\xi}') \tag{23a}$$

(integral equation)

The first two conditions are generalizations of well-known results for the spherical harmonics on R^3. The last equation is proved in the appendix.

Now we find a complete set of functions on $\bar{T}_3^+$ which are eigenfunctions of the operator given by Eqn. (17b). Here we restrict ourselves to the upper sheet of the hyperboloid in Fig. (2). Let us consider the eigenfunctions of Eqn. (17b) which separate as follows

$$H_{N\ell m}(\vec{\xi}) = Z_{N\ell}(\rho) Y_{\ell m}(\vec{\zeta})$$

where $Y_{\ell m}(\vec{\xi})$ is a spherical harmonic on the sphere, S_2. The Laplacian in our four-dimensional Minkowski space is defined in an open neighborhood having nonempty intersection with T_3^+ by

$$\Box = \frac{\partial^2}{\partial \xi_0^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial \xi_i^2} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{\Delta_{II}}{r^2} \tag{24}$$

where Δ_{II} is given by Eqn. (17b). As before, $H_{N\ell m}$ an eigenfunction of Δ_{II} is equivalent to $r^{\lambda}H_{N\ell m}(\vec{\xi})$ (defined in a neighborhood of M_4 having nonempty intersection with T_3^+) being a homogeneous function of $\vec{\omega} = r\vec{\xi}$ of degree λ with $\lambda = -1 + iN$ such that

$$\Box^4 r^{\lambda} H_{N\ell m}(\vec{\xi}) = 0 \tag{25}$$

A boundary condition which restricts N will be imposed on the functions, namely that $H_{N\ell m}(\vec{\xi})$ be a continuous function of smallest possible growth at infinity. This condition forces N to be real. These requirements define a differential equation which $Z_{N\ell}(\rho)$ must satisfy and furthermore it can be solved to determine $Z_{N\ell}(\rho)$ in terms of trigonometric functions. These $Z_{N\ell}(\rho)$ are given explicitly in Bander and Itzykson on p. 349 (Ref. 3). Also proved in that paper are the following three conditions:

$$\int_0^{\infty} dN \sum_{\ell m} H_{N\ell m}(\vec{\xi}_1) H_{N\cdot m}(\vec{\xi}_2) = \delta_{\text{hyperboloid}}(\vec{\xi}_1,\vec{\xi}_2) \tag{21b}$$

(completeness condition)

$$\int_{T_3^+} d\Omega_{T_3} H_{N_1\ell_1 m_1}(\vec{\xi}) H_{N_2\ell_2 m_2}(\vec{\xi}) = \delta_{\ell_1\ell_2}\delta_{m_1 m_2}\delta(N_1 - N_2) \tag{22b}$$

(orthogonality condition)

$$\int_{T_3^+} d\Omega_{T_3} \frac{H_{N\ell m}(\vec{\xi}\,')}{(1 + t^2 + 2t\vec{\xi}\vec{\xi}\,')} = 2\pi^2 \times \frac{t^{-1} \cos(N \log(t))}{N \sinh N} H_{N\ell m}(\vec{\xi}) \tag{23b}$$

(integral equation)

In Eqn. (23b) t is a complex variable in the cut plane from $-\infty$ to 0, log (t) is real for t real and positive and the argument of the expression $(1 + t^2 + 2t\vec{\xi}\vec{\xi}\,')$ is 0 for t real positive.

The two Eqns. (11a) and (23a) in case I and Eqns. (11b) and (23b) in case II need to be compared. Setting Eqn. (11a) equal to Eqn. (23a) we obtain

$$E = -\frac{\alpha^2}{2n^2} \qquad n = 1,2,3... \tag{26a}$$

which is the familiar formula for the energy spectrum of the hydrogen atom. The completeness condition--Eqn. (21a)--exhausts all the solutions of Eqn. (11a). Case II is more difficult and we will only quote the result here that Eqns.

(11b) and (23b) are consistent with each other only if

$$E = \frac{\alpha^2}{2N^2} \qquad N \in R \tag{26b}$$

A derivation of this result is given in Bander and Itzykson on p. 353. Because of the completeness relation of Eqn. (21b) we have exhausted all the solutions of Eqn. (11b).

Now that we have demonstrated the equivalence of the Schrödinger equation with the wave equations on S_3 and T_3 in the case of bound states and scattering states respectively, we proceed with the construction of the fiber bundle for the Kepler problem. The Schrödinger equation for two particles of mass m_1 and m_2 interacting through an inverse square central force is:

$$\left\{ \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} - \frac{\alpha}{r} \right\} \psi(\vec{r}_1, \vec{r}_2) = E\psi(\vec{r}_1, \vec{r}_2) \tag{27}$$

where P_1 and P_2 are the operators of momentum for the two particles and $r = |\vec{r}_1 - \vec{r}_2|$ ($\hbar$ has been set equal to unity). In a coordinate system where the center of mass is fixed at the origin we have

$$m_2 r_2 = m_1 r_1 \tag{28}$$

and furthermore $r = r_1 + r_2$. The classical system is illustrated in Fig. (3). The polar coordinates of m_1 are r_1, θ, ϕ and the polar coordinates of m_2 are r_2, $\theta + \pi$, $\phi + \pi$. Note from Eqn. (28) that:

$$\left.\begin{aligned} r_1 &= \frac{m_2}{m_1 + m_2}\, r \\ r_2 &= \frac{m_1}{m_1 + m_2}\, r \end{aligned}\right\} \qquad (29)$$

and so

$$\left.\begin{aligned} \vec{p}_1 &= m_1 \dot{r}_1 \vec{n} = \frac{m_1 m_2}{m_1 + m_2}\, \dot{r}\vec{n} = \mu \dot{r}\vec{n} \\ \vec{p}_2 &= -m_2 \dot{r}_2 \vec{n} = -\frac{m_1 m_2}{m_1 + m_2}\, \dot{r}\vec{n} = -\mu \dot{r}\vec{n} \end{aligned}\right\} \qquad (30)$$

Here $\vec{n}$ is the unit vector shown in Figure 3. From these equations, we have that $\vec{p}_1 = -\vec{p}_2 = \vec{p}$. Substituting this into Eqn. (27) and replacing $\psi(\vec{r}_1, \vec{r}_2)$ by $\psi(R)\psi(r)$ we obtain

$$\left\{ \frac{p^2}{2\mu} - \frac{\alpha}{r} \right\} \psi(\vec{r}) = E\psi(\vec{r}) \qquad (31)$$

Notice that all of the momentum in the center of mass coordinate system is relative: this is the equation for the relative or "internal motion" part of the Kepler problem.

In constructing the fiber bundle we let the base space, B, consist of the configuration space for the center of mass. The fiber is S_3 for the bound states and T_3 for the scattering states. It is connected with the momentum space coordinates for the relative motion by the equations of stereographic projection (Eqns. (7a) and (7b)). The fiber bundle for the bound states is the one associated to $P(R^3, SO(4))$:

$$E(R^3, F = SO(4)/SO(3), G = SO(4), P). \tag{32a}$$

The fiber bundle for the scattering states is the one associated to $P(R^3, SO(3,1))$:

$$E(R^3, F = SO(3,1)/SO(3), G = SO(4), P). \tag{32b}$$

They are illustrated in Figs. (1) and (2). The soldering condition (iv') is fulfilled by taking any isomorphism of the momentum space for the relative motion with the tangent space to the fiber at the point of contact.

The equation of motion in the fiber is

$$\Lambda_I \psi(\vec{\xi}) = \lambda \psi(\vec{\xi}) \tag{33a}$$

or

$$\Delta_{II} \psi(\vec{\xi}) = \lambda \psi(\vec{\xi}) \tag{33b}$$

depending upon whether we are dealing with case I or case II. Δ_I is the Laplacian on S_3 (Eqn. (17a)) and Δ_{II} is the Laplacian on T_3 (Eqn. (17b)). Note that λ is not the same as the energy eigenvalue, E, in Eqn. (31).

The infinitesimal generators of the orthogonal rotations in the $\xi_0\xi_i$ (i = 1,2,3) planes of Figs. (1) and (2) are given by the differential operators

$$L_{i0}^{+} = i\left(\xi_0 \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_0}\right) \qquad (i = 1,2,3) \qquad (34^{+})$$

$$L_{i0}^{-} = \left(\xi_0 \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_0}\right) \qquad (i = 1,2,3) \qquad (34^{-})$$

acting on $L_2(S_3)$ and $L_2(T_3^{+})$, respectively. The mapping defined by Eqn. (10a) effects a transformation from $L_2(S_3)$ onto a closed subset of $L_2(R^3)$ and that defined by Eqn. (10b) effects a transformation from a closed subset of $L_2(T_3^{+})$ onto a closed subset of $L_2(R^3)$. Acting on functions of the momentum space--elements of $L_2(R^3)$--for which p_0 is a constant we have[3)]

$$\begin{aligned} L_{i0}^{+} &= \frac{i}{p_0}\left[\frac{p^2 - p_0^2}{2}\frac{\partial}{\partial p_i} - p_i\left(\sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j}\right) - 2ip_i\right] \\ L_{i0}^{-} &= \frac{1}{p_0}\left[\frac{p^2 - p_0^2}{2}\frac{\partial}{\partial p_i} - p_i\left(\sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j}\right) - 2ip_i\right] \end{aligned} \qquad (35^{\pm})$$

for L_{i0}^{+} an orthogonal rotation and L_{i0}^{-} a hyperbolic one.

The differential operators:

$$L_{ij} = i\left(\xi_i \frac{\partial}{\partial\xi_j} - \xi_j \frac{\partial}{\partial\xi_i}\right) \qquad (i,j = 1,2,3) \tag{36}$$

expressed in terms of stereographic projection coordinates acting on momentum space functions for which p_0 is constant are given by:

$$L_{ij} = i\left(p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i}\right) \tag{37}$$

Now we will obtain quantum mechanical operator expressions for Λ_I and Λ_{II}. In order to obtain agreement with the Kepler problem we must choose $p_o=\sqrt{+2H}$(case I) and $p_o=\sqrt{-2H}$(case II), and then move p_o^2 to the right of $\frac{\partial}{\partial p_i}$ (see Bander and Itzykson, p. 3 Here H is the Schrödinger Hamiltonian for the relative motion given in Eqn. (31). We further make the substitutions $\frac{\partial}{\partial p_i} \to -iQ_i$, $p_i \to P_i$, where P_i and Q_i are the generators of the Heizenberg algebra. We then obtain for the quantum mechanical operators corresponding to the differential operators in Eqns. ($35^{\pm}$) and (37) the following:

$$L_{i0}^{+} = \frac{1}{\sqrt{-2H}}\left[\frac{1}{2}\,\varepsilon_{ijk}\{P_j,L_k\} - \frac{\alpha Q_i}{(\Sigma Q_i Q^i)^{1/2}}\right] \tag{38^+}$$

$$L_{i0}^{-} = \frac{-1}{\sqrt{+2H}}\left[\frac{1}{2}\,\varepsilon_{ijk}\{P_j,L_k\} - \frac{\alpha Q_i}{(\ Q_i Q^i)^{1/2}}\right] \tag{38^-}$$

$$(\{P_j,L_k\} = P_j L_k + L_k P_j)$$

and

$$L_i = \varepsilon_{ijk} P_j Q_k = L_{jk} \tag{39}$$

Using Eqns. (38^+), (38^-) and (39) we compute the Laplacian Δ_I and Δ_{II}:

$$\Delta_I = L^+_{i0} L^+_{i0} + L_i L_i = - \frac{\alpha^2}{2H} - 1 \tag{40a}$$

and

$$\Delta_{II} = L^-_{i0} L^-_{i0} - L_{ii} = \frac{\alpha}{2H} + 1 \tag{40b}$$

The energy eigenvalues given in Eqns. (26a) and (26b) are obtained by substituting the above expressions for Δ_I and Δ_{II} into Eqns. (33a) and (33b) and using the results on the SO(4) and SO(3,1) eigenfunctions derived above (see Eqns. (20) and (25)).

In the above analysis, we saw that it was possible to do away with the potential by dealing with motion in a fiber which is a curved space. The two-body problem in quantum mechanics can be given a fiber bundle description: let the base space be the center of mass coordinates and the fiber be the momentum space of the relative motion. The equation in the fiber is the momentum space Schrödinger equation (Eqn. (4)). By generalizing the fiber to a curved space and relating coordinates of the curved space to the momentum space coordinates of the relative motion, so that the coordinates in the fiber play the role of generalized coordinates,[5] it was possible to eliminate the potential. Instead we have for the equation of motion in the curved fiber, a free particle wave equation: Eqn. (40a) or Eqn. (40b) or the associated integral equations given by Eqn. (11a) or Eqn. (11b).

B. The Rotator Model

We now give a fiber bundle description of the rotator model of A. Bohm.[4] The fiber bundle for the description of the rotator turns out to be the same as for the scattering states of the Kepler problem. However, before we attack this problem, we will give a brief description of the rotator model which we are considering. A nonrelativistic quantum mechanical system with translational and rotational degrees of freedom has for its translation generators the momenta, P_i, and for its generators of rotations the angular momenta, J_i. They fulfill the commutation relations of the Lie algebra of the Euclidean group:

$$[P_i, P_k] = 0 \qquad [P_i, J_k] = i\varepsilon_{ik\ell} P_\ell \qquad [J_i, J_k] = i\varepsilon_{ik\ell} J_\ell \tag{41}$$

For a mass point without interaction the energy operator is

$$E = \frac{P^2}{2\mu} . \tag{42}$$

It is possible to replace the momenta by generalized momenta or "dynamical momenta" given by

$$B_i = P_i + \frac{\lambda}{2} (P_i P^i)^{-1/2} \epsilon_{ik\ell}\{P_\ell, J_k\} \qquad (\{P_\ell, J_k\} = P_\ell J_i + J_k P_\ell) \tag{43}$$

Using the commutation relations (Eqn. (41)), we can show that the B_i satisfy the commutation relations of the Lie Algebra of SO(3,1):

$$[B_i, B_j] = -\lambda^2 i \epsilon_{ijk} J_k \qquad [B_i, J_k] = i\epsilon_{ik\ell} B_\ell \qquad [J_i, J_k] = i\epsilon_{ik\ell} J_\ell . \tag{44}$$

λ is a real number. For the energy of this system we take

$$E = \frac{B^2}{2\mu} = \frac{1}{2\mu} \{P^2 + \lambda^2 (L^2 + 1)\} \tag{45}$$

which is (up to a constant additive factor) the energy operator for a particle executing translational and rotational motion. Assuming that the physical states are eigenstates of B^2, P^2 and J^2, we obtain for the energy

$$E_j = \langle \epsilon, j_3, j | \frac{B^2}{2\mu} | \epsilon, j_3, j \rangle = \frac{1}{2\mu} (\epsilon^2 + \lambda^2(1 + j(j+1)) \qquad (46)$$

Here ϵ is the eigenvalue of P^2 and j, j_3 are the eigenvalues of J^2 and J_3. Note that the second-order Casimir operator $\lambda^2 Q = B^2 - \lambda^2 J^2$ of SO(3,1) is related to the second-order Casimir operator of the Euclidean group, P^2, by:

$$P^2 = \lambda^2 Q - \lambda^2 . \qquad (47)$$

The fiber bundle for the description of the rotator is the bundle

$$E(R^3, F = SO(3,1)/SO(3), G = SO(3,1), P) \qquad (48)$$

The fiber is illustrated in Fig. (4). The ξ_0 direction is pure imaginary and the radius, R, is pure imaginary. The fiber can also be embedded in Minkowski space, M_4; there it is the two-sheeted hyperboloid shown in Fig. (2).

The way in which we combine the momentum in the base space with the momentum in the fiber in order to arrive at the formula for the "dynamical momenta" (Eqn. 43)) can be geometrically motivated as follows: consider the classical kinematics of a particle moving in the rotator fiber bundle. The classical linear momentum vector in the base

space for a free particle is: $p_i = m\dot{x}_i$ where $\dot{x}_i$ is the velocity of the particle. The classical angular momentum 6-vector in the fiber for a free particle is:
$L_{ab} = \mu_{ab}(\xi_a\dot{\xi}_b - \xi_b\dot{\xi}_a)$ with $\dot{\xi}_a$ being the velocity of the particle along the ξ_a direction in the fiber. The analogue of linear momentum in the fiber is

$$L_{i0} = \mu(\xi_0\dot{\xi}_i - \xi_i\dot{\xi}_0) \tag{49}$$

The total momentum of a particle moving in the fiber bundle is: $B_i = p_i + \frac{\lambda}{2} L_{0i}$. We take for the Hamiltonian of a free particle moving in the fiber bundle

$$H(x_i,\xi_i;\dot{x}_i,\dot{\xi}_i) = \frac{B^2}{2\mu} \quad . \tag{50}$$

According to this, p_i is not conserved since it not constant along the integral curves of H.

Now we derive the formula for the "dynamical momenta" given by Eqn. (43). We relate the coordinates in the fiber to those in the momentum space of the base space by stereographic projection:

$$\left.\begin{aligned} p_i &= \frac{2\xi_i}{1 + \xi_0/R} \\ \xi_i &= \frac{1}{1 + \frac{p^2}{4R^2}}\, p_i \\ \xi_0 &= \frac{R\left(1 - \frac{p^2}{4R^2}\right)}{\left(1 + \frac{p^2}{4R^2}\right)} \end{aligned}\right\} \qquad (51)$$

the measure on the sphere is given by:

$$d\Omega = \frac{1}{4} \frac{1}{\left(1 - \frac{p^2}{4R^2}\right)^3} d^3p \qquad (52)$$

Functions on the sphere and functions on momentum space are related by:

$$\phi(\vec{\xi}) = \frac{1}{2}\left[1 + \frac{p^2}{4R^2}\right]^2 \phi(\vec{p}) \qquad (53)$$

The generators of rotations in the i0 planes are given by Eqn. (34). Using Eqns. (41), (52), and (53) we calculate the action of these generators on functions of momentum space to be:

$$L_{i0} = \frac{-i}{2R}\left(\frac{p^2 - 4R^2}{2}\frac{\partial}{\partial p_i} - p_i p_j \frac{\partial}{\partial p_j} - 2p_i\right) \qquad (i = 1,2,3) \qquad (54)$$

Likewise for the generators of rotations in the ij planes given by Eqn. (36) we obtain

$$L_{ij} = i(p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i}) \qquad (i,j = 1,2,3) \tag{55}$$

acting on functions of the momentum space.

Next we obtain quantum mechanical operators for L_{i0} and L_{ij}. We make the substitutions $\frac{\partial}{\partial p_i} \rightarrow -iQ_i$ and $p_i \rightarrow P_i$ (P_i and Q_i are the generators of the Heisenberg algebra). Furthermore in order to obtain agreement between the expression for the L_{i0} (Eqn. (54)) and the "fiber part of the B_i given by Eqn. (43), we replace 2R by

$$iP = \sqrt{- \sum_{i=1}^{3} P_i P^i} = \sqrt{-P^2}$$

and also move it to the right of Q_i in Eqn. (54). We then obtain the following expression for the L_{i0}:

$$L_{i0} = \frac{i}{\sqrt{-p^2}} \{P^2 Q_i - P_i (P \cdot Q) - iP_i\}$$

$$= \frac{1}{\sqrt{p^2}} \varepsilon_{ijk} \{P_j, L_k\} . \tag{56}$$

which is the momentum operator in the fiber. Combining thi with the momentum operator in the base space we obtain for the total momentum operator for a particle moving in the fiber bundle.

$$B_i = P_i + \frac{\lambda}{2} L_{0i} = P_i + \frac{\lambda}{2} (P_i P^i)^{-1/2} \varepsilon_{ik\ell}\{P_\ell, J_k\} \qquad (57)$$

This is the formula for the "dynamical momenta" of the rotator model (Eqn. (43)). Similarly the quantum mechanical operators corresponding to the generators of rotations in the ij planes (Eqn. (55)) are given by:

$$L_i = L_{jk} = \varepsilon_{ijk} P_j Q_k \, . \qquad (58)$$

From the above analysis we find the interesting geometrical interpretation of the momentum P in Eqn. (43) to be that it represents -i/2 times the radius of the fiber.

Appendix: Proof of an Integral Equation

We now prove Eqn. (23a) using one of Green's identities:[15)]

$$\int_V (u\nabla^2 v - v\nabla^2 u)\,d\tau = \int_{\partial V} (u\vec{\nabla} v - v\vec{\nabla} u)\cdot\vec{n}\,d\sigma$$

where V is the volume of the region of integration and ∂V its boundary. Define the surface of integration in Green's formula as

$$S_\varepsilon = \{u : u^2 = 1, |u-v|^2 \geq \varepsilon\} \cup \{u : u^2 \leq 1, |u-v|^2 = \varepsilon\} = S \cup \gamma_\varepsilon$$

The surface is shown in Fig. 5. Since $\mathcal{Y}_{n\ell m}(u)$ and $\frac{1}{|u-v|^2}$ are harmonic in u everywhere except at u = v, the volume integral vanishes and Green's formula becomes

$$0 = \int_S \{\mathcal{Y}_{n\ell m}(u)\ \vec{\nabla}\frac{1}{|u-v|^2} - \frac{1}{|u-v|^2}\vec{\nabla}\mathcal{Y}_{n\ell m}(u)\}\vec{n}\,d\sigma$$

$$+ \int_{\gamma_\varepsilon} \{\mathcal{Y}_{n\ell m}(u)\ \vec{\nabla}\frac{1}{|u-v|^2} - \frac{1}{|u-v|^2}\vec{\nabla}\mathcal{Y}_{n\ell m}(u)\}\vec{n}\,d\sigma\,. \qquad (59)$$

(Here $\vec{u}$ is the variable point of integration and $\vec{n}$ is the outward normal on S_ε.)

Now we can deal with the singularity in the second integral

[15)] Sokolnikoff, Mathematics of Physics and Modern Engineering, p. 500, McGraw-Hill (1966).

as follows: note that on γ_ε

$$\vec{\nabla}\frac{1}{|u-v|^2}\cdot\vec{n} = -\frac{\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon^2}\right) = \frac{2}{\varepsilon^3}$$

so that

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\left\{\mathcal{Y}_{n\ell m}(u)\,\vec{\nabla}\frac{1}{|u-v|^2} - \frac{1}{|u-v|^2}\,\vec{\nabla}\mathcal{Y}_{n\ell m}(u)\right\}\vec{n}\,d\sigma$$

$$= \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\mathcal{Y}_{n\ell m}(u)\frac{2}{\varepsilon^3}\,d\sigma = 2\pi^2\mathcal{Y}_{n\ell m}(v) = 2\pi^2 Y_{n\ell m}(v). \tag{60}$$

We have used the fact that the 3-dimensional area of the hemisphere, γ_ε, is $\pi^2\varepsilon^3$ and in the last step we have used $|\vec{u}| = 1$.

If we multiply the first integral through by $|\vec{u}| = 1$, we can use Euler's theorem on homogeneous functions. We obtain

$$|\vec{u}|\;\vec{\nabla}\mathcal{Y}_{n\ell m}(u)\cdot\vec{n} = \vec{u}\;\vec{\nabla}\mathcal{Y}_{n\ell m}(u) = (n-1)\mathcal{Y}_{n\ell m}(u) = (n-1)Y_{n\ell m}(u)$$

$$\text{since } |\vec{u}| = 1 \tag{61}$$

and

$$|\vec{u}|\;\vec{\nabla}\frac{1}{|u-v|^2}\cdot\vec{n} = \vec{u}\;\vec{\nabla}\frac{1}{|u-v|^2} = -\frac{1}{|u-v|^2}. \tag{62}$$

Substituting Eqns. (60), (61) and (62) into Eqn. (59) (after having taken the limit as $\varepsilon\to 0$) yields

$$0 = 2\pi^2\,Y_{n\ell m}(v) + \int_S d\sigma\{-1-(n-1)\}\frac{Y_{n\ell m}(u)}{|u-v|^2}$$

Therefore

$$Y_{n\ell m}(v) = \frac{n}{2\pi^2}\int d\Omega_{S_3}\,\frac{Y_{n\ell m}(u)}{|u-v|^2}$$

which proves Eqn. (23a).

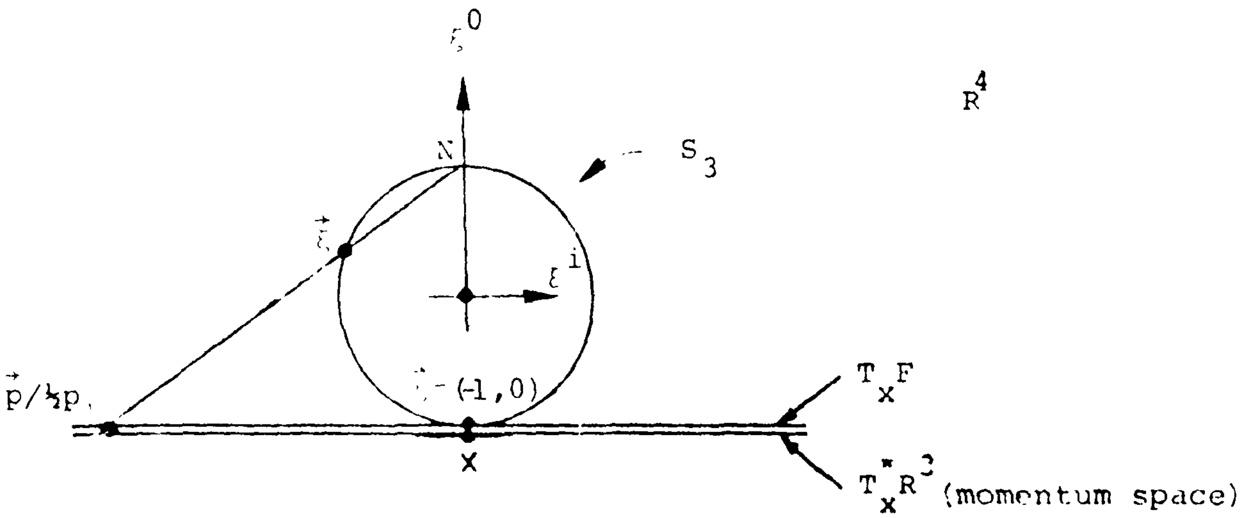

Fig. 1. The fiber bundle $E(R^3, S_3, SO(4), P)$ illustrating the soldering condition (iv') and steriographic projection of the three dimensional momentum space onto the unit sphere, $\xi^2 = 1$, minus the north pole, N, in the Euclidean space, R^4.

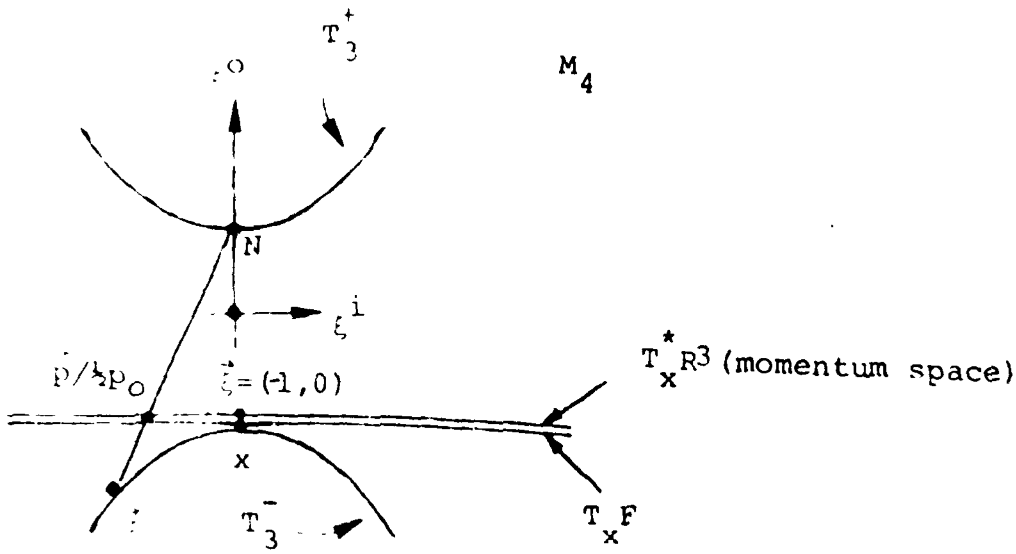

Fig. 2. The fiber bundle E(R^3, T_3^+ T_3^-, SO(3,1), P) illustrating the soldering condition (iv') and steriographic projection of the three dimensional momentum space onto the two sheeted-hyperboloid $\xi^2 = \xi_o^2 - \xi^i\xi_i = 1$ minus the point, N, in the Minkowski space, M_4.

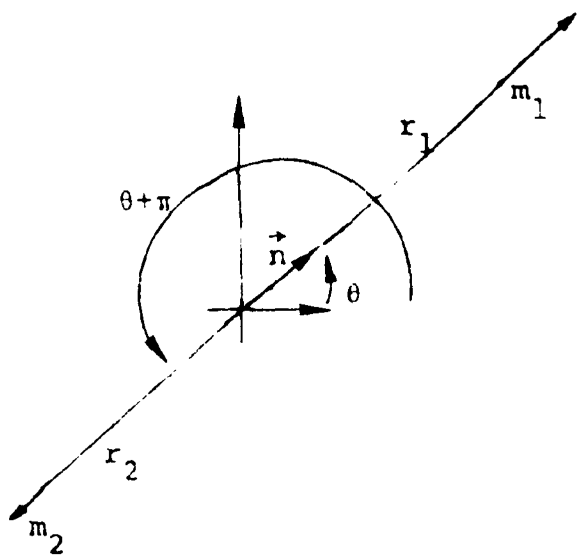

Fig. 3. The two body problem. For simplicity we have omitted one of the spatial directions. The center of mass is at the origin.

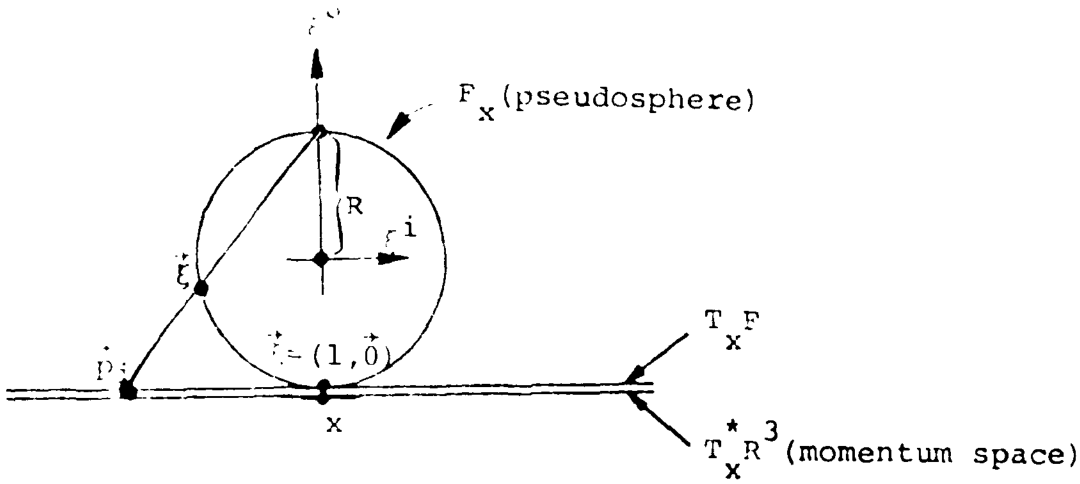

Fig. 4. The fiber bundle $E(R^3$, $F=SO(3,1)/SO(3)$, $G=SO(3,1)$, $P)$. The fiber is the sphere, $\xi^{0\,2} + \xi^{1\,2} + \xi^{2\,2} + \xi^{3\,2} = R^2$, in a flat space. The radius R and ξ^0 are pure imaginary.

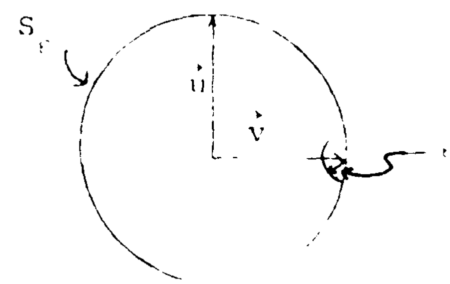

Fig. 5. The surface of integration used in Green's Formula.

Acknowledgments

The author wishes to express his sincere thanks to Dr. A. Bohm who suggested this problem and provided much needed encouragement during the time spent working on this problem. He would also like to acknowledge helpful discussions with Mr. H. AlHendi and Dr. B. Bouqeois. Finally he would like to thank the mathematics department and the Center for Particle Theory at the University of Texas at Austin for financial help during this period.