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Bäcklund Transformations, Conservation Laws and Linearization of the Self-Dual Yang-Mills and Chiral Fields

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Abstract

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Abstract

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Introduction

It has become increasingly clear that, besides its mathematical beauty, the Yang-Mills theory may provide the key to our understanding of strong interactions. With the recent experimental observation of gluon jets[2], the idea of non-Abelian gauge theory for strong interactions is brought one step further to reality. Despite many interesting theoretical and phenomenological observations, like confinement,[3] asymptotic freedom[4] and QCD perturbative studies[5], the non-Abelian gauge theory is far from fully solved.

In the past ten years or so, powerful mathematical tools have been developed in completely solving many two-dimensional non-linear differential equations[6], e.g. obtaining infinite number of conservation laws, soliton solutions and even the construction of S-matrix.[7] Interestingly it has been observed that many of these two-dimensional theories bear resemblances[8] to that of Yang-Mills theory, e.g. topological solutions, asymptotic freedom and confinement. In addition, it has been shown that if properly formulated, the equations for the self-dual Yang-Mills[9],[10] as well as the full Yang-Mills[11] resemble those of the two-dimensional chiral fields. Therefore it is hopeful that the mathematical tools used for the two-dimensional theories can also be developed for the four-dimensional non-Abelian gauge theory. Here I shall report some of the progress made in A. Ogielski,[12] M. K. Prasad and A. Sinha,[10] plus the new result of the linearization of the self-dual Yang-Mill fields.

In Section I, the Backlund transformation (BT) is discussed in the classical case of the Sine-Gordon equation, to which Backlund[6] introduced his transformation in 1875. It is then ... how infinite numbers of local currents can be obtained using the parameter in the BT. In Section II, we give a parallel discussion for the chiral field. In addition we show as an example how to linearize the chiral fields from the infinite number of non-local currents. In Section III, the properly formulated self-dual Yang-Mills equations are shown to have a two-parameter BT and an infinite number of non-local currents. Then the new result of linearizing the self-dual Yang-Mills equations is given. Finally we mention the hope given by the loop formulations(11) of using these techniques for solving the full Yang-Mills equation.

I. The Sine-Gordon equation---the BT and its applications

To understand what BT is, we give a concrete example. The Sine-Gordon equation (S-G) is a second-order differential equation,

$$\alpha_{,\zeta\eta} = \sin\alpha. \tag{I.1}$$

Its BT is a pair of coupled 1st-order differential equations:

$$\tfrac{1}{2}(\alpha' - \alpha)_{,\zeta} = \gamma \sin\tfrac{1}{2}(\alpha' + \alpha), \tag{I.2a}$$

$$\tfrac{1}{2}(\alpha' + \alpha)_{,\eta} = \gamma^{-1} \sin\tfrac{1}{2}(\alpha' - \alpha), \tag{I.2b}$$

where α and α' are functions of ζ, η; the subscript with a comma means differentiation. Given a function α, which is a solution to Eq. (I.1), α' satisfying Eqs. (I.2a) and (I.2b) will also satisfy Eq. (I.1). Thus BT can be loosely defined as coupled set of 1st-order differential equations which relate two functions such that each of the two functions satisfies a designated second-order equation. It is elementary to show Eqs. (I.2a) and (I.2b) have such properties:

$$\text{(I.2a)}_{,\eta} \rightarrow \tfrac{1}{2}(\alpha'-\alpha)_{,\zeta\eta} = \gamma \frac{1}{2}(\alpha'+\alpha)_{,\eta} \cos\tfrac{1}{2}(\alpha' + \alpha)$$

$$= \sin\tfrac{1}{2}(\alpha'-\alpha)\cos\tfrac{1}{2}(\alpha'+\alpha) \tag{I.3a}$$

$$\text{(I.2b)}_{,\zeta} \rightarrow \tfrac{1}{2}(\alpha'+\alpha)_{,\eta\zeta} = \frac{1}{\gamma}\ \frac{1}{2}(\alpha'-\alpha)_{,\zeta} \cos\tfrac{1}{2}(\alpha'-\alpha)$$

$$= \sin\tfrac{1}{2}(\alpha'+\alpha)\cos\tfrac{1}{2}(\alpha'-\alpha), \tag{I.3b}$$

$$\text{(I.3a)}+\text{(I.3b)} \rightarrow \alpha'_{,\zeta\eta} = \sin\alpha', \tag{I.4a}$$

$$\text{(I.3a)}-\text{(I.3b)} \rightarrow \alpha_{,\zeta\eta} = \sin\alpha. \tag{I.4b}$$

So α and α' individually ("strong" BT) satisfy the same ("auto" BT) second-order differential equation, the S-G equation; these two properties are characterized[13] as "strong" and "auto" respectively as written in the parentheses .

The first use of the BT is to generate new solutions from a given solution (which can be a very trivial kind) by solving first-order differential equation rather than the original second-order differential equations, e.g. given $\alpha=0$, a trivial solution to Eq. (I.1), we can solve Eqs. (I.2a), (I.2b) to obtain the non-trivial one soliton solution

$$\alpha' = 4\tan^{-1}[\pm\mathrm{Exp}(\gamma\zeta + \gamma^{-1}\eta + c)], \qquad (I.5)$$

where c is an integration constant. Multisoliton solutions can also be generated.[14]

The second use of the BT is to obtain infinite number of local conservation laws from the parameter γ. First we need to obtain a continuity-like equation involving both α and α': multiplying Eq. (I.2a) by $\frac{1}{2}(\alpha'-\alpha)_{,\eta}$ and Eq. (I.2b) by $\frac{1}{2}(\alpha'-\alpha)_{,\zeta}$ gives

$$\gamma[\cos\tfrac{1}{2}(\alpha'+\alpha)]_{,\eta} - \frac{1}{\gamma}[\cos\tfrac{1}{2}(\alpha'-\alpha)]_{,\zeta} = 0. \qquad (I.6)$$

Note that as $\gamma = 0$, $\alpha' = \alpha$ in Eq. (I.2a). So we expand α' near $\gamma = 0$, i.e.

$$\alpha' = \alpha_0 + \gamma\alpha_1 + \gamma^2\alpha_2 + \cdots \quad . \qquad (I.7)$$

Substituting Eq. (I.7) into the BT Eqs. (I.2a), (I.2b), we can solve for α_i,

$$\alpha_0 = \alpha, \quad \alpha_1 = 2\alpha_{,\zeta}\ , \ \alpha_2 = \alpha_{,\zeta\zeta} \cdots \quad . \qquad (I.8)$$

Substituting Eq. (I.8) into Eq (I.7) and then into Eq. (I.6) and collecting terms of the same power in γ, one can then obtain

(I.9)

(I.10)

infinite number of constraints, which relate α with higher and higher order of differentiations.[15] These infinite number of conservation equations, in the quantum mechanical version, gives severe constraints and implies no particle production in a scattering.[7] This forms the basis of constructing the S-matrix [7] for the S-G equation. Though the existence of both solitons and the infinite set of conservation equations must be related, it appears that the latter is more relevant for the quantum mechanical system.

Though physicists recently christened Eq. (I.1) as the Sine-Gordon equation, the equation was well known to the mathematicians in the 19th century. Eq. (I.1) defines a surface of constant Gaussian curvature -1, i.e. a pseudosphere, if α is taken to be the angle between two asymptotes.[16] The BT generates a new pseudosphere from the old one with the following characteristics[17]: they share common tangents, the distance between the two points sharing the common tangent, and the angle θ between the normals at the two points are kept constant. The free parameter γ in Eqs. (I.2a) and (I.2b) is related to θ by $\gamma = \cot\theta + \csc\theta$. $\theta = 90^{\circ}$ is a special case first given by Bianchi.[17] The geometric meaning of the first soliton generation by the BT is shown in Fig. (1). Starting with $\alpha=0$, a limiting case of a pseudosphere, the first soliton α' as given by Eq. (I.1) with $\gamma=1$ corresponds to the surface of the revolution of a tractrix.

Therefore, whenever a BT is constructed, mathematicians will ask its geometric interpretations and physicists will look for new soliton solutions and possible conservation laws if there are parameters in the BT.

II. Chiral fields---parametric BT, local and non-local conservation laws, and linearization

Since BT is so useful, one may ask if there is a systematic way of finding BT for a given second-order differential equation. As far as I know, the answer is negative. BT is found by guesswork. Here I shall discuss the BT with two parameters for the principal chiral fields (generalization to chiral field is trivial). The BT here is guessed and motivated from our work for the self-dual Yang-Mills field in four dimensions[10], which I shall discuss in the next section.

The principal chiral fields $g(\zeta,\eta)$ of group SU(N) are n x n matrix fields, which have the following Lagrangian density and constraints:

$$\mathscr{L} = \mathrm{Tr}(\partial_\zeta g)(\partial_\eta g^+) \text{ , with } g^+g = gg^+ = I. \qquad \text{(II.1)}$$

Defining

$$A_\zeta \equiv g^+\partial_\zeta g, \quad A_\eta \equiv g^+\partial_\eta g, \qquad \text{(II.2)}$$

equation of motion obtained from Eq. (II.1) is

$$\partial_\zeta A_\eta + \partial_\eta A_\zeta = 0. \qquad \text{(II.3)}$$

Notice here that A_ζ, A_η form the algebra of SU(N) and they can be considered as pure gauge potential due to $\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta] = 0$. Eq. (II.3) has the appearances of a continuity equation. Eq. (II.2) and Eq. (II.3) characterize the most important properties of the system, i.e. curvature less gauge potential and continuity like equation satisfied by the potential. These properties are shared by many non-linear differential equations including the properly formulated self-dual Yang-Mills equations, as we shall discuss in the next section.

Bäcklund transformation and local conservation laws: The BT we have constructed for the principal chiral fields is:

$$g'^{+}\partial_\zeta g' - g^{+}\partial_\zeta g = \partial_\zeta (g^{+}g'), \quad \text{(II.4)}$$

$$g'^{+}\partial_\eta g' - g^{+}\partial_\eta g = -\partial_\eta (g^{+}g') \quad \text{(II.5)}$$

with the constraint

$$g^{+}g' + g'^{+}g = 2\beta I,\ \beta \leq 1 \quad \text{(II.6)}$$

and

$$g^{+}g' = g^{+}g = 1. \quad \text{(II.7)}$$

β is a constant parameter. It is easy to show that Eq. (II.4) and (II.5) are the BT. $\text{(II.4)},_\eta + \text{(II.5)},_\zeta \rightarrow \partial_\eta A'_\zeta + \partial_\zeta A'_\eta - \partial_\eta A_\zeta - \partial_\zeta A_\eta = 0.$ (II.7) So A'_ζ and A'_η satisfy the equation of motion Eq.(II.3), if A_ζ, A_η do. Unlike the BT for the S-G equation, this is a "weak" BT. The condition Eq. (II.6) can be shown to follow from Eqs. (II.4) and (II.5).

Since there is a parameter in the BT, we can now try to derive local currents just like in the S-G case. For this purpose our experience taught us that it's better to re-write Eq (II.4), (II.5) in the following form:

$$2(1-\beta)\partial_\zeta (g'+g) = (g'-g)[(\partial_\zeta g^{+})g' + g'^{+}(\partial_\zeta g)], \quad \text{(II.8)}$$

$$2(1-\beta)\partial_\eta (g'-g) = -(g'+g)[(\partial_\eta g^{+})g' + g'^{+}(\partial_\eta g)]. \quad \text{(II.9)}$$

Then incorporating equation of motion Eq. (II.3), a continuity-like equation could be obtained,

$$(1-\beta)\partial_\zeta \{\mathrm{Tr}[(\partial_\eta g^{+})g' + g'^{+}(\partial_\eta g)]\} + (1+\beta)\partial_\eta \{\mathrm{Tr}[(\partial_\zeta g^{+})g' + g'^{+}(\partial_\zeta g)] = 0. \quad \text{(II.10)}$$

This is analogous to Eq. (I.6) for the S-G equation. Using the procedure demonstrated in Section I, local conservation laws can be derived by expanding around $\beta=1$.

We first discuss a special case of the O(3) σ-model. In this case,

$$g^{+} = g = \vec{q} \cdot \vec{\sigma} \quad \text{and} \quad \vec{q} \cdot \vec{q} = 1, \tag{II.11}$$

where $\vec{\sigma}$ are the Pauli matrices . The BT of Eqs. (II.8) and (II.9) and the constraint equation (II.6) become

$$2(1-\beta)\partial_\zeta(\vec{q}\,' + \vec{q}) \quad (\vec{q}\,' - \vec{q})[(\partial_\zeta \vec{g})\cdot\vec{g}\,'], \tag{II.12}$$

$$2(1+\beta)\partial_\eta(\vec{q}\,' - \vec{q}) = -(\vec{q}\,' + \vec{q})[(\vec{\partial}_\eta \vec{q})\cdot\vec{q}\,'], \tag{II.13}$$

and

$$\vec{q}\,' \cdot \vec{q} = \beta. \tag{II.14}$$

The continuity equation Eq. (II.10) becomes [18]

$$(1-\beta)\partial_\zeta[(\partial_\eta\vec{q})\cdot\vec{q}\,'] + (1+\beta)\partial_\eta[(\partial_\zeta\vec{q})\cdot\vec{q}\,'] = 0. \tag{II.15}$$

Define by

$$\begin{aligned} \tfrac{1}{2}(1-\beta) &\equiv \sin^2\theta = \theta^2 - \theta^4/3 + 2\theta^6/45 + \ldots , \\ \tfrac{1}{2}(1+\beta) &\equiv \cos^2\theta = 1 - \theta^2 + \theta^4/3 - 2\theta^6/45 + \ldots , \end{aligned} \tag{II.16}$$

and expand $\vec{q}\,'$ around $\theta \approx 0$,

$$\vec{q}\,' = \vec{a}_0 + \theta\vec{a}_1 + \theta^2\vec{a}_2 + \ldots \tag{II.17}$$

Substituting Eqs.(II.17) into Eq (II.12) and (II.13) and collecting coefficients of different powers in θ, one can obtain

$$\vec{a}_0 = \vec{q},\ \vec{a}_1 = \pm\partial_\zeta\vec{q}/|\partial_\zeta\vec{q}|,\ \vec{a}_2 = \partial_\zeta(\partial_\zeta\vec{q}/|\partial_\zeta\vec{q}|)/|\partial_\zeta\vec{q}| \ , \ \ldots , \tag{II.18}$$

Substituting Eq. (II.18) into Eq. (II.17) and then into Eq. (II.15), an infinite number of local constraints on q and its derivatives are obtained:

$$\partial_\eta[(\partial_\zeta q)^2] = 0,\ \partial_\eta\{[\partial_\zeta(\partial_\zeta\vec{q}/|\partial_\zeta q|)]^2\} = \partial_\zeta[(\partial_\zeta\vec{q})\cdot(\partial_\eta\vec{q})/|\partial_\zeta\vec{q}|] \qquad \text{(II.19)}$$

These are the well-known local conserved local currents first obtained by Pohlmeyer(18) by a very different method. In the quantum version, these local currents were shown by Polyakov(19) to survive, actually in much simpler forms, and put severe constraint of no-particle production in an interaction, thus providing a basis for the construction of S-matrix(20) for the σ-model.

Now come back the general chiral fields. Following the same procedure as in Eq. (II.16), (II.17) and (II.18), except now we are working with matrix:

$$g' = a_0 + \theta a_1 + \theta^2 a_2 + \dots \qquad \text{(II.20)}$$

Substituting into Eq. (II.8) and (II.9) and the constraint equations (II.6) and (II.7), consistent solutions for a_0 and a_1 were found

$$a_0 = g,\ a_1 = \sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}\ . \qquad \text{(II.21)}$$

Substituting Eqs. (II.21), (II.20) into Eq. (II.10) we obtained the first non-trivial conserved quantity

$$\partial_\eta\left(\mathrm{Tr}\sqrt{\partial_\zeta g^+ g_\zeta}\right) = 0, \qquad \text{(II.22)}$$

Using equation of motion Eq. (II.3) and Eq. (II.22) one can obtain for arbitrary n

$$\partial_\eta\left[\mathrm{Tr}\left(\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}\right)^n\right] = 0, \qquad \text{(II.23)}$$

i.e. the eigenvalues of $\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}$ are independent of η. For the next order, a_2 must satisfy the following matrix equations:

$$a_1^+\partial_\zeta a_1 = (\partial_\zeta g^+)a_2 - a_2^+(\partial_\zeta g),$$

$$g^+a_2 + a_2^+g = -4I, \tag{II.24}$$

$$a_1^+a_2 + a_2^+a_1 = 0,$$

where a_1 is given by Eq. (II.21). So far we have not been able to solve these equations for a_2. However, we have demonstrated that local conservation laws do exist for the chiral field.

Actually the BT we found has another parameter. The general BT is

$$g'^+\partial_\zeta g' - g^+\partial_\zeta g = \lambda\partial_\zeta(g^+g') \tag{II.25}$$

$$g'^+\partial_\eta g' - g^+\partial_\eta g = -\lambda\partial_\eta(g^+g'), \tag{II.26}$$

and the constraint $\lambda g^+g' + \lambda^* g'^+g = 2\beta I$; where β is a real constant, λ and its complex conjugate λ^* are also constants. The reader is referred to Ref. (10) for a more detailed exposition.

<u>Non-local conservation laws</u>: Besides local conservation laws, the chiral fields also have non-local conservation laws. The existence of such non-local currents for the σ-model was first obtained by Lüscher and Pohlmeyer.[21] Here I shall demonstrate using the method of Brezin, et al.[22] As I mentioned before, Eq. (II.3) is like a continuity equation. So let's consider A_ζ , A_η to be the first currents, i.e.

$$V_\zeta^{(1)} \equiv A_\zeta = \partial_\zeta\chi^{(1)} \ , \quad V_\eta^{(1)} \equiv A_\eta = -\partial_\eta\chi^{(1)}. \tag{II.27}$$

Such $\chi^{(1)}$ exists and can be obtained from the A's by integration because of the equation of motion Eq. (II.3). The higher currents are then obtained by an iteration procedure. Suppose the n^{th} current $V_\zeta^{(n)}$, $V_\eta^{(n)}$ exist, i.e.

$$\partial_\eta V_\zeta^{(n)} + \partial_\zeta V_\eta^{(n)} = 0, \text{ and } V_\zeta^{(n)} = \partial_\zeta \chi^{(n)},\ V_\eta^{(n)} = -\partial_\eta \chi^{(n)}. \tag{II.28}$$

Then the $(n+1)^{th}$ currents can be constructed from $\chi^{(n)}$ by

$$V_\zeta^{(n+1)} = \mathcal{D}_\zeta \chi^{(n)},\ V_\eta^{(n+1)} = \mathcal{D}_\eta \chi^{(n)}, \tag{II.29}$$

where

$$\mathcal{D}_\zeta \equiv \partial_\zeta + A_\zeta,\ \mathcal{D}_\eta = \partial_\eta + A_\eta \tag{II.30}$$

Now we need to prove $\partial_\eta V_\zeta^{(n+1)} + \partial_\zeta V_\eta^{(n+1)} = 0$.

$$\begin{aligned}
\partial_\eta V_\zeta^{(n+1)} + \partial_\zeta V_\eta^{(n+1)} &= (\partial_\eta \mathcal{D}_\zeta + \partial_\zeta \mathcal{D}_\eta)\chi^{(n)}, && \text{from Eq. (II.29)}\\
&= (\mathcal{D}_\zeta \partial_\eta + \mathcal{D}_\eta \partial_\zeta)\chi^{(n)} && \text{from Eq. (II.3)}\\
&= -(\mathcal{D}_\zeta V_\eta^{(n)} - \mathcal{D}_\eta V_\zeta^{(n)}) && \text{from Eq. (II.28)}\\
&= -(\mathcal{D}_\zeta \mathcal{D}_\eta - \mathcal{D}_\eta \mathcal{D}_\zeta)\chi^{(n-1)}, &&\\
&= 0, \text{ because of Eq. (II.2).} &&
\end{aligned}$$

Therefore the $(n+1)^{th}$ currents constructed from Eq. (II.29) are conserved To obtain the $(n+1)^{th}$ charge we need integration and differentiation of lower charges at different points in ζ and η:

$$\chi^{(n+1)} = \int^{\zeta} \mathcal{D}_{\zeta'} \chi^{(n)} d\zeta' = \int^{\zeta} \mathcal{D}_{\zeta'} [\int^{\zeta'} \mathcal{D}_{\zeta''} \chi^{(n-1)} d\zeta'' = \ldots . \tag{II.31}$$

Thus the term "non-local" is used for these conservation laws.

In the quantum mechanical version, Lüscher[23] showed that in the σ- model these non-conservation laws also imply no particle production, which is the basis for constructing the S-matrix.[20] However, so far the physical origin and meaning of these non-local currents for the chiral field have not been exploited.

<u>Linearization</u>: Now we want to show how to obtain the "linearized" equations, or sometimes named the inverse-scattering equations, for the chiral equations. These equations were known.[24] Here we like to demonstrate the method, which is new and will be used for the self-dual Yang-Mills fields in the next section.

From Eqs. (II.28) and (II.29) we obtain

$$\partial_\zeta \chi^{(n)} = \mathcal{D}_\zeta \chi^{(n-1)}, \tag{II.31}$$

$$\partial_\eta \chi^{(n)} = -\mathcal{D}_\eta \chi^{(n-1)}. \tag{II.32}$$

Multiply Eq. (II.31) by ℓ^n and sum,

$$\sum_{n=1}^{\infty} \ell^n \partial_\zeta \chi^{(n)} =_\lambda \sum \ell^n \mathcal{D}_\zeta \chi^{(n-1)}, \tag{II.33}$$

where ℓ is an arbitrary constant. Eq. (II.33) can be rewritten as

$$\partial_\zeta \left(\sum_{n=0}^{\infty} \ell^n \chi^{(n)} \right) = \ell \mathcal{D}_\zeta \left(\sum_{n=0}^{\infty} \ell^n \chi^{(n)} \right), \tag{II.34}$$

where the sum on the left-hand side of Eq. (II.33) can be extended to n=0 because $\chi^{(0)} = 1$. Now define

$$\Psi \equiv \sum_{n=0} \ell^n \chi^{(n)}, \tag{II.35}$$

which is a function of ζ, η, and ℓ. Eq. (II.34) becomes

$$\partial_\zeta \Psi = \ell \mathcal{D}_\zeta \Psi . \tag{II.36}$$

By similar procedure we obtain

$$\partial_\eta \chi = -\ell \mathcal{D}_\eta \Psi. \tag{II.37}$$

To claim that Eq. (II.36) and (II.37) are the linearized equations for the chiral fields, we need to show that the integrability condition of Ψ from Eqs. (II.36) and (II.37) implies the chiral field. Eqs. (II.36) and (II.37) can be rewritten as

$$\partial_\zeta \Psi = \ell(1-\ell)^{-1} A_\zeta \Psi, \qquad \text{(II.36)'}$$

$$-\partial_\eta \Psi = \ell(1+\ell)^{-1} A_\eta \Psi. \qquad \text{(II.37)'}$$

$$\partial_\eta \text{(II.36)'} + \partial_\zeta \text{(II.37)'} \rightarrow$$

$$\ell(1-\ell)^{-1}[(\partial_\eta A_\zeta)\Psi + A_\zeta \partial_\eta \Psi] + \ell(1+\ell)^{-1}[(\partial_\zeta A_\eta)\Psi + A_\eta \partial_\zeta \Psi] = 0. \tag{II.38}$$

Using Eqs. (II.36)' and (II.37)' in Eq. (II.38) and after simple manipulations, one obtains

$$\{\partial_\eta A_\zeta + \partial_\zeta A_\eta + \ell(\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta])\}\Psi = 0. \tag{II.39}$$

We see that the integrability of Ψ for arbitrary ℓ implies $\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta] = 0$, and $\partial_\eta A_\zeta + \partial_\zeta A_\eta = 0$. These are just the conditions of curvaturelessness of the gauge potential A and the continuity-like equation of Eq. (II.2) and (II.3). Notice that if we define $\lambda = \ell^{-1}$, Eqs. (II.36)' and (II.37)' are just the inverse-scattering equations of Zakharov and Mikhailov[24] for the chiral fields. Thus we see that the existence of conserved non-local currents is closely related to the fact that there is an arbitrary parameter in the linearized equations. This aspect should be further analyzed in order to shed light on the meaning of those non-local currents.

III. Self-dual Yang Mills Field -- BT, Non-local currents and linearization

After the historical discovery of the one instanton solution[25] to the self-dual Yang-Mills field in 4-dimensional Euclidean space, general multi-instanton solutions were constructed.[26] They are actually proven to be all the solutions giving finite, minimum actions of the Yang-Mills field.[27] However, since the construction of the Polyakov-'t Hooft[28] monopole, which in the Bogomolny, Prasad-Sommerfield[29] is a self-dual finite energy solution depending on three dimensions in the 4-dimensional Euclidean space, no finite-energy multi-monopole solutions have been found, neither the proof of their non-existence. It is therefore important to pursue other methods which may help to find such solutions. Further, it is interesting in its own right to exploit the possibility of making similar, successful mathematical developments for 4-dimensional fields analogous to those for the two-dimensional theories. Indeed we find that the similarities have been striking, e.g. the BT, the conservation laws, and now even the inverse-scattering formulation. Conversely, our studies of 4-dimensional theories also helped to make progress for the 2-dimensional theories, like the BT for chiral fields discussed in Sec. II.

After a lengthy reformulation,[30] the self-dual equations

$$F_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}, \qquad \text{(III.1)}$$

can be written as

$$B_y \equiv J^{-1}\partial_y J, \quad B_z \equiv J^{-1}\partial_z J, \qquad \text{(III.2)}$$

$$\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z = 0, \qquad \text{(III.3)}$$

where $y \equiv (x_1+ix_2)/\sqrt{2}$, $\bar{y} = (x_1-ix_2)/\sqrt{2}$, $z = (x_3-ix_4)/\sqrt{2}$, $\bar{z} = (x_3+ix_4)/\sqrt{2}$; and J is a $n \times n$ Hermitean matrix with $\det J = 1$. The problem of finding solutions to Eq. (III.1) becomes to find a n x n Hermitian matrix J satisfying Eq. (III.3) with det J = 1. The field curvature can be constructed by $F_{u\bar{v}} = -D^{-1}\partial_{\bar{v}}(J^{-1}\partial_u J)D$, where u,v = y,z and is any n x n matrix with det D = 1, and picking a different D corresponds to choosing a different gauge.

Bäcklund transformation: For. Eq. (III.2) a BT can be constructed[10]

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = e^{i\alpha}\partial_{\bar{z}}(J^{-1}J') , \qquad \text{(III.4)}$$

with the algebraic constraint

$$J'J^{-1} - J'^{-1}J = \beta I, \qquad \text{(III.5)}$$

where α, β are real constants. To show Eq. (III.4) with Eq. (III.5) is a BT, we first obtain the following equation

$$J^{-1}\partial_z J - J'^{-1}\partial_z J' = -e^{i\alpha}\partial_{\bar{y}}(J^{-1}J'), \qquad \text{(III.6)}$$

from Eq. (III.4), by taking Hermitian conjugate of Eq. (III.4), and then multiplying J'^{-1} to the left and J to the right, and finally using Eq. (III.5). Now differentiate Eq. (III.4) by $\bar{y}$ and Eq. (III.6) by $\bar{z}$, and then add

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) - \partial_{\bar{y}}(J'^{-1}\partial_y J') - \partial_{\bar{z}}(J'^{-1}\partial_z J') = 0.$$

So J' satisfies the self-dual equation Eq. (III.3) if J does. The BT Eq. (III.4) and (III.5) is thus "weak". It has two paramters[31] α, β. We could also show from Eq. (III.5) alone that the BT is non-auto. For example, starting with a self-dual J giving field of SU(2), J' gives a field of SU(1,1) and vice-versa. In general for a J giving a field of SU(n) with n > 2 using the algebraic equation Eq. (III.5) alone, we can only show that J' must not give a field of SU(n), but we do not know the final group explicitly without solving Eq. (III.4). So far no local conservation laws have been found from this BT, though it has two parameters. This is probably related to the fact that the BT is non-auto. The topological-solution-generating properties of the BT are yet to be studied.

Non-local conservation laws: Notice the striking similarity between the self-dual equations (III.2) and (III.3) and the chiral equations (II.2) and (II.3). Therefore non-local conservation laws can similarly be constructed. Consider B_y and B_z of Eq. (III.3) being the first conserved currents,

$$V_y^{(1)} \equiv B_y = \partial_{\bar z}\chi^{(1)}, \quad V_z^{(1)} \equiv B_z = -\partial_{\bar y}\chi^{(1)}, \tag{III.7}$$

$\chi^{(1)}$ exists because Eq. (III.3). Now suppose that the n^{th} currents exist, i.e.

$$\partial_{\bar y}V_y^{(n)} + \partial_{\bar z}V_z^{(n)} = 0, \tag{III.8}$$

$$V_y^{(n)} = \partial_{\bar z}\chi^{(n)}, \quad V_z^{(n)} = -\partial_{\bar y}\chi^{(n)}. \tag{III.9}$$

Then the $(n+1)^{th}$ currents are

$$V_y^{(n+1)} = \mathcal{D}_y\chi^{(n)}, \quad V_z^{(n+1)} = \mathcal{D}_z\chi^{(n)}, \tag{III.10}$$

where $\mathcal{D}_u \equiv \partial_u + B_u$, $u=y,z$.

Now show the $V^{(n+1)}$'s are conserved;

$$\partial_{\bar y}V_y^{(n+1)} + \partial_{\bar z}V_z^{(n+1)} = (\partial_{\bar y}\mathcal{D}_y + \partial_{\bar z}\mathcal{D}_z)\chi^{(n)}, \text{ from Eq. (III.10);}$$

$$= (\mathcal{D}_y\partial_{\bar y} + \mathcal{D}_z\partial_{\bar z})\chi^{(n)}, \text{ due to Eq. (III.3);}$$

$= -\mathcal{D}_y V_z^{(n)} + \mathcal{D}_z V_y^{(n)}$, using Eq. (III.8);

$= (-\mathcal{D}_y \mathcal{D}_z + \mathcal{D}_z \mathcal{D}_y)\chi^{(n-1)}$, using Eq. (III.10) again;

$= 0$, due to Eq. (III.2).

Therefore, the self-dual Yang-Mills fields have these non-local conservation laws. However their physical meaning is not yet clear.

<u>Linearization:</u> Using the method demonstrated in Sec. II, from Eqs. (III.9) and (III.10):

$$\partial_{\bar{z}}\chi^{(n)} = \mathcal{D}_y \chi^{(n-1)}, \tag{III.11}$$

$$-\partial_{\bar{y}}\chi^{(n)} = \mathcal{D}_z \chi^{(n-1)}. \tag{III.12}$$

Multiplying Eqs. (III.11),(III.12) by ℓ^n (ℓ being an arbitrary constant), summing over n and defining

$\Psi \equiv \sum_{\ell=0} \ell^n \chi^{(n)}$, we obtain

$$\partial_{\bar{z}}\Psi = \ell \mathcal{D}_y \chi , \tag{III.13}$$

$$-\partial_{\bar{y}}\chi = \ell \mathcal{D}_z \chi. \tag{III.14}$$

To show that these equations are indeed linearized equations for the self-dual Yang-Mills equation, we need to show that the integrability of Ψ from Eqs. (III.13),(III.14) gives Eqs. (III.2),(III.3). Eqs. (III.13),(III.14) can be rewritten as

$$(\partial_{\bar{z}} - \ell\partial_y)\Psi = \ell B_y \Psi, \tag{III.15}$$

$$-(\partial_{\bar{y}} + \ell\partial_z)\Psi = \ell B_z \Psi . \tag{III.16}$$

Differentiating Eq. (III.15) by $\bar{y}$ and Eq. (III.16) by $\bar{z}$, and after some simple manipulations, we obtain

$$\ell(\mathcal{D}_y \mathcal{D}_z - \mathcal{D}_z \mathcal{D}_y)\Psi + (\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z)\Psi = 0. \qquad \text{(III.17)}$$

For Eq. (III.17) to be true for all ℓ, we need $\mathcal{D}_y \mathcal{D}_z - \mathcal{D}_z \mathcal{D}_y = 0$, and $\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_{\bar{z}} = 0$, which are precisely Eqs. (III.2) and (III.3). Therefore, Eqs. (III.15),(III.16) are the linearized equations(32) for Eqs. (III.2) and (III.3). Work is in progress to find the solution generating properties of Eqs. (III.15),(III.16). Recently progress has been made in the quantum formulation starting from inverse scattering equations(33) in the two dimensions. Hopefully those methods can be applied here too.

Though much work is still needed, it is already quite surprising that many of the mathematical formulations for 2-dimensional theories can be carried over to 4-dimension via the self-dual Yang-Mills fields. Still our horizon should not end here. Our real goal is to solve the 4-dimensional non-Abelian gauge theory. Such a hope has been raised by the loop formulation,(11) i.e. the classical Yang-Mills theory can be formulated as chiral field in the loop space. It is foreseeable that the methods discussed here will be useful even for the full Yang-Mills theory.

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Figure Caption

Fig. 1: The surface of the revolution of a tractrix, i.e. the surface generated by the Backlund transformation of the Sine-Gordon equation from a straight line (a limiting case of a pseudosphere).

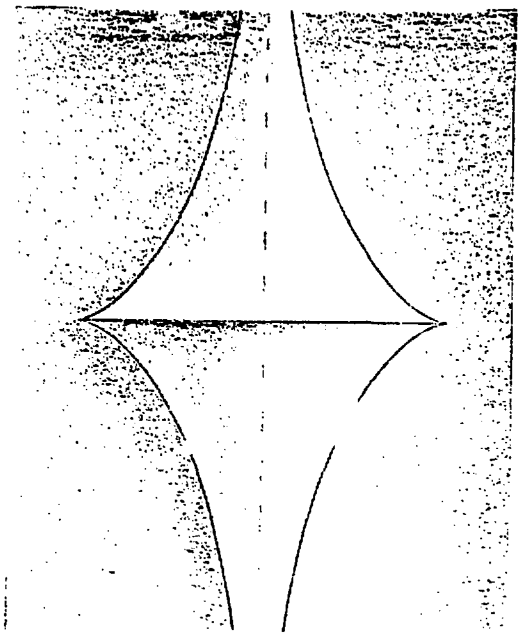

Figure 1