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A separable approach to the Bethe-Salpeter equation and its application to nucleon-nucleon scattering

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ABSTRACT

The Bethe-Salpeter equation is solved in closed form with the help of a four dimensional separable "potential". For possible applications to three-nucleon investigations we have fitted all nucleon-nucleon S-wave phase shifts in a sufficient way by this method; in addition we also present an example for a P-wave.

I. INTRODUCTION

In describing the interaction of two hadrons even at low energies relativistic effects should be taken into account[1]. To treat this problem in a reliable way we use the Bethe-Salpeter (BS) equation[2], the relativistic analogue to the Lippmann-Schwinger equation[3]. Due to the complexity of the BS equation, even in the ladder approximation, complicated numerical calculations are necessary[4]. Therefore in many investigations[5] the two-body Green's function is replaced by a suitably chosen function which reduces the four-dimensional BS equation to a three dimensional integral equation.

It is the goal of this paper to present a separable approach to the BS equation without changing the Green's function. The resulting amplitudes are among other advantages, suitable to describe the two-particle subsystem-interactions of relativistic three-body calculations[6] within the BS formalism.

In section II we derive the BS equation in the presence of separable interactions, in section III the analytic calculations are presented. Section IV shows the numerical fits to the nucleon-nucleon scattering phase shifts and section V summarizes our approach.

II THE BS EQUATION WITH SEPARABLE INTERACTIONS

The BS equation in momentum space can be written in the form:

$$T(\hat{q},\hat{q}';s) = V(\hat{q},\hat{q}') + \frac{i}{4\pi^3}\int d\hat{k}\, V(\hat{q},\hat{k})\, G(\hat{k},s)\, T(\hat{k},\hat{q}';s) \tag{1}$$

with

$$G(\hat{k},s) = [(a\hat{P}+\hat{k})^2 - m_1^2]^{-1}\,[(b\hat{P}-\hat{k})^2 - m_2^2]^{-1} \tag{2}$$

the free two-particle Green's function. Our notation for the four-dimensional relative momenta is the following:

$\hat{q} = (q_0,\vec{q}) = \frac{m_1\hat{q}_2 - m_2\hat{q}_1}{m_1+m_2}$, where $\hat{q}$ is the initial relative momentum and m_1, m_2 are the masses of the two particles with momenta $\hat{q}_1$ and $\hat{q}_2$ respectively; $\hat{k}$ and $\hat{q}'$ are the intermediate and final relative momenta defined analogously; $\hat{P} = \hat{k}_1 + \hat{k}_2 = (\sqrt{s},\vec{0})$ is the total c.m. momentum with $s = (\sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2})^2$ and $p = |\vec{p}|$ the so called mass-shell value. Further we have introduced $a = \frac{m_1}{m_1+m_2}$ and $b = \frac{m_2}{m_1+m_2}$ for abbreviation.

We use the partial wave expansion from Vasavada[7]:

$$T(\hat{q},\hat{q}';s) = \sum_l \frac{2l+1}{4\pi}\, p^{2l}\, P_l[\vec{e}_q\cdot\vec{e}_{q'}]\; T_l(q_0,q,q_0',q';s) \tag{3}$$

$$V(\hat{q},\hat{q}') = \sum_l \frac{2l+1}{4\pi}\, p^{2l}\, P_l[\vec{e}_q\cdot\vec{e}_{q'}]\; V_l(q_0,q,q_0',q') \tag{4}$$

to reduce the four-dimensional BS equation to a two-dimensional integral equation:

$$T_l(q_0,q,q_0',q';s) = V_l(q_0,q,q_0',q') + \tag{5}$$

$$+ \frac{i}{4\pi^3}\, p^{2l}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2\, dk\, V_l(q_0,q,k_0,k)\, G(k_0,k,s)\, T_l(k_0,k,q_0',q';s).$$

The amplitude $T_\ell(0,p,0,p;s) \equiv T_\ell(p)$ with all four legs on the mass and energy shell is connected with the phase shift by

$$T_\ell(p) = g_\ell(p)\, e^{i\delta_\ell(p)} \sin\delta_\ell(p) \qquad (6)$$

where

$$g_\ell(p) = -\frac{8\pi\sqrt{s}}{p^{2\ell}} \qquad (7)$$

is the inverse of the real part of the integral

$$I_\ell(p) = \frac{i}{4\pi^3}\, p^{2\ell} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2\, dk\; G(k_0,k,s). \qquad (8)$$

Note that the choice of the partial wave expansion of Vasavada has already incorporated the correct threshold behaviour of the amplitude for all partial waves.

In the nonrelativistic three-body theory separable potentials are used to describe the two-particle subsystem-interaction to keep numerical calculations as simple as possible. Relativistic three-body calculations, c.f. the theory of Amado et al.[8], use separable potentials for their two-body input, fitted with the Blankenbecler-Sugar equation[9].

We present now a separable approximation to the BS-equation, suitable for the relativistic four-dimensional three-body theory of Freedman, Lovelace, and Namyslowski[6].

With the ansatz

$$V_\ell(q_0,q,k_0,k) = v_\ell(q_0,q)\,\lambda\, v_\ell(k_0,k) \qquad (9)$$

the BS equation (5) can be solved in closed form:

$$T_\ell(q_0,q,q_0',q';s) = v_\ell(q_0,q)\, v_\ell(q_0',q') \,/\, D_\ell(p) \qquad (10)$$

with

$$D_\ell(p) = \lambda^{-1} - \frac{i}{4\pi^3}\, p^{2\ell} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 dk\, v_\ell^2(k_0,k)\, G(k_0,k;s). \qquad (11)$$

$v_\ell(q_0,q)$ is the covariant form factor with one free "range parameter" β (see (12)), and λ is the coupling parameter of a "separable potential".

To fit the free parameters of V_ℓ resp. T_ℓ in each partial wave to the scattering data we have to evaluate equ.(11).

III. SOLUTION OF THE BS EQUATION

The correct threshold behaviour of our amplitude, in view of Vasavada's partial wave expansion, allows the choice of the same analytic expression for v_ℓ for all partial waves:

$$v_\ell(k_0,k) = \frac{1}{(k_0^2 - k^2 - \beta^2)^{\ell}} \qquad (12)$$

For the integration over k_0 in equ.(11), we have to investigate the singularities of

$$J_\ell(k,s) = \int_{-\infty}^{\infty} dk_0\, v_\ell^2(k_0,k)\, G(k_0,k,s) \qquad (13)$$

in the complex k_0 plane. There are four poles coming from the Green's function

$k_{01} = -a\sqrt{s} + E_1 - i\varepsilon$, $k_{02} = -a\sqrt{s} - E_1 + i\varepsilon$, $k_{03} = b\sqrt{s} + E_2 - i\varepsilon$, $k_{04} = b\sqrt{s} - E_2 + i\varepsilon$,

and two poles originating from the formfactor

$k_{05} = e - i\varepsilon$, $k_{06} = -e + i\varepsilon$, where $E_i = \sqrt{k^2 + m_i^2}$ and $e = \sqrt{k^2 + \beta^2}$. Splitting the square of the formfactor according to

$$\frac{1}{k_0^2 - e^2 + i\varepsilon} = \frac{1}{2e}\left\{\frac{1}{k_0 - e + i\varepsilon} - \frac{1}{k_0 + e - i\varepsilon}\right\}, \tag{14}$$

$J_\ell(k,s)$ can be written in the form

$$J_\ell(k,s) = \frac{1}{2e}\int_{-\infty}^{\infty} dk_0 \frac{1}{[k_0 + a\sqrt{s} - E_1 + i\varepsilon][k_0 + a\sqrt{s} + E_1 - i\varepsilon][k_0 - b\sqrt{s} - E_2 + i\varepsilon][k_0 - b\sqrt{s} + E_2 - i\varepsilon]}$$

$$\times\left\{\frac{1}{k_0 - e + i\varepsilon} - \frac{1}{k_0 + e - i\varepsilon}\right\} \equiv GA_+ + GA_- \tag{15}$$

The singularities of GA_+ and GA_- in the complex k_0 plane are displayed in figure 1. If we close the contour in the lower half plane for GA_+ and in the upper half plane for GA_-, we avoid the poles from the "separable potential". Therefore we can calculate the integrals as sums of residues:

$$J_\ell(k,s) = \frac{1}{2e}\left[-2\pi i \sum R_- + 2\pi i \sum R_+\right], \tag{16}$$

where R_+ and R_- are the residues of the poles of the Green's function in the upper and lower half plane, respectively. As a result we obtain:

$$J_\ell(k,s) = \frac{\pi i\,\Omega_1}{2eE_1[s-(E_1+E_2)^2+i\varepsilon][s-(E_1-E_2)^2][(E_1+e)^2 - a^2 s]} +$$

$$+\frac{\pi i\,\Omega_2}{2eE_2[s-(E_1+E_2)^2+i\varepsilon][s-(E_1-E_2)^2][(E_2+e)^2 - b^2 s]} \tag{17}$$

with

$$\Omega_1 = 2(E_1^2 - E_2^2 + s)(E_1 + e) + 4asE_1 \tag{18}$$

and

$\Omega_2 = 2(E_2^2 - E_1^2 + s)(E_2 + e) + 4 b s E_2 .$

With equ.(17) we have performed the k_0 integration in equ.(11).

For the k-integration we use the relation

$$\frac{1}{s-(E_1+E_2)^2+i\varepsilon} = \frac{P}{s-(E_1+E_2)^2} - i\pi\,\delta\left(s-(E_1+E_2)^2\right) \qquad (20)$$

to get

$$\mathrm{Im}\, D_\ell(p) = \frac{p^{2\ell+1}}{16\pi\,\bar{e}\sqrt{s}} \left\{ \frac{1}{b\sqrt{s}-(\bar{E}_2+\bar{e})} + \frac{1}{a\sqrt{s}-(\bar{E}_1+\bar{e})} \right\} \qquad (21)$$

and

$$\mathrm{Re}\, D_\ell(p) = \lambda^{-1} + \frac{p^{2\ell}}{8\pi} P\!\!\int_0^\infty k^2 dk \frac{\Omega_1 E_2 [b^2 s-(E_2+e)^2] + \Omega_2 E_1 [a^2 s-(E_1+e)^2]}{e E_1 E_2 [s-(E_1+E_2)^2][s-(E_1-E_2)^2][(E_1+e)^2-a^2 s][b^2 s-(E_2+e)^2]}$$

where $\bar{E}_i = \sqrt{p^2+m_i^2}$ and $\bar{e} = \sqrt{p^2+\beta^2}$. (22)

In the equal mass case our formulas (21,22) simplify to

$$\mathrm{Im}\, D_\ell(p) = - \frac{p^{2\ell+1}}{8\pi\,\bar{e}^2\sqrt{s}} \qquad (23)$$

and

$$\mathrm{Re}\, D_\ell(p) = \lambda^{-1} - \frac{p^{2\ell}}{2\pi^2} P\!\!\int_0^\infty k^2 dk \frac{2E+e}{eE[s-4E^2][\frac{s}{4}-(E+e)^2]} . \qquad (24)$$

IV. APPLICATION TO NUCLEON-NUCLEON SCATTERING

To fit the parameters λ and β in each partial wave to the scattering data we distinguish three cases:

i) no sign change in the phase shift and no pole in the amplitude

ii) one sign change in the phase shift and no pole in the amplitude

iii) one sign change in the phase shift and one pole in the amplitude.

These differences will be handled similarly to the three-dimensional case[9] by using different forms of λ.
In case i) λ is chosen to be a constant: This is sufficient for most of the higher partial waves. To show the accurracy of our approach we give as an example the $^4P_4$ phase and compare our theoretical phase shift with the result of the phase shift analysis of Arndt, Hackmann, and Roper[10]. The parameters of our fit are listed in table 1, the corresponding phase shift is plotted in figure 2. Although we have used such a simple separable model with only two free parameters, agreement of our results with the experimental data is sufficient.

Due to the importance of S-waves in three-body calculations we want to discuss these partial waves in more detail. In all cases the fits were done again to the phase shift analysis of Arndt, Hackmann, and Roper[10]; and to the corresponding scattering lengths; for the $^4S_0$ partial wave we fitted to their values for n-p and p-p phase shifts.

To reproduce the sign change in the phase shifts (case ii)) at the position of the experimental value p_0 we use an energy dependent expression $\lambda(s_0-s)$, where s_0 is the energy value corresponding to p_0. Table 2 contains, in addition to the parameters λ , p and s_0 , the comparison of the scattering lengths of our calculations of the $^1S_0$ waves with the experimental values. The phase shifts are plotted in figures 3 and 4.

Case iii) of our investigations occurs in the n-p $^3S_1$ partial wave, where in addition to the zero of the T-matrix, related to the zero of the phase shift, our T-matrix must have a pole at the bound state energy M_D^2 with M_D = 9.5051 fm^{-1} (the deuteron mass). To handle this problem we set $\lambda(s) = \frac{s_0-s}{s-m_0^2}$ in equ.(11), with m_0 as an additional free parameter and get:

$$D_\ell(p) = \lambda^{-1}\frac{s-m_0^2}{s_0-s} - \frac{i}{4\pi^3}\, p^{2\ell}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 dk\; v_\ell^2(k_0,k)\, G(k_0,k;s). \qquad (25)$$

If we choose m_0 in such a way that $D_\ell(M_D) = 0$:

$$m_0^2 = M_D^2 - \lambda(s_0-M_D^2)\frac{i}{4\pi^3}\, p^{2\ell}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 dk\; v_\ell^2(k_0,k)\, G(k_0,k,M_D^2) \qquad (26)$$

and subtract equ.(26) from (25), we obtain an analytic expression for $T_\ell(p)$ which guarantees the pole at the bound state energy:

$$D_\ell(p) = \lambda^{-1}\frac{s-M_D^2}{s_0-s} - \frac{i}{4\pi^3}\, p^{2\ell}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 dk\; v_\ell^2(k_0,k) \times$$

$$\times \left[G(k_0,k,s) - \frac{s_0-M_D^2}{s_0-s}\, G(k_0,k,M_D^2)\right]. \qquad (27)$$

The parameters λ, β, s_0, together with the scattering lengths for the n-p $^3S_1$ partial wave, are given in table 2; the phase shift is plotted in figure 5.

V. SUMMARY

With a simple separable ansatz for the four-dimensional "potential" we have solved the BS equation in closed form and have applied this formalism to the most important partial waves for 3-body calculations. Our results for the phase shifts and scattering lengths as compared with the experimental nucleon-nucleon data are sufficiently good.

This separable approach is successfully applicable also to other scattering systems, c.f. to the pion-nucleon scattering[11]. As a consequence of our approach relativistic three-body calculations for p-d and π-d scattering within the Bethe-Salpeter approach should be possible.

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TABLE CAPTIONS

Table 1. Parameters of our fit to the $^1P_1$ n-p scattering data.

Table 2. Parameters of our fits to the $^1S_0$ n-p, $^1S_0$ p-p, and $^3S_1$ n-p scattering data.

TABLE 1.

partial wave	λ [1]	β [fm^{-1}]
$^1P_1$ n-p	-376.6440	2.3822

TABLE 2.

partial waves	λ	β [fm^{-1}]	s_o [fm^{-2}]	$a_{theor.}$ [fm]	$a_{exp.}$ [fm]
^1S_o n-p	292.8430 [1]	7.9786	104.1040	-23.794	-23.71
^1S_o p-p	139.7458 [1]	4.4502	102.7261	- 7.673	- 7.823
$^3S_1$ n-p	-21815.61 [fm^{-2}]	10.7602	105.598	4.42	5.42

FIGURE CAPTIONS

Fig. 1(a). Singularities of GA_+ in the complex k_0 plane.

(b). Singularities of GA_- in the complex k_0 plane.

Fig. 2. Phase shift for the $^1P_1$ n-p partial wave. The circles denote the result of the phase shift analysis of ref. 10.

Fig. 3. Phase shift for the $^1S_0$ n-p partial wave. The circles denote the result of the phase shift analysis of ref. 10.

Fig. 4. Phase shift for the $^1S_0$ p-p partial wave. The circles denote the result of the phase shift analysis of ref. 10.

Fig. 5. Phase shift for the $^3S_1$ n-p partial wave. The circles denote the result of the phase shift analysis of ref. 10.

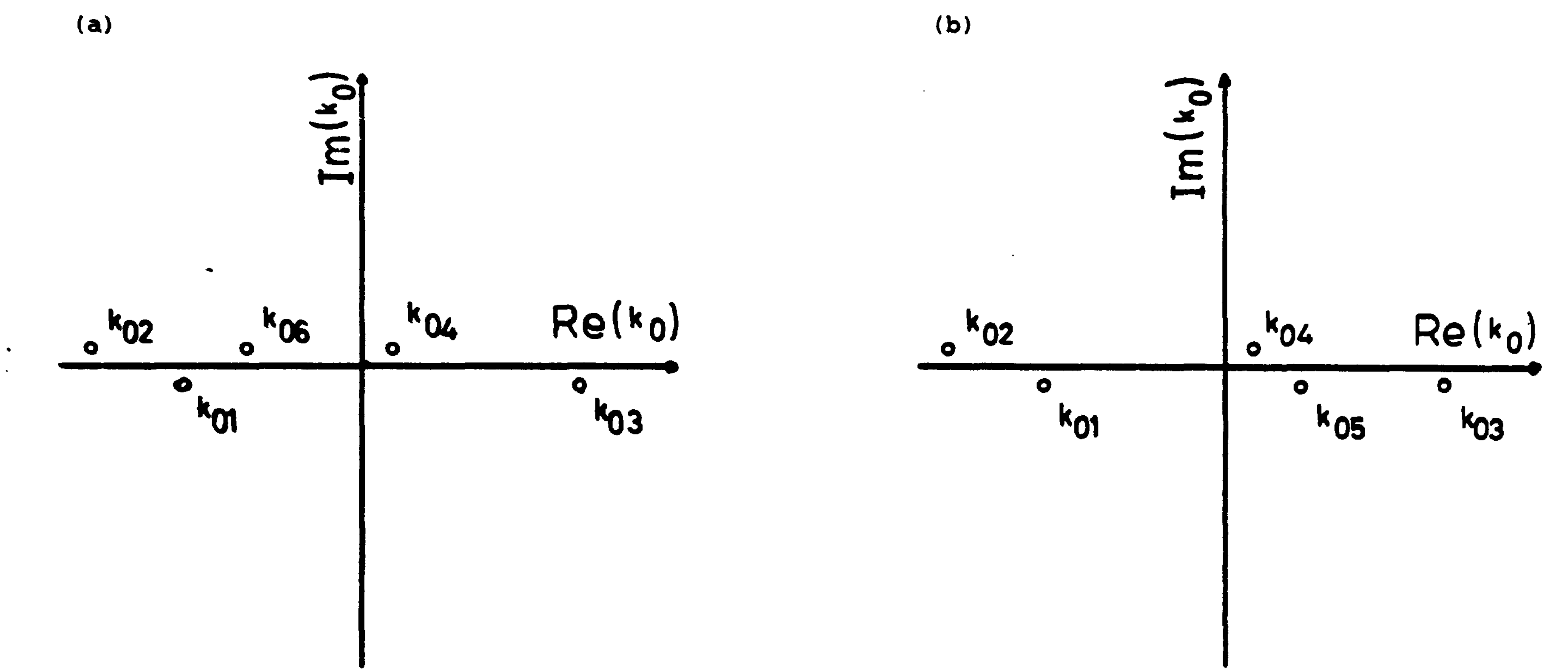


FIGURE 1

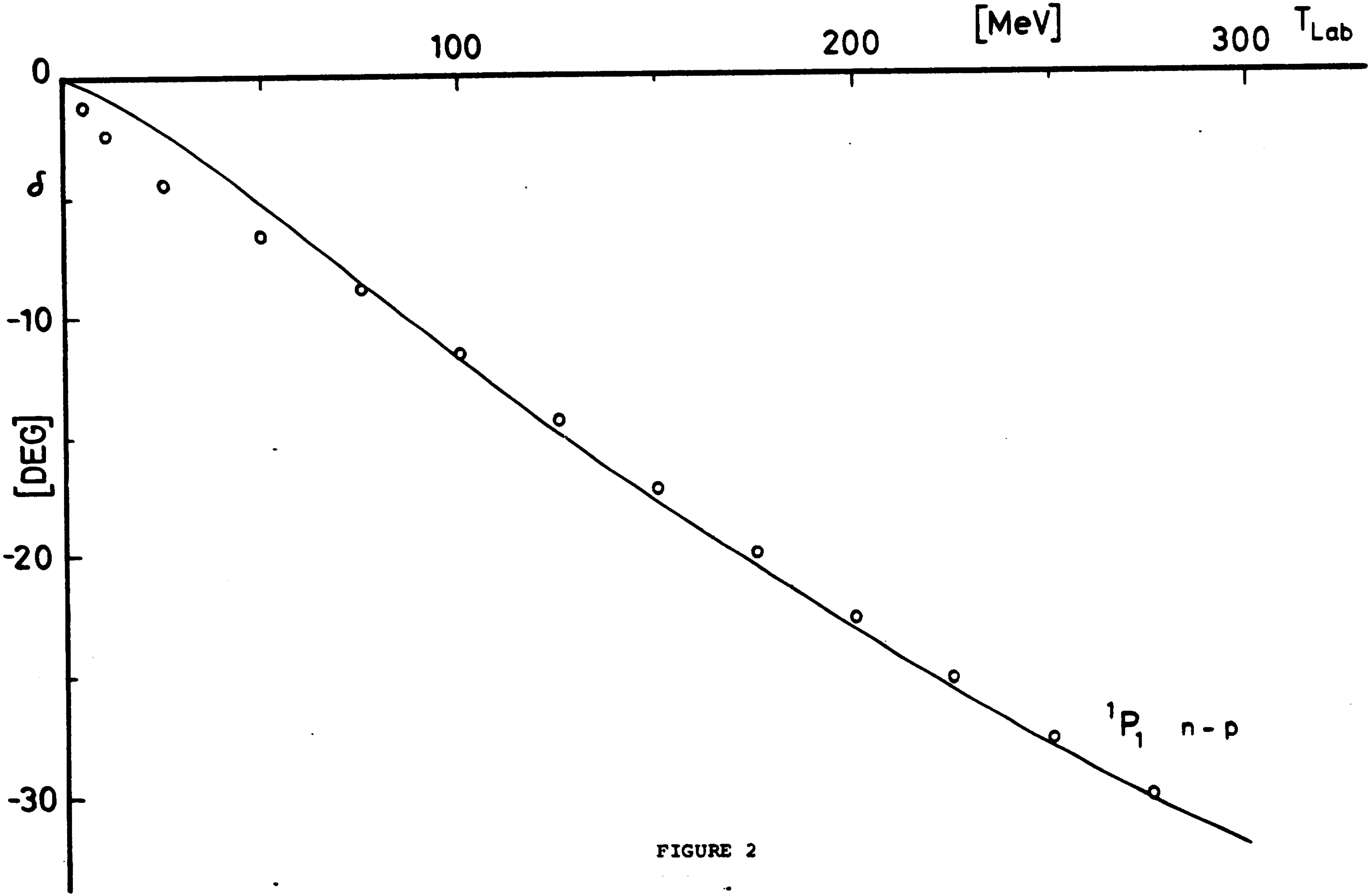
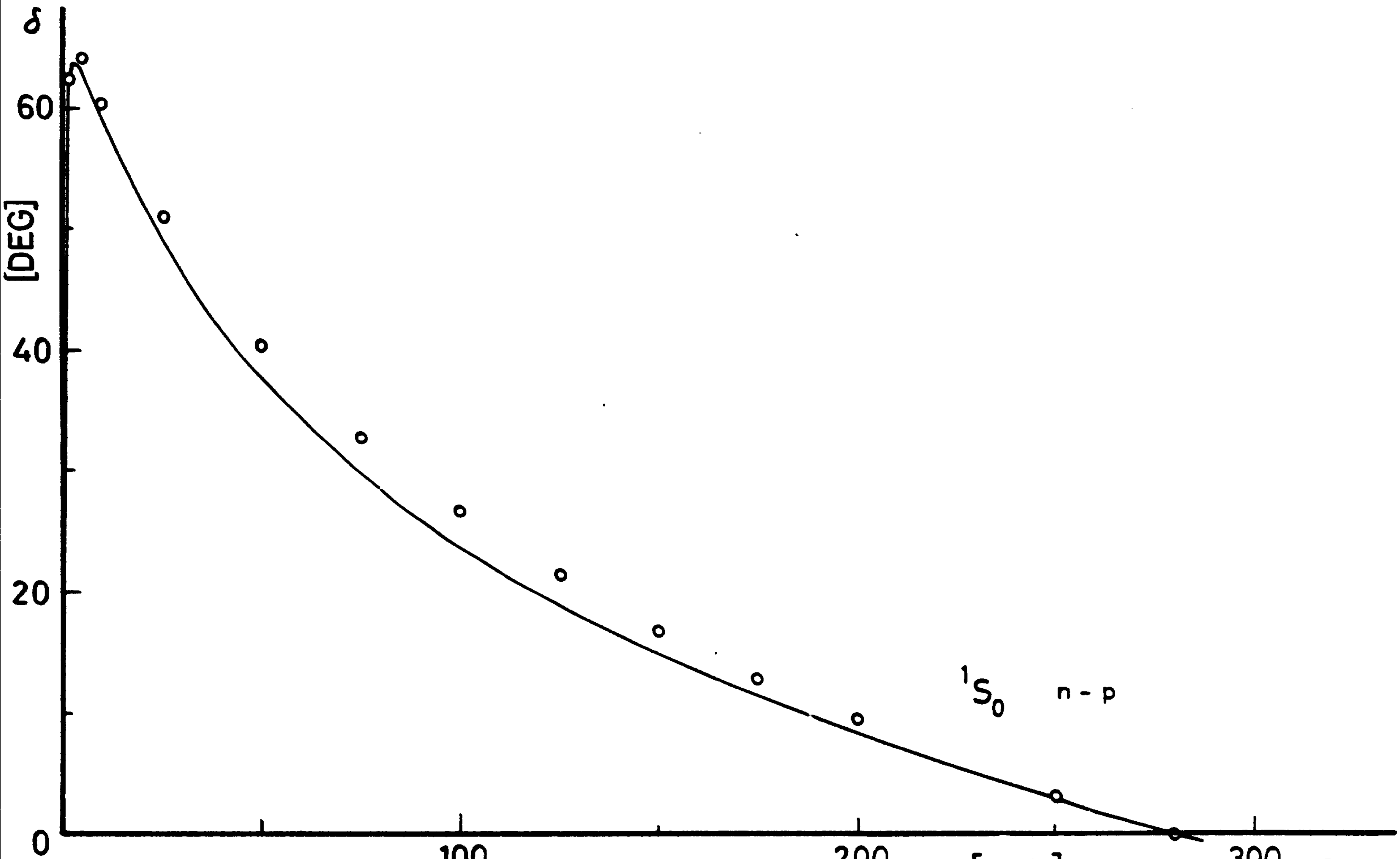


FIGURE 2



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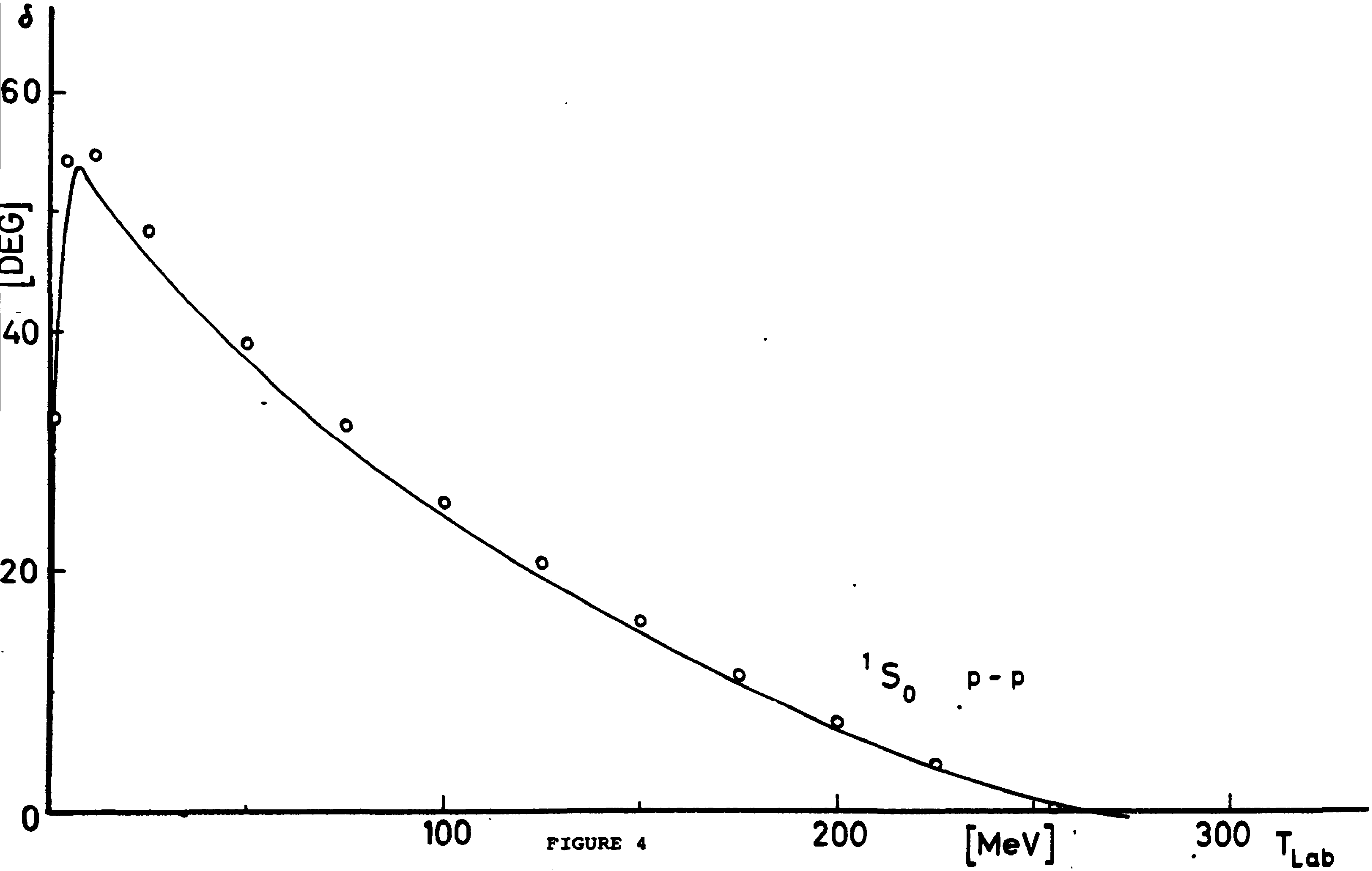

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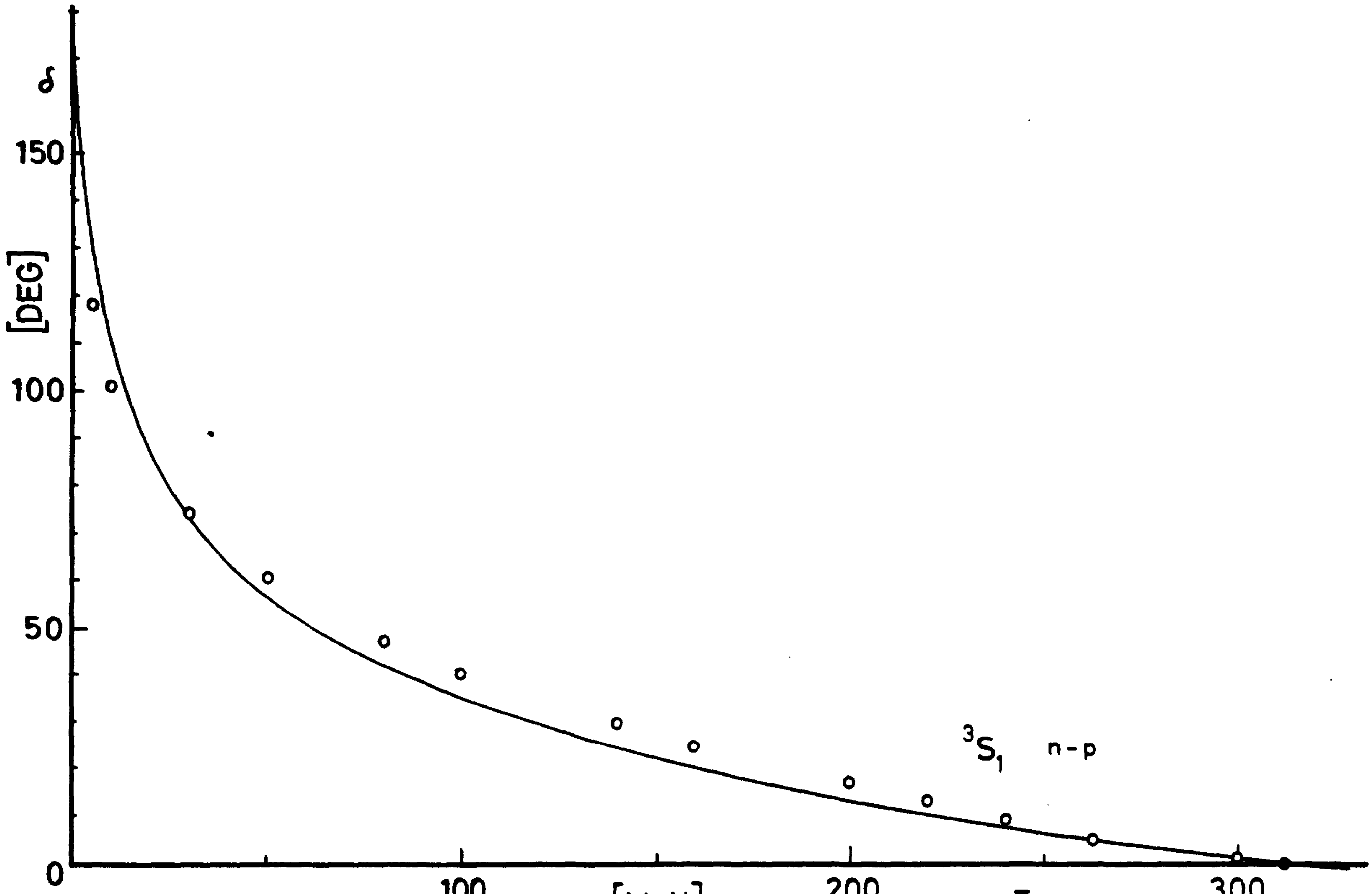

FIGURE 5