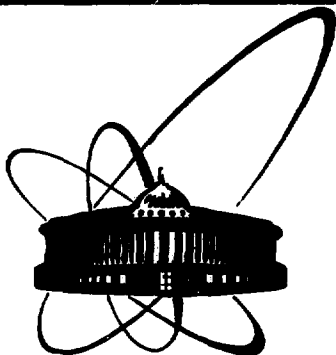


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J.Löffelholz

**EUCLIDEAN FIELDS:
VECTOR MESONS AND PHOTONS**

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Леффельхольц Ю.

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Евклидовые поля: Векторные мезоны и фотоны

Рассматривается свободное поперечное векторное поле частиц масс $m \geq 0$. Модель относится к обычной теории квантовых полей свободного векторного мезонного поля и свободного электромагнитного поля посредством расширения от поперечных к произвольным пробным функциям. Мы описываем одночастичные состояния в поперечной калибровке и их локализацию. Докажем физическую положительность. Мы даем евклидов подход к фотонному полю в сферическом мире, используя дилатационную инвариантность и инверсии.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

Löffelholz J.

E5 - 12780

Euclidean Fields: Vector Mesons and Photons

We study free transverse vector fields of mass $m \geq 0$. The model is related to the usual free vector meson and electromagnetic quantum field theories by extension of the field operators from transverse to arbitrary test functions. We describe the one particle states in transverse gauge and their localization. We prove reflexion positivity and derive free Feynman-Kac-Nelson formulas. We give an Euclidean approach to a photon field in a spherical world using dilatation covariance and inversions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

0. Introduction and content

It was Schwinger in 1958 who first argued that Euclidean Q.F.T. may be a "possible avenue for future development of field theory". Now, 1979, it has already a long history. Stimulated by the progress in scalar $P(\phi)_2$ -theory from 1972, the techniques exploited by Nelson and others have been used to describe in the Euclidean region also Fermi and other higher-spin fields even if gauge fields are included in the interactions.

The basic idea is to pass from Minkowski to Euclidean space-time which brings the structural simplification of positive-definite metric and, in the case of Boson models we will discuss, the main technical advantage of commuting random variables. However, once having the Euclidean formulation of a quantum field theoretical problem it yields an interesting mathematical model itself, and we can forget about the "physical background". In fact, the reflection positivity - together with Euclidean covariance and a regularity condition - always guaranties the backward step to Minkowski space-time.

In particular, the free vector meson and electromagnetic quantum field theories have unique Euclidean formulations via the reconstruction of Osterwalder and Schrader. They were given by Yao, Gross, Guerra and others. Independently, in 1974 we have found the quantum and Euclidean models for the photon field in terms of transverse potentials and derived them from corresponding massive theories as limits $m \downarrow 0$. We avoided the well known problem of quantizing the free Maxwell's equations, restricting

the vector potentials to transversal test functions in the Lorentz or Coulomb gauge.

Applying the above idea to the massive case we have defined also the quantum and Euclidean models for the free vector meson field in the transverse gauge. The main result of our paper is that reflection positivity and reflexivity hold for $m \geq 0$, even for the Euclidean models on a lattice.

However, the Markoff property can be recovered exactly for the Euclidean photon field theory. We have studied a nice analogy to classical magnetostatics and end with the following interpretation of reflection positivity:

Let $\{m_i\}_{i \in I}$ a finite set of magnetic dipoles localized in the open right half-space $\Lambda_+ = \{r = (r^0, \underline{r}) : r^0 > 0\}$ and $\overset{e}{A}(h)$, $h = \sum_I \nabla x(m_i \delta_{a_i})$, the related Euclidean random field. Then

$$\langle \overset{e}{A}(h) \overset{e}{A}(\theta_0 h) \rangle = - \sum_{i, j \in I} m_i \cdot K(a_i - \theta_0 a_j) \theta_0 m_j \geq 0,$$

because it is just the expression for the Coulomb interaction energy of the set of electric dipoles $\{m_i\}_{i \in I}$ with their mirror images with respect to the hyperplane $S_0 = \{r : r^0 = 0\}$. Above θ_0 denotes reflection at S_0 and K the coupling matrix of the dipoles.

The paper consists of three parts which are organized as follows: (cF. JINR preprints E5-I2779, I2781 for the parts II and III)

I. Vector Mesons and Photons: 1. One-particle states in the transverse gauge, 2. Localization, Euclidean and dilatation covariance, 3. Reflection positivity and Hamiltonian semi-groups,

II. Quantum and Random Fields: 1. Vector meson field in the Stückelberg gauge and photon field in terms of $F^{\mu\nu}$, 2. Second quantization, 3. Transversal self-interaction,

III. Lattice models: 1. Transversal lattice fields, 2. Steady currents and magnetic dipoles, 3. Conclusions.

1. One-particle states in the transverse gauge

Let be $\{n_\mu: \mu=0,1,2,3\}$ the Cartesian basis in R^4 and $a, b \rightarrow ab = g^{\mu\nu} a_\mu b_\nu = -a^0 b^0 + \underline{a} \cdot \underline{b}$ the indefinite metric. In the momentum space we have the upper mass hyperboloids $V_+^m = \{p=(w, \underline{p}): w = +\sqrt{\underline{p}^2 + m^2}\}$, $m > 0$, and the forward light cone $V_+^0 = \{p=(k, \underline{p}): k = +|\underline{p}|\}$. For $m \geq 0$ we introduce the tangent planes with respect to points $p \in (V_+^m \setminus \beta)$, $\beta = (m, \underline{0})$,

$$(1) \quad T_p = \{h \in C^4: ph = -wh^0 + \underline{p} \cdot \underline{h} = 0\}.$$

Note that $T_p \cong C^3$. Moreover, T_p is well defined if $m > 0$. We construct a helicity basis $\{v_\lambda: \Lambda v_\lambda = \lambda v_\lambda, \lambda=0, \pm 1\}$ on T_p as follows. For given $\underline{n} \in R^3$ non-collinear in \underline{p}

$$(2) \quad v_0 = (1, \underline{p}w/k^2), \quad v_{\pm 1} = (\underline{n} \cdot \underline{p}(\underline{p} \cdot \underline{n})/k^2 \pm i p x_n)/k.$$

We now define the one-particle Hilbert space in the transverse gauge for the free vector meson quantum field of mass $m > 0$ as the direct integral

$$(3) \quad M_m = \int_{V_+^m}^{\oplus} T_p \frac{d^3 p}{2p_0}.$$

As an operator in M_m Λ is called helicity and the square $C = \Lambda^2$ Coulomb projection, where

$$(4) \quad C^{\mu\nu}(p) = \begin{cases} g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} & \mu, \nu = 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

They allow a unique decomposition $\tilde{h} = \tilde{v} \oplus \tilde{l}$, $\tilde{v} \in M_m^C$, where

$$(5) \quad M_m^C = \text{ran } C = \{\tilde{h} \in M_m: \tilde{h}^0 = 0\}$$

and $\tilde{l} \in \ker C$. The restriction to V_+^m defines a unitary equivalence

$$(6) \quad M_m^C \cong \int_{\mu}^{\oplus} L^2(R^3, \frac{d^3 p}{2w}) \cap \{\tilde{h}: -wh^0 + \underline{p} \cdot \underline{h} = 0 \text{ a.e.}\}.$$

In particular $\|1\| = m \frac{V_0}{V} \|L^2(\frac{d^3p}{2w})$, hence for $m=0$ -
 - i.e. for the one-photon space $L^2(\frac{d^3p}{2w})$ - in the Lorentz gauge we
 put the factor space

$$(7) \quad M_{ph} = M_m / M_m^0, \quad m=0,$$

and in the Coulomb gauge $M_{ph}^C = M_m^C$, $m=0$. We write shortly M
 for M_m and M_{ph}^C , respectively. By the conjugation

$$(8) \quad h \rightarrow (h^*)^M(p) = g^{M\nu} \tilde{h}^\nu(-p)$$

we select a $*$ -real part M_r . For $m > 0$ we have dense sub-
 spaces $S^T(V_m^m)$ in M_m of smooth transverse test functions \tilde{h} on V_m^m
 which satisfy $\tilde{h}(\beta)=0$, and similar $\tilde{h}(0)=0$ for $\tilde{h} \in S^T(V_+^0)$.

Gross has identified M_m with a Hilbert space of weak vector-
 valued solutions of the Proca field equations and M_{ph} with a
 Hilbert space of Maxwells equations in vacuum [Gr]. Using the
 L_+^1 -invariant tempered measure $dw_+^m = \theta(p^0) \delta(p^2+m^2) d^4p$ in R^4 and
 the projections $\tilde{h}(p): C^4 \rightarrow T_p$

$$(9) \quad \tilde{h}^{M\nu}(p) = g^{M\nu} + \frac{p^M p^\nu}{m^2},$$

we can redefine the one-vector-meson space as $M_m = L^2(V dw_+^m)$
 not specifying any gauge.

Therefore in the Euclidean formulation it seems natural to
 introduce for $m \geq 0$ the Hilbert spaces

$$(10) \quad N_m = L^2(\frac{d^4p}{p^2+m^2}), \quad \tilde{h}^{M\nu}(p) = \delta^{M\nu} - \frac{p^M p^\nu}{p^2},$$

where $p^2 = p^2 + (p^0)^2 \geq 0$ and for $p \neq 0$ $E(p)$ are the projections
 onto the Euclidean transverse planes $T_p = \{h \in C^4: p \cdot h = 0\}$. Passing
 to spherical coordinates $p = (\varrho, \omega) \in R^4$, $\varrho = |p|$, the Euclidean
 counterpart of definition (3) is $N_m(\varrho) = \int_{S_\varrho} \tilde{h}_p^M d\omega$, where
 $S_\varrho = \{p: |p| = \varrho\}$. However

$$(11) \quad N_m = \pi^2 / 2 \int_0^\infty \oplus N_m(\varrho) \frac{\varrho^3 d\varrho}{\varrho^2 + m^2},$$

i.e. N_m is reducible with respect to the Euclidean group.

We write shortly N for $N_m, m \geq 0$. The natural conjugation in N is given by the map $h \rightarrow \check{h}(p) = \overline{h(-p)}$. We have a dense subspace S^T in N_m - independent of the mass m - of smooth transverse test functions h in R^4 which satisfy point-wise $p \cdot \check{h}(p) = 0$ and $\check{h}(0) = 0$. The set

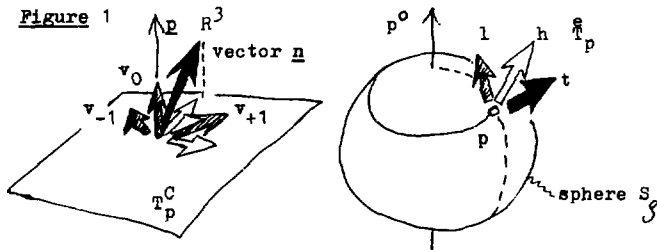
$$(12) \quad N^C = \{h \in N: h^0 = 0\}$$

is a closed subspace in N and it defines a projection $C = C^e$. Let be $N(s) = L^2(R^4, \frac{d^4p}{2^4\pi^2})$ Nelsons one-particle Hilbert space for the free Eucl. $p^2 + m^2$ scalar field of mass $m \geq 0$. Then

$$(13) \quad J_+^0: \check{u} \rightarrow (1, -pp^0/k, 2) \frac{k\check{u}}{ip^0+k}, \quad k = +/p/,$$

is a unitary map from $N(s)$ onto $M^0 = \ker C^e$.

Figure 1



In the next section we pass to the description of the one-particle states in the coordinate representation, by the formal Fourier transform in R^n

$$(14) \quad h^M(x) = (2\pi)^{-n/2} \int_{R^n} e^{ip \cdot x} \check{h}^M(p) d^n p.$$

Using the correspondence $w \rightarrow (-\Delta + m^2)^{1/2}$ in R^3 , we can identify $M(s) = L^2(R^3, \frac{d^3p}{2^3\pi})$ with the dual of the Sobolev space $H_{+1/2}(R^3)$ and we denote the norm by $\| \cdot \|_{-1/2}$. Similar $p^2 \rightarrow -\Delta$ in space-time R^4 . We write $\| \cdot \|_{-1}$ for the norm in $N(s)$, even for $m=0$.

2. Localization, Euclidean and dilatation covariance

By Fourier transform we identify the Euclidean one-particle spaces N_m , $m \geq 0$, for the free Euclidean vector meson fields of masses m with the Hilbert spaces of tempered vector-valued distributions

$$(15) \quad \sum_{\mu}^{\oplus} S(h^{\mu}) \cap \{h = (h^0, \underline{h}) : \nabla \cdot h = 0, \|h^{\mu}\|_{-1}^2 = \int_{R^4} |h^{\mu}(p)|^2 \frac{d^4 p}{p^2 + m^2} < \infty\}$$

We introduce a localization in N - different from the usual one - as follows: For given closed subsets Λ in R^4 we let

$$(16) \quad N_{\Lambda} = N \cap \sum_{\mu}^{\oplus} S(\Lambda)^{\mu} = \{h \in N : \text{supp } h^{\mu} \subseteq \Lambda\}.$$

Those subspaces N_{Λ} are closed in N and we associate with the above localization the family of projections $\{E_{\Lambda} : \Lambda \text{ closed}\}$. If N_{Λ} then also $N_{S=\partial\Lambda}$ is non-trivial, where $\partial\Lambda$ denotes the boundary of Λ . We have $N = \bigcup_{S} N_S$, however, in contrast to the scalar case $N_S = N$ for all S .

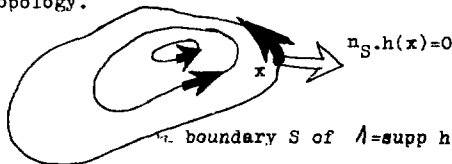
$$(17) \quad \bigcup_{t \in R} N_{S_t} = N^{\perp}, \quad S_t = \{x = (x^0, \underline{x}) : x^0 = t\}$$

is a proper subspace. More general, by Gauss' law the elements h in N_{Λ} should have vanishing normal component $h_{n_S} = 0$ with respect to the boundary surface $S = \partial\Lambda$. N is localizable in the sense that the set of smooth transverse test functions

$$(18) \quad \{h = \Delta E_{\mu} u : u \in S(\mathcal{O}), \mathcal{O} \text{ open}\}$$

is just S^{\perp} which is dense in N . Fixing \mathcal{O} with closure Λ , the Hilbert space completion $N(\mathcal{O})$ in N coincides with N_{Λ} . Note that the operator ΔE in (18) is local and continuous in the Schwartz topology.

Figure 2



The localization of particle states in quantum field theory is still problematic. The Wightman approach uses a suitable family of projection-valued measures on the \mathcal{G} -algebra of Borel sets in R^3 and their eigenstates in the corresponding one-particle Hilbert space for a free field. In this sense vector mesons are localizable.

However, the photon position operator Q^C in M_{ph}^C has non-commuting components and Fleming argued maximal localizability of photons to be on hyperplanes. But contrary to that conjecture, Jadczyk and Jancewicz found generalized helicity states of photons on a curve C in R^3 \underline{h}_λ^C , $\lambda = \pm 1$, for C a straight line and a circle. Due to (2) they are

$$(19) \quad \underline{h}_{\pm 1}^C = (\Lambda_{\pm 1}) \Lambda U \underline{t}^C, \\ Q^C = UC \underline{x} CU^{-1} = i \nabla + i p \underline{x} \underline{\Sigma} / k^2, \quad (\underline{\Sigma}_\alpha)^{\mu\nu} = \varepsilon^{\alpha\mu\nu}$$

where $\underline{t}^C \in \sum_3^{\oplus} S(C)'$ is a normalized tangential vector field on C and $U: M_{ph}^C \rightarrow L^2(\mathbb{C}d^3x)$ a unitary mapping.

In the next section we prove that for $m \geq 0$

$$(20) \quad J_{\underline{h}}^{\underline{h}} = (0, \underline{\delta}_0 \underline{\Theta} \underline{h})$$

defines an isometric embedding of M^C in N with $\text{ran } J_{\underline{h}} = N_{\underline{\delta}_0}$, hence we can identify the photon states described above with the elements $J_{\underline{h}}^{\underline{h}} \in N$. It would be nice to derive them from the localization in the Euclidean model.

Next we describe some covariance properties of the one-particle spaces. For $m > 0$ the Hilbert space M_m carries an irreducible unitary representation $g \rightarrow U_g$, $g = (a, L) \in iL_+^{\uparrow}$, of the inhomogeneous restricted Lorentz group defined formally by

$$(21) \quad (U_g^{\underline{h}})^{\mu}{}_{\nu}(p) = e^{iap} L^{\mu\nu} h(L^{-1}p), \quad p \in V_+^m.$$

In particular $U_t = e^{-itH_0}$ denotes the one-parameter group of time evolution with the flat Hamiltonian $H_0 = +\sqrt{p^2 + m^2}$. Time and space reversal are excluded from L_+^{\uparrow} . Let be Θ_μ , $\mu = 0, 1, 2, 3$ reflections at the hyperplanes $S_\mu = \{x \in R^4: x^\mu = 0\}$.

The helicity operator Λ is a Casimir for $m=0$. Since Λ anti-commutes with space reflection $U_{\underline{0}}$ the one-photon space M_{ph}^C is irreducible for the representation

$$(22) \quad g \rightarrow CU_g C^{-1}, \quad g \in iL_+ U_{\underline{0}}$$

$$U_{\underline{0}}: M^{+1} \rightarrow M^{-1},$$

where C^{-1} denotes the inverse of the unitary map from the Hilbert space M_{ph} of one-photon states in the Lorentz gauge to M_{ph}^C . The full covariance group in M_{ph}^C is the conformal group.

For $m \geq 0$ the Euclidean one-particle space N_m carries a reducible unitary representation $g \rightarrow T_g$, $g=(a, \vec{h}) \in iO(4)$, of the inhomogeneous Euclidean group defined formally by

$$(23) \quad (T_g h)^{\mu}(x) = R^{\mu\nu} h^{\nu}(R(x-a)),$$

and $T_{\theta\kappa}$ are also well defined unitary self-adjoint operators in N . In particular for reflections at S_0

$$(24) \quad (T_{\theta} h)^{\mu}(x) = g^{\mu\nu} h^{\nu}(-x^0, \underline{x})$$

we write shortly θ_0 . The localization in N is related with covariance by

$$(25) \quad T_g E_{\Lambda} T_g^{-1} = E_{\Lambda_g}, \quad g \in iO(4) \cup \{\theta_{\mu}\}$$

Let $E_{\pm 0}$ denote the projections in N onto the subspaces spanned by elements with support in the half-spaces

$$(26) \quad \Lambda_{\pm} = \{x \in R^4: x^0 \gtrless 0\}$$

Λ_{-} and on $S_0 = \Lambda_{+} \cap \Lambda_{-}$, respectively. Then $\theta_0 E_{\pm 0} = E_{\mp}$, however, the reflexivity $\theta_0 E_0 = E_0$ is a non-trivial property. Note that the intersection of the relativistic and Euclidean groups, $iO(3) = L_+^{\uparrow} \cap iO(4)$, consists exactly of all transformations which leave the hyperplane $S_0 \cong R^3$ invariant.

For $m=0$ the full covariance group in N is the Euclidean conformal group. We discuss dilatations $x \rightarrow \lambda x$, $0 < \lambda < \infty$,
 inversions $\theta_I: x \rightarrow \frac{x}{|x|^2}$ and reflections at the spheres
 $S_R = \{x \in \mathbb{R}^n : |x|=R\}$

$$(27) \quad \theta_R: x \rightarrow \left[\frac{R}{|x|} \right]^2 x.$$

For space-time \mathbb{R}^n , $n \geq 3$, on the domain $\sum_{j=0}^n C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we introduce the diagonal operators

$$(28) \quad T_\lambda h(x) = \lambda^{\frac{n+2}{2}} h(\lambda x), \quad \theta_\lambda h(x) = |x|^{-(n+2)} h(\theta_I x)$$

and $\theta_R = \theta_{T_\lambda} \theta_I T_\lambda^{-1} \theta$, $\lambda=R$, where $\theta: x \rightarrow -x$. Note that

$$(29) \quad \theta_I T_\lambda \theta_I = T_\lambda^{-1}.$$

The Fourier transform gives $T_\lambda \hat{h}(p) = \lambda^{1-n/2} \hat{h}(\lambda^{-1} p)$, hence the operators of dilatations conserve the transversality $p \cdot \hat{h}(p) = 0$ and build a multiplicative unitary group in N , $m=0$, while for $m>0$ they rescale the mass parameter, too.

Since θ_I does not commute with the projection E , we shall study these operators first in $N' = \sum_{j=0}^n \mathcal{G}_N(s)$ with the scalar product given by the kernel

$$(30) \quad \frac{\delta_{\mu\nu}}{c_n |x-y|^{n-2}},$$

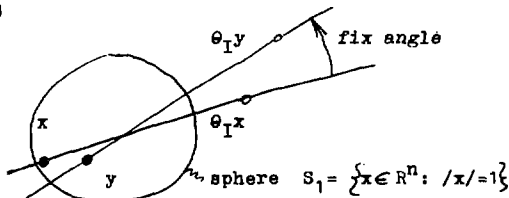
where c_n is the area of the unit sphere in \mathbb{R}^n .

Lemma 1 θ_I is a unitary self-adjoint operator in N' .

Proof: On the dense set $\{h = n^m e^{i(x \cdot D)^2} D(x) e^{-2|x|^2/2}\}$, where D is any differential operator with constant coefficients in \mathbb{R}^n , θ_I is well defined and symmetric. The fact that θ_I is isometric depends on $d^n(\theta_I x) = (-1)^{n+1} |x|^{-2n} d^n x$ and

$$(31) \quad |\theta_I x - \theta_I y| = \frac{|x-y|}{|x| \cdot |y|} \quad \text{for } x \neq y. \quad \text{Q.E.D.}$$

Figure 3



Of course for $a \in \mathbb{R}^n$, $\theta_I^T a \theta_I$ are unitary operators in N' , $m=0$, of special Euclidean conformal transformations. Since dilations leave the hyperplane S_0 invariant

$$(32) \quad U_\lambda^C = J_0^* T_\lambda J_0$$

are unitary operators in M_{ph}^C . In the scalar case, let us denote by $j_0: \mathbb{R}^n \rightarrow \mathcal{S}_0 \otimes \mathbb{C}$ the isometric embedding of $L^2(d\mathbb{P}/k)$ in $N(\mathbb{S})$ with $\text{ran } j_0 = N_{S_0}(\mathbb{S})$. Then in coordinate representation and for space-time \mathbb{O} of dimension $n=d+1 \geq 3$, we find

$$(33) \quad j_0^* \theta_I j_0: u \rightarrow |x|^{-(d+1)} u\left(\frac{-x}{|x|^2}\right).$$

This result coincides with the unitary representation of inversion on the one-particle Hilbert space of the free scalar massless field given by Swieca and Völkel.

Note that θ_I and θ_R are well defined in the subspace $U \cap N_{S_R}$ of N spanned by elements with vanishing radial $k > 0$ component. In particular we have $\theta_R^* E_{S_{R_1}} \theta_R = E_{S_{R_2}}$, $R_1, R_2 = \mathbb{R}^2$, and we have the inversion property

Lemma 2

$$(34) \quad \theta_R^* E_{S_R} = E_{S_R}, \text{ for } m \geq 0.$$

Proof: For $h \in N_{S_R}$ the transversality $\nabla \cdot h = 0$ implies just $x \cdot h = 0$, hence \mathbb{R} passing to spherical coordinates $x = (r, \underline{\omega}) \in \mathbb{R}^n$, $h = J_R \underline{t} = (0, \delta_R \underline{t})$.

Q. E. D.

3. Reflection positivity and Hamiltonian semi-groups

First we remark that the localizations of the transverse Euclidean one-particle states in Λ_+ , S_0 and in some other closed subsets of R^4 we can describe in momentum representation, and we need not refer to coordinate representation at all.

Using a version of the Paley-Wiener-Schwartz theorem, we find that the components $\hat{h}^m(\cdot, p)$ for $h \in N_+$, $m > 0$, have analytic continuations to the complex half-plane $C_+ = \{p^0; \text{Im } p^0 > 0\}$ a.e. with respect to the Lebesgue measure d^3p . More precisely

$$(35) \quad \hat{h}^m(\cdot, p) \in \text{HL}^2\left(\frac{dp^0}{(p^0)^2 + w^2}\right), \quad w = +\sqrt{p^2 + m^2},$$

in particular $\hat{h}^m \in L^2(d^3p/2w)$, where $\hat{h}^m(p) = \hat{h}^m(iw(p), p)$ denotes the restriction of the Laplace transform of $h^m(x)$ to V_+ . A similar result we get for $m=0$.

Then by continuity arguments, for $h \in N_{S_0} = N_+ \cap N_-$

$$(36) \quad \hat{h}^m(p) = \sum_{\substack{\text{finite} \\ c^m=0,1,2,\dots}} c^m \hat{h}^m(p),$$

which in coordinate representation expresses the fact that $h^m(\cdot, \underline{x})$ is a finite derivative of $\delta_0(x^0)$.

From $\|\hat{h}^m\| < \infty$ follows $c^m=0, m=0,1,2,3$. From the transversality $\hat{h}^1(p)=0$ a.e. we derive $h^0=0$. Then we get $\|\hat{h}^3\| = \|\hat{h}^2\|_{-1/2}$, $\beta=1,2,3$ and $p \cdot \hat{f}=0$. Therefore, in momentum representation, any element $\hat{h} \in N_{S_0}$ has the form $\hat{h} = J_0 \hat{f} = (0, 1\theta \hat{f})$, $f \in M^C$.

Lemma 3 $E_+ \theta_0 E_+ \geq 0$.

Proof: For real $h \in N_+$ by the Cauchy integral formula

$$(37) \quad \langle h, \theta_0 h \rangle_N = \int d^3p \int_{p^0+iw}^{dp^0} \left[\frac{g^m \hat{h}^m(p) \hat{h}^m(\bar{p})}{p^0+iw} \right] \\ = \mathcal{P} \int d^3p/w \left(-\hbar^0 / 2 + \hbar^2 / \hbar \right).$$

The property $\nabla \cdot h = 0$ implies

$$(38) \quad w/h^0 = /p \cdot \underline{h} / \leq w/\underline{h} / \quad \text{a.e.,}$$

and since θ_0 is symmetric in the real part N_r , (37) generalizes to complex h as well. With the decomposition $h = \underline{t} \oplus 1$ we can rewrite the nominator as

$$(39) \quad \underline{t}, \underline{t}^0 = / \sqrt[4]{\rho} \cdot 2^{N_0} (1^0)^0, \quad \forall = (1, -\rho p^0 / k^2),$$

hence

$$(40) \quad \langle h, \theta_0 h \rangle = \sum_N \sum_{\beta=1,2,3} \|t^\beta\|^2 + m^2 \|k^{-1} 1^0\|^2. \quad \text{Q.E.D.}$$

From the formula (40) we expect $E_+ \theta_0 E_+ \rightarrow E_{S_0}$ as $m \downarrow 0$. By an argument of Hegerfeldt this may happen if S_0 and only if $E_+ E_- = E_{S_0}$, $m=0$. Conversely, for $m > 0$ the Markoff property with respect to the hyperplane S_0 is valid in the subspace N^C only. Note that the discussion on page 12 explains the reflexivity $\theta_0 E_{S_0} = E_{S_0}$, $m \geq 0$.

Moreover, $E_{S_0} = C e_{S_0}$ since C and e_{S_0} commute in N' and

$$(41) \quad N_{S_0} = \{ h \in N' : h^0 = 0, \nabla \cdot h = 0 \text{ and } \text{supp } h \subset S_0 \}.$$

Lemma 4 The localization $\Lambda \rightarrow N_\Lambda$ defined by formula (16) satisfies the Markoff property, exactly for $m=0$.

Proof: We shall prove the Markoff property with respect to S_0 , however, our proof generalizes to any smooth hypersurface. Let be φ a vector test function with components $\varphi^M \in S(\Lambda_+^{\text{int}})$. Then $E\varphi \in N$, and for given $h \in N_-$

$$(42) \quad (E_+ h, \varphi) = \langle E_+ h, (-\Delta + m^2) E\varphi \rangle_N$$

vanishes exactly for $m=0$, since then $(-\Delta + m^2) E\varphi$ has support in Λ_+^{int} , too. Therefore, $E_+ E_- = E_-(E_+ E_-) = E_{S_0}$, $m=0$.

Q.E.D

In the momentum representation the Markoff property for $m=0$ results from the fact that $p^{2MMV}(p)$ is a polynomial.

Naturally there arises the question of a possible orthogonal local decomposition in N with respect to a given sufficiently smooth hypersurface. For the hyperplane S_0 - and similar for any other by Euclidean covariance - let us regard the subspaces $N_{>} = N_+ \ominus N_{S_0}$ and $N_{<}$ with the corresponding projection operators $E_{>} \ominus E_{S_0}$ in N , E_{\geq} , where $E_{<} = \theta_0 E_{>} \theta_0$.

For $m=0$ the problem is solved by the identity

$$(43) \quad E_{>} \oplus E_{<} \oplus E_{S_0} = 1_N,$$

which results from the Markoff property. However, for $m>0$ (40) implies that $\langle 1, \theta_0 1 \rangle_N \neq 0$ for some $1 \in N_+^0$. Hence in this case $N_{>}$ and $N_{<}$ are not orthogonal to each other and there is no such kind of partition of unity.

Explicitly, using that the Fourier transform of $\frac{\delta(x)}{p^{2+m}}$ for $m=0$ is just $\frac{4\pi/x}{p^{2+m}}$

$$(44) \quad \langle 1, \theta_0 1 \rangle_N = - \iint_{R^3 \times R^3} \frac{dx dy}{|x-y|} \int_{-\infty}^{+\infty} 1^0(s, \underline{x}) 1^0(-s, \underline{y}) ds = 0.$$

Having reflection positivity in the Euclidean one-particle space N we can start the Osterwalder Schrader reconstruction. First we regard the "physical subspaces" in N which we shall denote by N_{M_m} and $N_{M_{ph}^C}$, respectively.

Lemma 5 Let us define for $m \geq 0$ $N_{M_m} = N_+ \ominus \{h: E_{-h}=0\}$. Then $N_{M_{ph}^C} = N_{S_0}$ and using $J_+^0 \uparrow_{N_{S_0}}(s) =: J_{S_0}^0$

$$(45) \quad N_{N_m} = N_{S_0} \oplus \text{ran } J_{S_0}^0, \text{ for } m > 0.$$

Proof: The Markoff property in N^C implies that $\{t \in N_+^C: E_{-t}=0\} = N_{>}^C$, hence $N_{M_m^C} = N_{S_0}$ for $m \geq 0$.

Then $\text{ran } J_+^0 \Big|_{N_+}(s) = N_+ \cap \ker G$ gives for $l = J_+^0 u$

$$(46) \quad E_+ \theta_0 E_+ \tilde{l} = m^2 J_+^0 \frac{v}{(w+p/\rho)}^2, \text{ supp } u \subset \Lambda_+.$$

For $m > 0$ (46) vanishes exactly if $\tilde{u}(i w, p) = 0$ a.e., and the restriction of the operator $(E_+ \theta_0 E_+)^{1/2}$ to N_M^0 corresponds to the map $\tilde{u} \rightarrow \frac{m \cdot \tilde{u}}{w+p/\rho}$ in $L^2(d^3 p/2w)$. Performing the Fourier Euclidean time variable transformation in the we find for $l \in N_M^0$

$$(47) \quad l(x^0, p) = (1, -i p \nabla^0 / k^2) \vartheta(x^0) e^{-p/x^0} / \rho / \tilde{u}(p). \quad \text{Q.E.D.}$$

The lemma 5 allows an isometric embedding of the relativistic one-particle space M into the Euclidean. However, it is more convenient to regard the canonical map $W: N_+ \rightarrow M$ defined by the relation

$$(48) \quad \langle h, \theta_0 h \rangle_N = \|Wh\|_M^2.$$

Lemma 6

$$(49) \quad \tilde{Wh} = (-ih^0, \underline{h})^v, \text{ for } m > 0.$$

Proof: Formally, W is analytic continuation from T_p^e to T_p , $p \in V_+^m$ and p fix. Using

$$(50) \quad W: (1, -pp^0/k^2) \frac{ik}{p^0+ik} \rightarrow (1, pw/k^2) \frac{-ik}{w+k}$$

we see that $\langle l, \theta_0 l \rangle_N = m^2 \|k^{-1} l^0\|^2$. Moreover, W correctly changes the metric, $-1/2$ transversality and conjugation from relativistic to Euclidean.

Alternatively, W will define a map to M_{ph} in the Lorentz gauge and for $m > 0$ to the factor space $L^2(P^m/dv_+^m)$, starting from the factor space

$$(51) \quad N_+ / \{h: E_- h = 0\} \quad \text{Q.E.D.}$$

In the section 2 we remarked that those Euclidean rotations, translations and reflections which leave the hyperplane S_0 invariant - here shortly denoted by $\underline{g}=(\underline{a}, \underline{R}, \underline{\theta})$ - induce unitary representations $\underline{g} \rightarrow U_{\underline{g}}$ in M_m . Using the canonical map W we get

$$(52) \quad U_{\underline{g}} W = W T_{\underline{g}} \text{ in } N_+.$$

Lemma 7 For Euclidean time translations $g_t: S_0 \rightarrow S_t$, the relation (52) generalizes to $U_{g_t} W = W T_t$, for $t=is$ with $\text{Im } s < 0$. Hence for $h, h' \in N_+$

$$(53) \quad \langle h', \theta_0 T_t h \rangle_{N_m} = (W h', e^{-tH_0} W h)_{M_m}, \quad t \geq 0.$$

For $m=0$

$$(54) \quad U_{\underline{g}, \lambda}^C = J_0^* \pi_{\underline{g}, \lambda} J_0, \\ e^{-t/H_0^C} = J_0^* T_t J_0 \quad (\text{free Feynman-Kac-Nelson formula}),$$

in M_{ph}^C . Moreover, the dilatations T_λ generate a continuous multiplicative semi-group of self-adjoint contractions in the subspace $N_{S_R} = \{h \in N: \nabla \cdot h = 0 \text{ and } \text{supp } h \subset S_R\}$

$$(55) \quad D_\lambda = E_{S_R} T_\lambda, \quad \lambda \geq 1.$$

Proof: The first part of the lemma simply generalizes Nelsons reconstruction of the free scalar Hamiltonian to the vector model. Since T_t and e_{S_0} commute with the projection C and $E_+^C \theta_+ E_+^C = E_{S_0} = G e_{S_0}$, (53) follows from

$$(56) \quad E_+ \theta_0 E_+ T_t^{-1} = \frac{m^2}{(w+p/2)^2} J_{S_0}^0 (T_t u)^v,$$

for $1 = J_{S_0}^0 u, \quad \hat{u} \in L^2(d^3p/2w)$.

The fact that $\{D_\lambda: \lambda \geq 1\}$ - and similar for $\lambda \leq 1$ - defines a self-adjoint semi-group in N_{S_R} , $m=0$, follows from the Markoff and inversion properties. Q.E.D.

In a forthcoming publication we will prove that in the parametrization $\lambda = e^{ct/R}$, $t \geq 0$, the semi-group can be expressed in the form

$$(57) \quad D_t = e^{-\frac{tH}{R}}, \quad H_R \geq \frac{2\hbar c}{R}.$$

Summary: In part I of our paper we have shown that the notion of transverse gauge is a good concept for the description of one-particle states of the free vector meson quantum field of mass $m > 0$ and the photon field, respectively. We derived the Hamiltonian semi-groups from a corresponding Euclidean model.

In the parts II and III we shall discuss the main properties of Euclidean transverse random fields. (JINR-preprints E5-I2779, I2781)

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