27
4/6/80
2 Fed NTIS

UCID-18579

Adiabatic Compression and Radiative Compression of Magnetic Fields

MASTER

C. H. Woods

February 12, 1980

Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under Contract W-7405-Eng-48.

Adiabatic Compression and Radiative Compression of Magnetic Fields

C. H. Woods

Lawrence Livermore Laboratory, University of California, Livermore, California 94550

Abstract

Flux is conserved during mechanical compression of magnetic fields for both nonrelativistic and relativistic compressors. However, the relativistic compressor generates radiation, which can carry up to twice the energy content of the magnetic field compressed adiabatically. The radiation may be either confined or allowed to escape.

1. Adiabatic Compression

The term adiabatic means slow compared to associated processes, and the sense in which we use the term here is that the compressor is slow compared to the speed of light. If any part of the compressing surface moves at a speed comparable with c, the magnetic field being acted upon not only appears increased in magnitude, but takes on the character of radiation, which must be reflected by the surface. In this case, the energy of the original magnetic field and the work done by the compressor are partially converted into radiation, and we call the process radiative compression.

It is useful to compare the two types of compression in the simple, two-dimensional model of a compressor shown in Fig. 1. The initial magnetic field B_0 points along the y-axis (out of the paper), and is confined by an internal sheet current K_0 flowing perpendicular to the y-axis. The region of confinement extends to $\pm\infty$ in the y-direction, and all derivatives $\frac{\partial}{\partial y}$ are zero. The walls, including the piston and its contact, are perfectly conducting.

During adiabatic compression, the magnetic flux is conserved, which implies that moving the piston slowly to position z will yield the magnetic field

$$B = \frac{L}{L-z} B_0 \quad . \tag{1}$$

The mechanical work required to move the piston through the "stroke" S, as depicted in Fig. 2, is

$$W_{adiabatic} = \frac{A}{8\pi}\int_0^S \frac{L^2}{(L-z)^2}\, B_0^2 dz$$

$$= \left(\frac{AL}{8\pi}B_0^2\right)\left(\frac{S}{L-S}\right) \quad , \qquad (2)$$

where A is the area of the piston and L is the original length of the cavity. The ratio of the final stored magnetic energy to the initial energy is equal to the compression ratio L/(L-S).

2. Radiative Compression

Radiative compression has its origin in the boundary conditions which must be imposed at the piston. In the case of a stationary or slowly moving, perfectly conducting surface, the usual boundary conditions are that the tangential component of $\vec{E}$ and the normal component of $\vec{B}$ must vanish. In the case of a rapidly moving piston, moving at a speed βc not necessarily small compared to the speed of light, the usual boundary conditions apply, but they must be applied in an inertial reference frame k' which moves along with the piston. Thus one must deal with the tangential and normal components of $\vec{E}'$ and $\vec{B}'$, respectively, the fields measured by observers in k'.

The laboratory reference frame k already shown in Fig. 1 is stationary, and the piston moves from the origin to the right along the z axis. Let the frame k' coincide with the frame k at time $t = t' = 0$. The point $z' = 0$ is then always at the surface of the piston, and the boundary conditions there become

$$E_x'(t', x', y', 0) = 0 \qquad (3a)$$

$$E'_y(t', x', y', 0) = 0 \tag{3b}$$

$$B'_z(t', x', y', 0) = 0 \quad . \tag{3c}$$

The primed values needed in Eqs. (3) are related to the unprimed field intensities by the well-known transformation[1]

$$E'_x = \gamma(E_x - \beta B_y) \tag{4a}$$

$$E'_y = \gamma(E_y + \beta B_x) \tag{4b}$$

$$E'_z = E_z \tag{4c}$$

$$B'_x = \gamma(B_x + \beta E_y) \tag{4d}$$

$$B'_y = \gamma(B_y - \beta E_x) \tag{4e}$$

$$B'_z = B_z \tag{4f}$$

in which $\gamma = 1/\sqrt{1-\beta^2}$. The components E_y, B_x, E_z and B_z are zero everywhere in the present problem, so that the boundary conditions (3b) and (3c) are automatically satisfied, and only Eqs. (4a) and (4e) are required. Now, Eq. (4a) suggests that the preexisting magnetic field B_0, which points along the y-axis, will lead to a value for E'_x equal to $-\beta\gamma B_0$. This is, in fact, the electric intensity which some observers in k' will see. But not all of them, at least not for all times t'. Near the surface of the piston, for example, E'_x must vanish. How can this be brought about from Eq. (4a), when the initial field E_x is everywhere zero? The answer is that E_x does not continue to be zero everywhere, but only to the right of the plane $z = ct$, and that we must construct a wavefield, propagating along the z-axis, which allows the boundary conditions (3) to be satisfied.

The physical origin of the wavefield which we need to satisfy the boundary condition (3a) can best be understood by thinking of the piston as a reflector. Observers in k' to the right of the plane z = ct interpret the preexisting field quite differently than stationary observers. For them we have

$$\left(E'_x\right)_0 = -\beta\gamma B_0 \tag{5a}$$

and

$$\left(B'_y\right)_0 = \gamma B_0 \quad . \tag{5e}$$

Their observations carry the subscript zero, because they arise from the original field B_0. For these observers, the Poynting vector is

$$\begin{aligned}(\vec{S}')_0 &= \frac{c}{4\pi} E' \times H' \\ &= \frac{c}{4\pi} \left(E'_x\right)_0 \left(B'_y\right)_0 \hat{e}_z \\ &= -\frac{c}{4\pi}\beta\gamma^2 B_0^2 \hat{e}_z \quad .\end{aligned} \tag{6}$$

Thus, energy is being carried backward relative to k'. The Poynting vector can be decomposed into two parts according to

$$\begin{aligned}(\vec{S}')_0 &= -\frac{c}{4\pi} \beta^2\gamma^2 B_0^2 \\ &\quad -\frac{c}{4\pi} \beta(1-\beta)\gamma^2 B_0^2 \quad .\end{aligned} \tag{7}$$

This decomposition corresponds to the only possible interpretation which the observers in k' can give to the original magnetic field B_0: For them, there exists a wave-like structure propagating in the negative-z' direction with electric field $-\beta\gamma B_0$ along the x'-axis and magnetic field $\beta\gamma B_0$ along the y'-axis, and also a remnant magnetic field $(1 - \beta)\gamma B_0$ (which vanishes in the limit $\beta \to 1$) pointing along the y' axis. These observers also see sheet current in the walls increased by the factor γ,

$$(K')_0 = \frac{1}{4\pi}\gamma B_0 \qquad , \tag{8}$$

as well as induced surface charge

$$\begin{aligned} (\sigma')_0 &= \frac{1}{4\pi}(E'_x) \\ &= -\frac{1}{4\pi}\beta\gamma B_0 \end{aligned} \tag{9}$$

at the lower wall, with an equal positive surface charge at the upper wall. The presence of surface charge may seem paradoxical in view of the fact that charge is conserved under Lorentz transformations. We need only recall that the charge density ρ forms the time component of the four-vector current density $J^\alpha = (c\rho, \vec{J})$, and therefore does generally change in going from one frame to another. The surface charge and sheet current may be regarded mathematically as δ-functions of x', and they transform exactly like $c\rho$ and $\vec{J}$.

The piston, being in frame k', must press against the remnant magnetic field, and, as a reflector, it must reflect the wave-like structure coming toward it. The reflected wave exhibits an electric field in the opposite direction, with intensity $+\beta\gamma B_0$, so that the boundary condition (3a) is now satisfied, and we may write for all observers in k' in the interval $0 < z' < ct'$, including those at the surface of the piston,

$$(E'_x)_1 = 0 \quad , \tag{10}$$

where the subscript 1 refers to conditions created after the first reflection. On the other hand the reflected wave exhibits a magnetic field which is in the same direction, with intensity $\beta\gamma B_0$ along the y'-axis. The magnetic field at the piston after reflection is therefore

$$\begin{aligned}(B'_y)_1 &= (1-\beta)\gamma B_0 + 2\beta\gamma B_0 \\ &= (1+\beta)\gamma B_0 \quad ,\end{aligned} \tag{11}$$

where we have included the remnant magnetic field, and doubled the magnetic field associated with the wave.

The reflected wave moves forward along z' at the velocity of light. Since we are neglecting the problem of accelerating the piston, the wave is presumed to have a sharp leading edge and a flat profile, corresponding to constant speed of the piston. Over the interval $0 < z' < ct'$ the sheet current is now

$$(K')_1 = \frac{1}{4\pi}(1+\beta)\gamma B_0 \tag{12}$$

and the surface charge vanishes,

$$(\sigma')_1 = 0 \quad . \tag{13}$$

When the forward-going wave reaches the end $z = L$ of the cavity it will, of course, reflect again, and we could carry the analysis further, using even subscripts to represent conditions after reflection from the fixed wall, and odd subscripts after reflection from the piston. We shall not do this here. However, with what we have already calculated we can make some observations about radiative compression. In the laboratory frame k, in the space interval $\beta ct < z < ct$, the corresponding intensities are readily found from the transformation Eqs. (4a and 4e). From Eq. (4a), we have

$$(E_x)_1 - \beta(B_y)_1 = 0 \quad , \tag{14}$$

while from Eq. (4e) we have

$$(B'_y)_1 = (1+\beta)\gamma B_0 = \gamma[(B_y)_1 - \beta(E_x)_1] \quad . \tag{15}$$

We find, by solving these last equations,

$$(E_x)_1 = \frac{\beta}{1-\beta}B_0 \tag{16}$$

and

$$(B_y)_1 = \frac{1}{1-\beta}B_0 \quad . \tag{17}$$

Let us now calculate the work done by the piston under radiative compression. We know that in the frame k' it presses against the magnetic field $(1+\beta)\gamma B_0^2$, and the force required is therefore

$$(f'_z)_1 = \frac{ALB_0^2}{8\pi}(1+\beta)^2\gamma^2 \quad . \tag{18}$$

However, in the frame k' the piston appears to do no work, because it is not in motion relative to k', and we must consider the force as observed in the frame k.

The transformation of a force vector $\vec{f}$ from one Lorentz frame to another is slightly involved, because the correct definition of the contravariant four-vector force involves not only the components of $\vec{f}$ but the state of motion of the object against which the force is exerted. If that state of motion is characterized in frame k by the velocity vector $\vec{u}$, and in frame k' by the velocity vector $\vec{u}'$, neither of which have, in general, any relation to the velocity of k' relative to k, the four-vector force F^α may be expressed in frame k as[2]

$$F^\alpha = \left(\frac{\vec{f} \cdot \vec{u}}{c^2\sqrt{1-\left(\frac{u}{c}\right)^2}} , \frac{\vec{f}}{c\sqrt{1-\left(\frac{u}{c}\right)^2}} \right) , \qquad (19)$$

and in frame k' as

$$F'^\alpha = \left(\frac{\vec{f}' \cdot \vec{u}'}{c^2\sqrt{1-\left(\frac{u'}{c}\right)^2}} , \frac{\vec{f}'}{c\sqrt{1-\left(\frac{u'}{c}\right)^2}} \right) , \qquad (20)$$

where $u = |\vec{u}|$, and $u' = |\vec{u}'|$. The components of F^α in k are related to the components in k' by the transformation equations[3]

$$F_t = \gamma(F'_t + \beta F'_z) \qquad (21a)$$

$$F_x = F'_x \qquad (21b)$$

$$F_y = F'_y \qquad (21c)$$

$$F_z = \gamma(F'_z + \beta F'_t) , \qquad (21d)$$

where $F^0 \equiv F_t$, $F^1 \equiv F_x$, $F^2 \equiv F_y$, and $F^3 \equiv F_z$. Now, in the particular case at hand, the velocity of the piston with respect to k' is, by the definition of k', zero, so that $\vec{u}' = 0$. Therefore $F_t' = 0$, and the four-vector force may be written simply as $\left(0, \frac{1}{c} \vec{f}'\right)$. From Eq. (21d) we find

$$F_z \equiv \frac{f_z}{c\sqrt{1-\left(\frac{u}{c}\right)^2}} = \gamma\left(\frac{1}{c} f_z' + \beta \cdot 0\right) \qquad . \tag{22}$$

In view of the special circumstance that $u = \beta c$, Eq. (22) reduces to

$$f_z = f_z' \qquad . \tag{23}$$

This result may seem either trivial or incorrect, but the implication of Eq. (23) is far from trivial: The force f_z required to move the piston is

$$(f_z)^1 = \frac{AB_0^2}{8\pi} (1+\beta)^2 \gamma^2 \qquad , \tag{24}$$

and the mechanical work done by the piston is found by multiplying the distance which it moves in the frame k by the magnetic force which it experiences in k', namely, $\frac{AB_0^2}{8\pi}(1+\beta)^2\gamma^2$. Without Eq. (23) one might be tempted to equate f_z to the quantity $\frac{AB_0^2}{8\pi} \left(\frac{1}{1-\beta}\right)^2 B_0^2$, related to the magnetic field as observed in k, and given by Eq. (17). But that would be incorrect, and a detailed balance of energy which we shall later consider would not be possible.

In calculating the work done by the piston, we must consider carefully whether or not the forward-going wave has had time to reflect from the fixed end of the cavity and again reach the piston. It should be noted that many reflections during the compression phase are equivalent to adiabatic compression, each reflection increasing the magnetic field, and alternate reflections creating and destroying the electric field, so that the terminal result is energy stored only in the magnetic field.

We wish to consider purely radiative compression involving the time interval $0 < t < \frac{L}{c}$, i.e., before the forward-going wave reaches the fixed end of the cavity. The piston moves during this time to the position βL. For clear comparison with adiabatic compression, it is useful to assume the stroke S to be of such length that $S = \beta L$, in which case the work done by the piston is

$$W_{radiative} = \frac{ALB_0^2}{8\pi} \frac{\frac{S}{L}\left(1+\frac{S}{L}\right)}{1-\frac{S}{L}} \quad . \qquad (25)$$

The total energy now compressed into the space interval $S < z < L$ is equal to the sum

$$\frac{ALB_r^2}{8\pi} + W_{radiative} = \frac{ALB_0^2}{8\pi} \frac{1+\left(\frac{S}{L}\right)^2}{1-\left(\frac{S}{L}\right)} \qquad (26)$$

of the original energy contained in the magnetic field B_0 and the mechanical work. In the limit $\frac{S}{L} \to 1$, the accumulated energy represented by Eq. (26) approaches twice the amount

$$\frac{ALB_0^2}{8\pi} + W_{adiabatic} = \frac{ALB_0^2}{8\pi} \frac{1}{1-\frac{S}{L}} \qquad (27)$$

available under adiabatic compression.

The energy accumulated in the electromagnetic field at the time the piston reaches the point z = S can also be deduced from Eqs. (16) and (17):

$$\frac{A(L-S)}{8\pi}\left[\left(E_x\right)_1^2 + \left(B_y\right)_1^2\right] = \frac{ALB_0^2}{8\pi} \frac{1+\left(\frac{S}{L}\right)^2}{1-\left(\frac{S}{L}\right)} \quad , \qquad (28)$$

which is consistent with the expression (26) obtained by considering the work done by the piston.

The energy stored in the form of radiation is

$$\frac{A(L-S)}{8\pi}\left[2\left(E_x\right)_1^2\right] = \frac{ALB_0^2}{8\pi} \frac{2\left(\frac{S}{L}\right)^2}{1-\frac{S}{L}} \quad . \qquad (29)$$

Comparison of Eqs. (26) and (29) shows that, in the limit $\frac{S}{L} \to 1$, all of the stored energy exists in the form of radiation. If the current distribution at the fixed end of the cavity is rearranged in such a way that the cavity can be left open, the radiation will be emitted as a pulse.

Although the generation of radiation by the mechanism described above may not have immediate practical applications because of the difficulty of creating very fast reflectors, there may be astrophysical contexts in which fast moving plasma interacts with the cosmic magnetic field to produce significant radiation.

ACKNOWLEDGMENTS

The possibility that a relativistic piston could transform the energy of a magnetic field into radiation was pointed out to the author by J. S. Pettibone of this laboratory.

REFERENCES

1. J. D. Jackson, Classical Electrodynamics (John Wiley & Sons, Inc., New York. N.Y., 1975), p. 552.
2. L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1970), p. 28.
3. Ibid., p. 14.

/vg

FIGURE CAPTIONS

1. A conducting "piston" is at position $z = 0$, and the initial magnetic field B_0 points out of the plane of the paper. A sheet of current K_0 flows around the region of confinement perpendicular to the y-axis, which is normal to the plane of the paper.

2. Adiabatic compression. Because of flux conservation the magnetic field B_y when the piston is at position $z = S$ is $B_y = \frac{L}{L-S} B_0$.

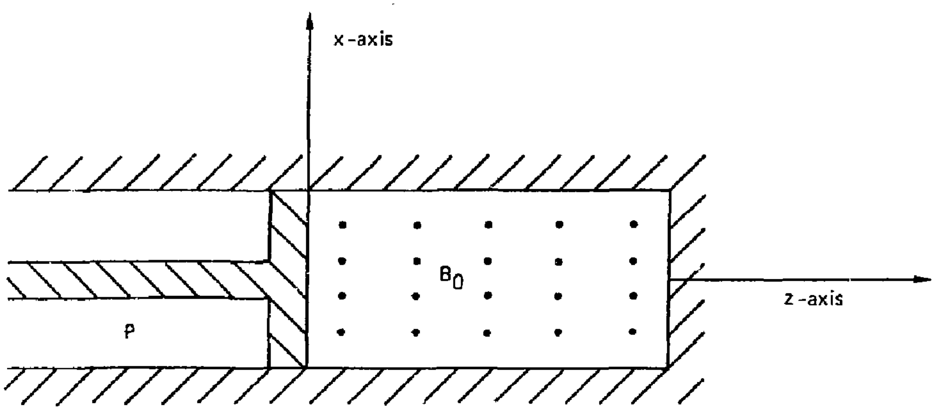

Figure 1

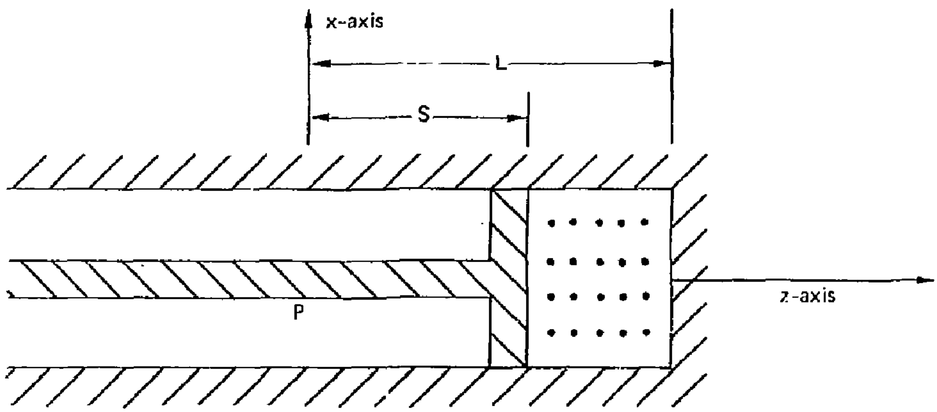

Figure 2