KFKI-1980-26

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ANALYTIC OBSERVABLES IN NUCLEAR PHYSICS

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

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Nuclear Physics A

HU ISSN 0368 5330
ISBN 963 371 664 3

ABSTRACT

The analytic dependence of quantum mechanical observables on their variables is discussed by using the assumption that the corresponding probability amplitudes are analytic functions. The properties of the cross section in the energy plane for spinless particles as well as the properties of the most general polarization observables in the cosϑ plane are considered in detail. In particular, the strength of the transfer pole for polarization observables is given. The practical possibilities to extract the different orbital momentum components of nuclei by extrapolation techniques are briefly outlined.

АННОТАЦИЯ

Предпологая, что соответствующие амплитуды вероятности являются аналитическими функциями от своих аргументов, доказан аналитичность квантовомеханических измеряемых величин. Были изучены аналитические свойства поперечного сечения безспиновых частиц в энергетической плоскости и свойства общих поляризационных величин в плоскости cosϑ. В частности, для поляризаций задана формула для мощности обменного полюса. Обсуждены практические возможности извлекания вершинных констант ядер для разных орбительных моментах с помощью метода эмпирического продолжения.

KIVONAT

Jelen munkában kimutattam, hogy a kvantummechanikai megfigyelhető mennyiségek változóik analitikus függvényei, feltéve hogy a megfelelő valószinüségi amplitudók is azok. A hatáskeresztmetszetnek spin nélküli részek esetén az energiasikon és a legáltalánosabb polarizációs mennyiségeknek a cosϑ sikon mutatott viselkedését tanulmányoztam, és az utóbbi esetben megadtam az átadási pólus erősségét. Röviden vázoltam az empirikus folytatás módszerének lehetőségeit atommagok különböző pályamomentumu komponenseinek elkülönitésére is.

1. INTRODUCTION

The characteristic feature of quantum mechanics is that through its equations it provides probability amplitudes (wave functions, scattering amplitudes and so on), while the directly observable quantities are given by expressions which include the amplitudes and their complex conjugates; in the simplest case these are in the form of absolute values. The analytic dependence of the amplitudes on their variables is a well studied problem, for cases of interest in nuclear physics see, for instance, refs.[1-5]. Not only the energy and cosϑ dependence of the scattering amplitudes is studied, but the dependence on such "exotic" parameters as the complex angular momentum as well. On the other hand, the analyticity of observable quantities, such as the cross section and polarization observables, is usually not considered. As the measurements directly provide these quantities, their properties are of great practical importance. Only very recently the analytic properties of the cross section (for a review see refs.[6-8]) and polarizations [9-13] found an application in the extrapolation to the trans' pole in the cosϑ variable. The importance of studying this problem, however, is not solely for practical purposes, the need to complete the theory justifies the activity.

Most probably the reason for neglecting this problem is the slightly misleading fact that the complex conjugate of an analytic function is not analytic at all, it usually serves as an example of a nowhere analytic function in books on the theory of analytic functions (see ref.[14] for instance). In view of this the complex conjugation should be given some special sense. In a trivial way this is done in section 2. Using redefined "complex conjugates" one can easily prove the analyticity of observables provided that the probability amplitudes are analytic functions. Judging from the numerous applications, such a proof should be known to many physicists, and it is at least implicit in dispersion relation studies[2]. The somewhat detailed presentation offered here is for the sake of the reader's convenience*. In section 3 the analyticity properties of the cross section in the energy plane are studied. To my knowledge this approach has not received any attention so far, though the continuation through an inelastic threshold shows some pe-

*Nevertheless, even in such rare cases when this problem is mentioned[7], one can detect viewpoints which differ.

culiar features. In the present paper the basic emphasis is on the analytic properties of the polarization observables in the cosϑ plane, these properties are discussed in section 4. The exploitation of the analyticity of polarizations has just begun[9-13] and it is important to see the general features. Their kinematic singularities on the edges of the physical region are studied, including a proof of the regularity of the cross section there. The strength of their second order transfer pole is also given. It has practical importance, because if one continues the polarization to the transfer pole it is possible to separate different orbital momentum components of various virtual decay channels of nuclei. Since the applicability conditions are practically the same as for the continuation of the cross section, based on the experience gained there one can outline the possibilities of this approach too (section 5).

Because of the heterogeneous material, the style of presentation is different in different parts of the paper. In the first part (sections 2-4), where the aim is to discuss general properties, it tends to be introductory; in the second part (section 4), the aim is to give as concise presentation of straightforward but lengthy calculations as is possible because the emphasis is on the details of the results; the third part (section 5), can only be more or less a mere statement of the author's view on the continuation problem.

2. ANALYTIC OBSERVABLES

As has been mentioned, if one wishes to prove the analytic nature of observables it is necessary to modify the operation of taking the complex conjugates. For this purpose it is not difficult to prove that if $f(z)$ is regular in the neighbourhood of the $[a,b]$ interval on the real axis, then an analytic function exists which takes on the values of $f^*(z)$, $z\in[a,b]$. It is called the conjugate function generated by $f(z)$. A possible singularity of it is not nearer to any point of the $[a,b]$ interval than the nearest singularity of $f(z)$.

The existence of the conjugate function can be proved in a straightforward way. If $f(z) = u(x,y)+i\cdot v(x,y)$ is a regular function of $z = x+i\cdot y$, that is, it satisfies the Cauchy-Riemann relations[14]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

in the neighbourhood of a given point $z = z_o^* = x_o - i\cdot y_o$, then $f^*(z^*) = u(x,-y) - i\cdot v(x,-y)$ considered as a function of z (that is, $f^*(z^*) = f^*(z^*(z))$) also satisfies these relations in the neighbourhood of $z = z_o = x_o + i\cdot y_o$. For real $z\in[a,b]$ the function $f^*(z^*)$ takes on the values of $f^*(z)$, and thus the existence of the conjugate function is proved. The limitation on its singularities follows from the condition that z_o^* should be inside the regularity region of $f(z)$. In particular, an analytic continuation of the conjugate function along the real axis is possible until the first singularity of f(z), the continued function has the values of $f^*(z)$ on the real axis between two singularities of f(z), so it is independent of the actual choice of the [a,b] interval.

As the different variables used in physics are usually constructed in such a way that they are real in their physically accessible region, in expressions for observables it is sufficient to replace the complex conjugates of the probability amplitudes by the conjugate functions and the possibility of an analytic continuation is ensured. The singularities of the conjugate functions are clearly connected with those of the generating functions, therefore it is not difficult to determine the singularities of the observables. Later on examples will be given for them.

3. ANALYTICITY PROPERTIES OF THE CROSS SECTION IN THE ENERGY PLANE

In this section the properties of the cross section as a function of the energy variable are discussed. As a first orientation, threshold properties are studied. For simplicity, spinless particles are considered and the type of the cross section is not specified. It can be eleastic scattering as well as reaction cross section at a fixed angle or with given partial waves.

It is supposed that around the thresholds the amplitude t(E), the square of the absolute value of which gives the cross section σ(E), is a regular function of all open and closed channel wave numbers[1,3] (therefore Coulomb interactions are excluded). The usual square root type threshold singularities in the energy plane are the consequence of the somewhat inappropriately chosen energy variable. It is suitable to consider the amplitude as a function of only that channel wave number k which has the threshold at the given energy. The other wave numbers are understood to be given by the energy conservation law with the usual choice of their signs[1]. For energies above their threshold the wave numbers are positive real, whereas below their threshold they are positive imaginary. The positive imaginary axes are physically accessible, except for the wave number of the initial and/or the final channel.

The second basic property of the amplitude assumed here is that it is <u>real</u> below an <u>elastic</u> threshold[3]. Therefore

$$t^*(k^*) = t(-k) \tag{2}$$

in a circle around the origo in which the amplitude is regular. For positive imaginary values of k this formula states only that the amplitude is real, but both sides being analytic functions of the k variable one can continue it for complex k too.

One can easily prove that the cross section given by an elastic amplitude is a regular function of the k^2=E variable. By using formula (2) one gets that

$$\sigma(k) \equiv t(k)\ t^*(k^*) = t(k)\ t(-k) = \sigma(k^2). \tag{3}$$

It gives the regularity in the $E=k^2$ plane around the origo, i.e. the square root type branch points of $t(E)$ and $t^*(E^*)$ mutually compensate each other. Continuing the regular cross section along the real axis, at negative energies one reaches the physically inaccessible imaginary axis of the k variable and one can extract information on the bound or virtual bound state poles or other singularities there. Note that for negative energies the continued cross section is not the square of the absolute value of the amplitude, the first order amplitude poles generate first order poles for the cross section too. This fact causes some difficulty in extracting the residue in the amplitude pole using cross section data. One should explicitly know the value of $t(-k)$ at $k=k_{pole}$. In addition, transformations of the type $t(k)\to t(k)e^{iak}$ leave the cross section unchanged whereas the values of $t(k)$ are changed. Therefore additional information on the amplitude should be used. It is always available, since there are two essentially independent sources, viz. the unitarity and the time reversal symmetry. The actual form of the relation one needs depends on what type of scattering is considered. In the case of forward scattering, for instance, it is the optical theorem which removes the ambiguity.

For a threshold which is not the elastic one the cross section ceases to be a regular function of the energy. A continuation of the physically measurable cross section on the positive parts of the real and imaginary axes of the k variable to the corresponding negative parts is possible in accordance with the general properties of the continuation process discussed in section 2 and the continued cross section is the square of the absolute value of the amplitude (apart from possible kinematic factors, which are not considered here at all). But a continuation from one of the axes to the other yields a complex valued function there, which is the consequence of the essential k dependence. Therefore beyond and below the threshold one has two independent regular functions of the k variable. This mosaic-like structure of the cross section is in complete accordance with the well known jump of the derivative in the k variable through the threshold (Wigner cusp[1,3]).

The above discussed threshold properties ensure the possibility of extracting bound state characteristics by empirical continuation of the elastic cross section. For the practical possibilities, see the example briefly discussed in ref.[8]. It is important to realize that for such purposes only data up to the first inelastic threshold can be used.

4. ANALYTIC PROPERTIES OF POLARIZATIONS IN THE $\cos\theta$ PLANE

In this section the properties of the most general polarization observables are studied. For the cross section the empirical continuation to the second order transfer pole in the non-physical region have proved to be a useful tool for extracting structure information[6-8]. Therefore, the aim here is to investigate those properties of the polarizations, the exploitation of which makes a similar continuation possible. These are: i) the analyticity of polarization observables, ii) the kinematic singularities on the edges of the physical region, iii) the strength of the second order exchange pole.

According to the most useful paper of Simonius[15)] the polarization observables for the A(x,y)B reaction are defined as

$$t^{k_x q_x k_A q_A}_{k_y q_y k_B q_B} = \mathrm{Tr}\left\{\left(\tau^{(y)}_{k_y q_y} \otimes \tau^{(B)}_{k_B q_B}\right) T \left(\tau^{(x)}_{k_x q_x} \otimes \tau^{(A)}_{k_A q_A}\right)^{+} T^{+}\right\} =$$

$$= \sum_{\substack{M_x M_A M_y M_B \\ M'_x M'_A M'_y M'_B}} \langle M'_y|\tau^{(y)}_{k_y q_y}|M_y\rangle\langle M'_B|\tau^{(B)}_{k_B q_B}|M_B\rangle\langle M_y M_B|T|M_x M_A\rangle \times \tag{4}$$

$$\times \langle M'_x|\tau^{(x)}_{k_x q_x}|M_x\rangle^{*}\langle M'_A|\tau^{(A)}_{k_A q_A}|M_A\rangle^{*}\langle M'_y M'_B|T|M'_x M'_A\rangle^{*},$$

where the matrix elements of the τ_{kq} operators are given by

$$\langle M'|\tau_{kq}|M\rangle = (-1)^{I-M}\sqrt{\hat{I}}\; C^{kq}_{IM',\, I-M} \; . \tag{5}$$

The reaction amplitude is denoted by $\langle M_y M_B|T|M_x M_A\rangle$, the spin and its projection of the corresponding particle by I and M. The usual Clebsh-Gordan coefficients ($\underset{\sim}{I}_1+\underset{\sim}{I}_2 = \underset{\sim}{k}$) are denoted by $C^{kq}_{I_1 M_1 I_2 M_2}$, while $\hat{I} \equiv 2\,I+1$.

The $\langle M_y M_B|T|M_x M_A\rangle$ reaction amplitudes are normalized in such a way that

$$t^{00\,00}_{00\,00} = \frac{d\sigma}{d\Omega} \; . \tag{6}$$

Note that the definition given here differs from the usual one[15)] by the lack of the dividing cross section. The use of such non-normalized polarizations gives essential advantages[10)].

If one replaces the complex conjugates of the amplitudes by the conjugate functions, as was discussed in section 2, then the polarizations become analytic functions. Their singularities are to be derived from the singularities of the amplitudes. Now the reason for omitting the normalizing cross-section is clear: its zeroes generate additional poles the positions of which are not defined by quantities of kinematic character (like masses, binding energies and so on).

4.1 Kinematic singularities on the edges of the physical region

It is well known that the amplitudes might have singularities of purely kinematic origin (i.e. because of the transormation properties associated with the spins of the involved particles) on the edges of the physical region at $\cos\theta = \pm 1$. In the non-relativistic case the invariant amplitude representation of Bilenky et al.[16)] makes it possible to study this problem in the most general case, as their invariant amplitudes are themselves regular at $\cos\theta =$ $= \pm 1$ (see also refs.[17,18)]).

According to Bilenky et al. the $F(j_i\ j_f\ \ell_i\ \ell_f\ I;\ E,\ z)$ invariant amplitudes define the reaction amplitude as follows[16)]

$$\langle M_y M_B|T|M_x M_A\rangle = \sum_{\substack{j_i j_f\ I\ \ell_i+\ell_f=r \\ \nu_i \nu_f\ M}} F(j_i j_f \ell_i \ell_f\ I;\ E,\ z) \times C^{j_i\nu_i}_{I_x M_x I_A M_A}\ C^{j_f\nu_f}_{I_y M_y I_B M_B}\ C^{j_f\nu_f}_{j_i\nu_i IM}\ \mathcal{Y}^*_{\ell_i\ell_f IM}(\underline{n}_i,\underline{n}_f) \tag{7}$$

$$\mathcal{Y}_{\ell_i\ell_f IM}(\underline{n}_i,\underline{n}_f) = \sum_{\mu_i\mu_f} i^{\ell_i+\ell_f}\ C^{IM}_{\ell_i\mu_i\ell_f\mu_f}\ Y_{\ell_i\mu_i}(\underline{n}_i)\ Y_{\ell_f\mu_f}(\underline{n}_f) \tag{8}$$

where $j_i(j_f)$ and $\nu_i(\nu_f)$ are the initial (final) channel spins and their projections, $\underline{n}_i(\underline{n}_f)$ is the direction of the relative momentum in the initial (final) channel, E is the CM kinetic energy and $z=\cos\theta=\underline{n}_i\cdot\underline{n}_f$. The value of r is I or I+1 and it is always even if the product of the internal parities is the same in the initial and final states; otherwise it is odd.

It is a straightforward matter to study the kinematic singularities. One has to insert formula (7) into the general expression for polarization observables of formula (4) and one then has to study the obtained expression at $\cos\theta=\pm1$ using the regularity of the invariant amplitudes at these points.

The presentation of the lengthy calculation is omitted here, but the way of summation can easily be restored with some experience in angular momentum algebra, as only the well known and most elementary properties of the corresponding quantities[19,20)] are used. That part of the result which is of interest for the kinematic singularities is as follows

$$t^{k_x q_x\ k_A q_A}_{k_y q_y\ k_B q_B} = \sum_{\substack{I\ j_i j_f\ \ I'\ j_i' j_f' \\ \ell_i+\ell_f=r\ \ \ell_i'+\ell_f'=r'}} \ldots F(\ell_i\ell_f j_i j_f\ I;\ E,\ z) \times F^*(\ell_i'\ell_f' j_i' j_f';\ E,\ z) \times \sum_{\substack{M\ \ K_i\ K_f\ K \\ M'\ Q_i\ Q_f\ Q}} \ldots \tag{9}$$

$$\begin{Bmatrix} I_x & I_x & k_x \\ I_A & I_A & k_A \\ j_i & j_i' & K_i \end{Bmatrix} \begin{Bmatrix} I_y & I_y & k_y \\ I_B & I_B & k_B \\ j_f & j_f' & K_f \end{Bmatrix} \begin{Bmatrix} j_i & j_i' & K_i \\ j_f & j_f' & K_f \\ I & I' & K \end{Bmatrix} \times$$

$$\times\ C^{K_i\ Q_i}_{k_A q_A k_x q_x}\ C^{K_f\ Q_f}_{k_B q_B k_y q_y}\ C^{K\ Q}_{I'-M'\ IM}\ C^{K\ Q}_{K_f - Q_f K_i Q_i} \times$$

$$\times\ \mathcal{Y}^*_{\ell_i\ell_f\ IM}(\underline{n}_i,\underline{n}_f)\ \mathcal{Y}_{\ell_i'\ \ell_f'\ I'M'}(\underline{n}_i,\underline{n}_f)$$

where the various sign factors and factors of the $\hat{I}$ type are omitted, the usual 9j symbols are denoted in brackets.

Choosing the z axis parallel either to $\underline{n}_i$ or to $\underline{n}_f$ (which gives essentially the same result) and using the behaviour of the spherical harmonics

$$Y_{\ell m}(\vartheta,\varphi) = e^{i\varphi} \times (\sin\vartheta)^{|m|} \times P_{\ell-|m|}(\cos\vartheta) \ , \tag{10}$$

where $P_{\ell-|m|}(\cos\vartheta)$ is a polynomial of the degree of $\ell-|m|$, one can easily show that

$$\mathcal{J}^{*}_{\ell_i \ell_f IM}(\underline{n}_i\ \underline{n}_f)\ \mathcal{J}_{\ell_i' \ell_f' I' M'}(\underline{n}_i\ \underline{n}_f) \sim (1-\cos^2\vartheta)^{(|M|+|M'|)/2} \ . \tag{11}$$

As for the projection quantum numbers M and M' formula (9) gives

$$M-M' = q_x + q_A - q_y - q_B \ , \tag{12}$$

one can use in formula (12) the equality

$$|M| + |M'| = |M - M'| + 2n \ , \tag{13}$$

where $n=|M|,|M'|$ or 0 depending on the sign and mutual relation of M and M', which gives

$$t^{k_x q_x\ k_A q_A}_{k_y q_y\ k_B q_B} \sim (1-\cos^2\vartheta)^{|q_x+q_A-q_y-q_B|/2} \ . \tag{14}$$

Here the factor of $(1-\cos^2\vartheta)^n$ is omitted as no selection rule forbids the value of n=0. his formula defines the behaviour of the polarizations on the edges of the physical region at $\cos\vartheta=\pm1$. If $q_x+q_A-q_y-q_B$ equals zero, one has a nonzero limit there (a special subcase is the cross section); other even values give zeroes of the corresponding order. Odd values generate branch points of square root type, i.e. with a two-level structure. The values of a given polarization observable on these two levels differ only in their sign, and the singularities of the invariant amplitudes (the dynamical singularities) appear on both sheets.

It is important to note that the kinematic singularities given by the basic formula (14) are <u>not</u> the leading terms of expansion series, but the full singularitities themselves: dividing the polarization by the right hand side one gets a regular function at $\cos\vartheta=\pm1$.

4.2 Contribution of the second order exchange pole

As polarizations are bilinear functions of the amplitudes, the transfer pole of the latter generates a second and a first order pole. The first order pole comes from the interference with the contribution of other mechanisms (i.e. with the background) and therefore the residue in it depends on the background amplitude too. The second order pole comes from the product of the pole amplitudes, therefore its strength depends entirely on the structure of the involved nuclei, just as for the cross section.

The method of determining its strength will be by calculating the contribution of the pole amplitudes in the physical region and investigating its analytic structure during a continuation from inside the physical region towards the transfer pole[6]. Finally, special cases of some practical importance are discussed.

4.2.1 Pole contribution in the physical region

The pole singularity of the amplitude was investigated in refs.[17,18] and a kinematically correct expression for the amplitudes was given there. If one considers the transfer reaction A(x,y)B, x=y+a, B=A+a (see fig.1) and uses the notations

$$\underset{\sim}{k}_i = k_i\,\underset{\sim}{n}_i;\quad \underset{\sim}{k}_f = k_f\,\underset{\sim}{n}_f \tag{15a}$$

for the CM relative wave numbers, E_i and E_f for the kinetic energies in the initial and final channels;

$$\begin{aligned}\underset{\sim}{p}_x &= p_x\,\underset{\sim}{n}_x = -\frac{m_y}{m_x}\underset{\sim}{k}_i + \underset{\sim}{k}_f = -\underset{\sim}{k}'_i + \underset{\sim}{k}_f \\ \underset{\sim}{p}_B &= p_B\,\underset{\sim}{n}_B = -\underset{\sim}{k} + \frac{m_A}{m_B}\underset{\sim}{k}_f = -\underset{\sim}{k}_i + \underset{\sim}{k}'_f\end{aligned} \tag{15b}$$

for the relative wave numbers in the x and B vertices;

$$\kappa_x = (2\mu_{ya}\,\varepsilon_x)^{1/2},\quad \kappa_B = (2\mu_{Aa}\,\varepsilon_B)^{1/2} \tag{15c}$$

for the relative wave numbers in the vertices $i\kappa_x$ and $i\kappa_B$ which correspond to the binding energies ε_x and ε_B, where $\mu_{ij} = m_i\,m_j/(m_i + m_j)$ is the reduced mass;

$$z_p = (k_i^2 + k_f'^2 + \kappa_B^2)/2k_i k_f' = (k_i'^2 + k_f^2 + \kappa_x^2)/2k_i'\,k_f \tag{15d}$$

for the location of the pole;
then the so called "pure" pole amplitudes are[18]

$$\begin{aligned}\langle M_y M_B | T^{(p)} | M_x M_A\rangle &= \frac{\text{const}}{z_p - z}\sum_{M_a} \\ &\left\{\sum_{\ell_x\mu_x j_x\nu_x}(p_x/i\kappa_x)^{\ell_x}\,G^{(x)}_{\ell_x j_x}\,C^{j_x\nu_x}_{\ell_x\mu_x I_a M_a}\,C^{I_x M_x}_{j_x\nu_x I_y M_y}\,Y_{\ell_x\mu_x}(\underset{\sim}{n_x})\right\}\times \\ \times &\left\{\sum_{\ell_B\mu_B j_B\nu_B}(p_B/i\kappa_B)^{\ell_B}\,G^{(B)}_{\ell_B j_B}\,C^{j_B\nu_B}_{\ell_B\mu_B I_a M_a}\,C^{I_B M_B}_{j_B\nu_B I_A M_A}\,Y^*_{\ell_B\mu_B}(\underset{\sim}{n_B})\right\}.\end{aligned} \tag{16}$$

In the vertices the coupling scheme of the type $\underset{\sim}{\ell}_B+\underset{\sim}{I}_a = \underset{\sim}{j}_B$, $\underset{\sim}{j}_B+\underset{\sim}{I}_A = \underset{\sim}{I}_B$ is used, as it is the usual one. Other coupling schemes are also possible. The following calculations can be performed using any of them, too. The $G_{\ell j}=G_{\ell j}(i\kappa)$ vertex constants are the values of the $G_{\ell j}(p)$ formfactors in the pole (see refs.[17,18]), and have the property $G_{\ell j}=\pm i^{\ell}|G_{\ell j}|$. The normalization constant of the pure pole amplitude is not given here; all the corresponding details can be found in refs.[6,17,18].

Using formulae (4) and (17) one can calculate, in a straightforward way, the contribution of the second order pole. With the complete normalization the result is

$$t^{k_x q_x \; k_A q_A}_{k_y q_y \; k_B q_B} = \frac{5}{8\pi^2} \frac{m_a^2 c^4}{E_i \; E_f} \frac{k_f}{k_i} \frac{1}{(z-z_p)^2} \frac{1}{\hat{I}_A \; \hat{I}_x} \times \tag{17}$$
$$\times \sum_{KQ} V^{(*)}_{KQ} (B,A) \; V_{KQ} (x,y),$$

where

$$V^{(*)}_{KQ}(B,A) = (-1)^{-k_A+q_A}[(\hat{I}_B)^3 \; \hat{I}_A \hat{k}_B \hat{k}_A/4\pi]^{1/2} \times$$
$$\times \sum_{\ell_B j_B \ell'_B j'_B} (-1)^{-j_B+j'_B} (\hat{\ell}_B \hat{\ell}'_B \hat{j}_B \hat{j}'_B)^{1/2} (p_B/i\kappa_B)^{\ell_B+\ell'_B} \; G^{(B)}_{\ell_B j_B} \; G^{(B)*}_{\ell'_B j'_B} \times$$
$$\times \sum_{\substack{K_B L_B \\ Q_B \Lambda_B}} (\hat{K}_B)^{1/2} \begin{Bmatrix} I_B & I_B & k_B \\ I_A & I_A & k_A \\ j_B & j'_B & K_B \end{Bmatrix} \begin{Bmatrix} j_B & j'_B & K_B \\ \ell_B & \ell'_B & L_B \\ I_a & I_a & K \end{Bmatrix} C^{L_B 0}_{\ell_B 0 \; \ell'_B 0} \times \tag{18}$$
$$\times C^{K_B Q_B}_{k_A -q_A \; k_B q_B} \; C^{KQ}_{L_B \Lambda_B \; K_B Q_B} \; Y^{*}_{L_B \Lambda_B} (\mathbf{n}_B) \; ,$$

and an analogous expression (x~B,y~A) is valid for $V_{KQ}(x,y)$ with $Y_{L_x \Lambda_x}(\mathbf{n}_x)$ corresponding to $Y^{*}_{L_B \Lambda_B}(\mathbf{n}_B)$. The $G_{\ell i}$ vertex constants are given in $fm^{1/2}$, while $t^{k_x q_x \; k_A q_A}_{k_y q_y \; k_B q_B}$ in mb/sr in accordance with formula (6). For cross sections an equivalent formula was given in ref.[6]. From formulae (17)(18) it is easy to derive formula (49) of ref.[6] by noting that in this case $K_B = K_x = 0$, $L_B = L_x = K$, $\Lambda_B = \Lambda_x = Q$ and by using the addition theorem for the spherical harmonics.

Other ways of summation were also performed, but since the results contained a larger number of intermediate quantum numbers formula (17) is optimal. If one uses a different coupling scheme in the vertices, namely of the type[18] $\mathbf{I}_A+\mathbf{I}_a = \mathbf{s}_B$, $\mathbf{s}_B+\boldsymbol{\ell}_B = \mathbf{I}_B$, a similar way of summation can be used because the structure of the coupling scheme for the polarizations is equivalent.

Apart from the relations following from the vertex coupling scheme (and from the trivial inequalities $k_i \le 2I_i$, $i = x,y,B,A$) there is a series of triangular inequalities following from formulae (17), (18):

$$\Delta(\ell_i\ \ell_i'\ L_i) \qquad \ell_i+\ell_i',\ L_i = \text{even}$$
$$\Delta(j_i\ j_i'\ K_i) \qquad \Delta(k_m\ k_n\ K_m) \tag{19}$$
$$\Delta(K_i\ L_i\ K\ ) \qquad \Delta(I_a\ I_a\ K\ )\ ,$$

where $i=x,B$ and $(m,n)=(x,y)$, (B,A). If these conditions cannot be satisfied, the sums in formulae (17), (18) are empty, consequently the second order exchange pole has zero strength. For a discussion of cases of practical importance, see below.

4.2.2 Continuation towards the transfer pole, its strength

The next step is to study the structure of the pole contribution as an analytic function of $z=\cos\vartheta$, while it is continued from inside the physical region towards the transfer pole. First of all, being a contribution to the complete polarization, it should have the kinematic singularities given by formula (14) on the edges of the physical region. Secondly, before reaching the pole, there are certain points in the nonphysical region where various quantities in the formula have branch points. One can show that the pole contribution is regular there. Thirdly, because of the above difficulties, it is useful to give the practical rules to calculate the pole strength.

To be definite, the coordinate system is chosen in such a way that the z axis is parallel to $\underline{k}_i$ and the y axis is perpendicular to the reaction plane. In this case the $\cos\vartheta=\underline{n}_i\cdot\underline{n}_f$ dependent quantities $p_B, \underline{n}_B, p_x, \underline{n}_x$ are given by (see formula (15b))

$$p_B^2 = k_f'^2 + k_i^2 - 2k_f'\,k_i\,\cos\vartheta \tag{20a}$$

$$\cos\vartheta_B = (k_f'\,\cos\vartheta - k_i)/p_B \tag{20b}$$

$$\sin\vartheta_B = k_f'\,\sin\vartheta/p_B\ , \tag{20c}$$

and similar relations hold for p_x^2, $\cos\vartheta_x$, $\sin\vartheta_x$. In the physical region the square root of a positive quantity is considered to be positive, therefore p_B and p_x are positive there.

On the edges of the physical region ($\cos\vartheta = \pm1$, $\sin\vartheta = 0$) one has $\cos\vartheta_B = \pm1$ $\cos\vartheta_x = \pm1$ and $\sin\vartheta_B = \sin\vartheta_x = 0$. Therefore the product $Y_{L_x\Lambda_x}(\underline{n}_x)\cdot Y^*_{L_B\Lambda_B}(\underline{n}_B)$ in formula (17) shows a singularity of $(\sin\vartheta)^{|\Lambda_x|+|\Lambda_B|}$, as it follows from formula (10). By using exactly the same means as formula (14) was derived by, one can show that this behaviour is equivalent to that given by formula (14), i.e. the pole contribution has the necessary kinematic singularities on both edges of the physical region. With the usual choice of the square root levels, $\sin\vartheta$, $\sin\vartheta_B$ and $\sin\vartheta_x$ all have negative (positive) imaginary values on the upper (lower) edge of the cut for $\cos\vartheta$ values larger than 1, as it follows from the equality $\sin^2\vartheta+\cos^2\vartheta = 1$. For other quantities ($p_B$, p_x, $\cos\vartheta_B$, $\cos\vartheta_x$), the point $\cos\vartheta = 1$ is not critical.

When continuing formula (17) further towards the transfer pole, there is a critical point where $p_B^2 = 0$. The discussion for the case $p_x^2 = 0$ is quite the same, therefore it is omitted. For p_B this point is a square root type branch point, whereas for $\cos\vartheta_B$ and $\sin\vartheta_B$ there is an infinite limit too. Nevertheless, for the pole contribution this point is not a singularity. Because both $\ell_B+\ell_B'$ and L_B are even and $\ell_B+\ell_B' \geq L_B$ (see formula (19)), $(p_B/i\kappa_B)^{\ell_B+\ell_B'} Y_{L_B\Lambda_B}(\underset{\sim}{n}_B)$ is a finite and regular function* of $\cos\vartheta$ on both levels of $\sin\vartheta$, as it follows from formula (10).

Therefore for larger values of $\cos\vartheta$ one can use both branches of p_B for calculating the contribution and, in particular, the strength of the second order pole; the result is independent of it. In one way it is more suitable to use the lower edge of the cut,as it gives positive imaginary values for p_B; in particular, p_B equals $i\kappa_B$ in the pole, which is the physical value.

To evaluate the pole strength, the following formulae can be useful in calculating various quantities in the transfer pole ($\cos\vartheta = z_p$, see formula (16c)).

$$\sin^2\vartheta + z_p^2 = 1, \qquad \sin\vartheta = i\cdot u, \qquad u>0 \tag{21a}$$

$$p_B/i\kappa_B = p_x/i\kappa_x = 1 \tag{21b}$$

$$\cos\vartheta_B = (k_f' \, z_p - k_i)/i\kappa_B, \qquad \cos\vartheta_x = (k_f \, z_p - k_i')/i\kappa_x \tag{21c}$$

$$\sin\vartheta_B/\sin\vartheta = k_f'/i\kappa_B, \qquad \sin\vartheta_x/\sin\vartheta = k_f/i\kappa_x \, . \tag{21d}$$

In all cases the lower edges of the cuts in the $\cos\vartheta$ plane ($\cos\vartheta = 1$, $q_B^2 = 0$, $q_x^2 = 0$) are used. It is a straightforward matter to insert these quantities into formulae (17),(18), therefore this procedure is omitted here. It is important to emphasize that the lower edge of the $\sin\vartheta$ cut is used, as the strength can have a different sign on the other sheet.

4.2.3 Special cases of practical importance

Until now the empirical continuation of the cross section was successful only in the case of a single nucleon transfer[8]. One can expect a similar field of applicability for the continuation of polarizations too[10], therefore it is interesting to discuss the effect of the selection rules given by formulae (19) for $I_a = 0$, 1/2 (which includes single nucleon and α particle transfer). Mostly in practice the bombarding particle is polarized and at most the polarization of the outgoing light particle is measured. The existence of a vertex where each k (for example, k_x and k_y in the case of a pick-up reaction) is zero implies that K=0, that is, formula (17) is factorized and additional (compared with the cross section) information can be extracted

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*Note that for similar reasons the pole amplitude is also regular at this point[18] therefore no singularity is present even in the interference term with the background amplitude, which is not considered here.

only for the polarized vertex[10]. If in the polarized vertex one k equals zero, then only even values are possible for the other. The fact of factorization presents a strong limitation on the usefulness of the empirical continuation of the polarizations as the number of possible vertices with a given polarized bombarding particle is not large, especially if one takes into account the applicability condition of the method[8]. Therefore, polarized targets are necessary to extract detailed structure information for a large number of states by continuing polarization observables.

Tensor polarized (k=2) bombarding beams of deuterons and ^{6}Li particles are widely available. For single nucleon and α particle transfer the $(\vec{d},p)$, $(\vec{d},n)$, $(\vec{d},t)$, $(\vec{d},{}^3He)$, $(\vec{d},{}^6Li)$ and $({}^6\vec{Li},d)$ reactions can give detailed information on the structure of the corresponding particle, namely they make it possible to separate the kinematically possible $\ell=0$ and $\ell=2$ vertex components. For the ratio of the second order pole strength of the polarizations (strictly speaking, of the analysing powers $t^{2q\,00}_{00\,00}$ or $t^{00\,2q}_{00\,00}$) to that of the cross section one can deduce from formulae (17),(18) that

$$(\text{strength } 2q)/(\text{strength } 00) =$$

$$= 2(4\pi/5)^{1/2}\, Y^*_{2q}(\underset{\sim}{n})\{s\cdot G_0G_2 - G_2^2/\sqrt{8}\}/\{G_0^2+G_2^2\}, \qquad (22)$$

where $\underset{\sim}{n}$ is the direction of the corresponding vertex momentum in the pole given by formula (21); the value of s is -1 for the $(\vec{d},t)$, $(\vec{d},{}^3He)$ reactions, whereas it is +1 for the other cases listed above; the $G_{\ell j}$ vertex constants with the vertex orbital momentum $\ell=0$ and $\ell=2$ (and with a uniquely defined value for j) are denoted by G_0 and G_2. For the special case of $\vec{d}$-p elastic scattering this formula was derived in ref.[9]. The denominators $G_0^2+G_2^2$ come from the cross section, the numerators $G_0G_2+G_2^2$ come from the tensor polarization. Therefore the cross section pole is an incoherent sum of the squares of the $\ell=0$ and the $\ell=2$ poles whereas the tensor polarization pole is the sum of the square of the $\ell=2$ pole and the interference of the $\ell=0$ and $\ell=2$ poles.

The purely kinematic factor of $Y^*_{2q}(\underset{\sim}{n})$ plays a very important role. As an example, one can discuss the reaction $d(\vec{d},p)t$. The cross section is symmetric, the large forward peak is dominated by stripping of the bombarding deuteron; the backward peak is dominated by pick-up of a neutron from the target deuteron (see the literature cited in ref.[8]). If one of the deuterons is polarized, no symmetry exists and the kinematic factor $2(4\pi/5)^{1/2}\, Y^*_{20}(\underset{\sim}{n})$ is -23.6 for the stripping process as opposed to 1.04 for the pick-up at E_d=13.0 MeV, for instance. It can thus be seen that there is a strong kinematic enhancement for the stripping of the polarized deuteron, which makes it easier to extract the deuteron vertex constants in the case of roughly equal d state components of the deuteron and the triton.

Because of the essential role of the kinematic factor, its value was calculated for $(\vec{d},p)$, $(\vec{d},t)$ and $(\vec{d},h)$ reactions on all stable nuclei in the mass region from ^{1}H to ^{31}P at various bombarding energies (E_d=10-52 MeV). For $(\vec{d},t)$ and $(\vec{d},h)$ reactions there is practically no target dependence, the values are very near to one. But for $(\vec{d},p)$ reactions there are large fluctuations: the typical value is somewhat over 10. Combining the values of the kinematic factor with the available information on the cross section continuation results[8] one can favour the continuation of the $d(\vec{d},p)t$ polarizations[10]; it is expected to give the best result for both the deuteron and the triton d state components.

5. EMPIRICAL CONTINUATION OF POLARIZATIONS IN THE $\cos\theta$ PLANE

In this section some problems of extracting the transfer pole strength by extrapolating the polarizations are discussed. It is assumed that the reader is acquainted with the general theory of empirical continuation and in particular with the content of ref.[8], where the continuation of the cross section is discussed in detail. It is to some extent premature to draw definite conclusions on the continuation of the polarizations based on the applications published so far[9-13] because of their very limited number. On the other hand, the experience gained during the continuation of the cross section[8] and the results of some unpublished analyses of the author make it possible to outline some necessary conditions, the fulfilment of which implies a correct result. The mentioned analyses use preliminary and unpublished data, but this should not alter the general features of the results. It would hardly be appropriate to with-hold the information provided by them if one has the aim of contributing towards the avoidance of possible incorrect applications.

In this section only one particle is considered to be polarized and this is the bombarding particle. Therefore "polarization" in this section means analysing power. In accordance with the usual convention $T_{kq}(z)$ denotes the normalized polarization, i.e. that defined by formula (4) and divided by the cross section. The non-normalized polarization studied so far is denoted by $\sigma(z)T_{kq}(z)$.

The possibility to continue the polarizations up to the transfer pole and to extract the latter's strength (after the removal of the kinematic singularity given by formula (14), of course) has been always clear (see. for instance, ref.[21]). The continuation of the $T_{22}(z)$ tensor polarization of the d-p scattering was proposed and attempted by Amado, Locher and Simonius[9]. They argued that the q=2 polarization is the most suitable one to continue because the kinematic zeroes essentially given by the factor $Y^*_{2q}(\underset{\sim}{n})$ in formula (22) are not present in this case and it is therefore a smoother function that is to be continued. They concluded that the feasibility of the method is demonstrated by their analysis. But the <u>normalized</u> polarization was continued, which apart from the Coulomb singularity at the forward edge of the physical region has two poles at complex values of $\cos\theta$ near the minimum of the cross section[22], therefore it is not possible to continue it to the exchange pole, which is well outside the convergence ellipse[8)*]. That this was not clear from the analysis itself is the consequence of the unsatisfactory accuracy of the experimental data used there. Independently, but somewhat later, the properties (kinematic singularities and strength of the second order pole) of the polarizations with one polarized particle were studied and reported in ref.[10)]; the continuation of non-normalized tensor polarizations $\sigma(z)\ T_{2q}(z)$ was proposed with q=0,1,2. It was argued that the applicability condition for the continuation of the cross section and polarizations should be roughly the same; and taking into account that the q=0 polarizations are measured with a smaller relative error at extreme angles, preference was given to them. Using accurate $\vec{d}$-p data it was demonstrated in ref.[11)] that the continuation scheme proposed by Amado, Locher and Simonius[9)] is not effective enough. Subsequently, without explaining the reasons underlying the failure, the continuation of $\sigma(z)\ T_{22}(z)$ was proposed[12)]. The extracted G_2/G_0 value for the deuteron was in agreement with the values known from other sources. A number of problems arises, however, such as the absence of the application of the well known optimal conformal mapping (and thus the necessary divergence of the extrapolation series due to the Rutherford singularity, see ref [8)]), the sharp contrast of the very small number (N=3-4) of the significant terms to the large number (N=9-10) of terms for the continuation of the cross section (see the literature cited in ref.[8)]), the usage of the theoretical pole strength for the cross section instead of that found empirically by continuing the cross section (the two quantities differ considerably because of Coulomb effects in the pole[22,8)]), and so on. It is not the aim of this paper to discuss these problems; on the whole, one tends to think that the agreement is only fortuitous. The very same scheme was applied for data at higher energies by Conzett et al.[13)], where the above mentioned problems are not so critical. Nevertheless, one can agree with the conclusion of ref.[13)] that more sophisticated analysis should be made in this case too.

The basic problem of the empirical continuation method is how to check that the extracted significant terms are sufficient to describe the function to be continued not only in the physical region but at the extrapolated point too. It was pointed out[8)] that in the case of continuation in the cosθ plane a simple and effective method to solve this problem is provided by considering the partial wave expansion of the amplitude and using the assumption that due to the short range character of the effective interaction only the partial wave amplitudes with orbital momenta $\ell \leq L \approx k \cdot R$ (k is the relative wave number, R is the distance at which the wave function of the transferred particle deviates from its Hankel-type asymptotics) are not defined by the pole. For the sake of simplicity, for a scalar amplitude t(z), that is, for particles without spin, one has

*This does not mean that normalized polarizations cannot be continued, but for this purpose one should use rational fraction approximations[22)] rather than polynomial ones.

$$t(z) = \sum_{\ell=0}^{L} \tilde{t}_\ell \, P_\ell(z) + \frac{g}{z-z_p} \, , \tag{23}$$

where $\tilde{t}_\ell$ denotes the deviation from the pole contribution. Using this formula one gets for the cross section

$$(z-z_p)^2 \, \sigma(z) = (z-z_p)^2 |t(z)|^2 = g^2 + (z-z_p) \, P_{2L+1}(z) \tag{24}$$

where $P_{2L+1}(z)$ is a polynomial of the degree of 2L+1. It is quite clear that to be able to extract the correct value of g^2 in a fit which uses an arbitrary polynomial system $\{B_n(z)\}$

$$(z-z_p)^2 \, \sigma(z) \simeq \sum_{n=1}^{N} A_n \, B_n(z) \tag{25}$$

one should have

$$N = 2 \, L + 3 \tag{26}$$

significant terms in order to describe $(z-z_p)P_{2L+1}(z)$ correctly.

This simple argumentation can easily be modified for more complicated cases. As discussed above, for the cross section the basic contribution comes from the $\ell=0$ amplitude (the G_2/G_0 value is supposed as being small), while the polarizations are essentially given by the interference of the $\ell=0$ and $\ell=2$ amplitudes. It is well known that for transfer processes the initial and final channel orbital momenta can differ by the vertex orbital momentum only[17]. Therefore the necessary number of terms for the cross section $\sigma(z)$ and for the polarizations $\sigma(z)T_{2q}(z)$ can differ at most by two. The empirical material allows one to establish that this possibility is attained only for $\sigma(z)T_{20}(z)$; the other polarizations need

$$N_q = N_\sigma + 2 - q \tag{27}$$

significant terms, where N_σ is the number of terms for the cross section found empirically or given by formula (26). This rule should be given a theoretical proof too. It is clear that this rule concerns only the dominant G_0G_2 term in formula (22). In order to extract the additional G_2^2 term reliably, one might need even more significant terms.

If one considers the number of significant terms only, then the continuation of the q=2 polarizations seems to be more favourable. But one should take into account that the q=0 polarizations are measured more accurately since their relative error for the most forward region is much smaller, simply because of the kinematic behaviour $(\sin\theta)^{|q|}$ (see formula (14)). The higher number of necessary terms and the better accuracy compensate each other to a great degree. Nevertheless, the q=0 polarization still provided results with a slightly lower statistical error.

If one uses polynomial approximations, the scheme of the analysis is proposed to be as follows. First of all the analysis of the cross section $\sigma(z)$ is necessary. One defines from it N_σ (see formula (27)) and one can check this by the methods described in ref.[8)] (such as $N_\sigma \simeq 2kR+3$, or a peripheral model fit to the data and so on) and/or by comparing the extracted pole strength with values known from other sources. Having the value of N_σ one analyses all the available tensor polarizations $\sigma(z)T_{2q}(z)$ with q=0,1,2 and one should then check whether the number of significant terms is in agreement with formula (27). If so, the results should agree with each other and they can be considered reliable. In the case of elastic scattering the application of a conformal mapping is trivially needed to deal with the Rutherford singularity and it is desirable to check, by the methods described in ref.[8)] (subtraction of the Rutherford cross section from the cross section, using suppresion factors and so on), that the contribution of the Rutherford singularity in the transfer pole is negligible.

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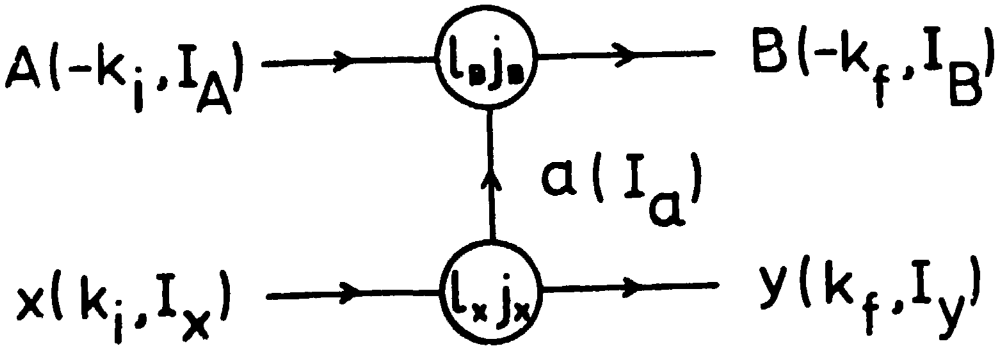

Fig. 1. The pole graph

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Budapest, 1980. május hó