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EDDY CURRENT HEATING OF IRREGULARLY SHAPED PLATES BY SLOW RAMPED FIELDS*

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MASTER

Summary

Theorems are presented for estimating eddy current heating of irregularly shaped plates by a perpendicular ramped field. The theorems, which are derived from two complementary variational principles, give upper and lower bounds to the eddy current heating. Illustrative results are given for rectangles, isosceles triangles, sectors of circular annuli, rhombuses, and L-shaped plates. A comparison is made with earlier work of Sikora et al.[1]

Introduction

Recently, Sikora et al.[1] calculated eddy current heating of thin conducting plates of various shapes by a perpendicular field. They assumed that the magnetic field created by the eddy currents is negligible in comparison with the time-dependent part of the external field. Their method is to introduce the stream function, which satisfies Poisson's equation, and then to solve Poisson's equation with a variational principle, the maximum of which is the eddy current heating. They give approximate formulas for rectangles, isosceles triangles, and sectors of circular annuli.

This paper starts where Sikora's does, namely, with the Poisson equation obtained by introducing the stream function. But instead of just a maximum principle, we introduce a pair of complementary variational principles,[2] one of which is a minimum and the other a maximum. Two such complementary principles give not only an estimate of the eddy current heating, but a bound on the error of the estimate as well.

Because the eddy current heating is the extreme value of the functionals in both principles, both can be used to obtain high order estimates of the eddy current heating (Ritz method[3] and Kantorovich method[4]). But in addition, the complementary variational principles can be used to prove a variety of interesting and powerful theorems for estimating eddy current heating. These theorems are of two different kinds, comparison theorems and geometrization theorems. Comparison theorems compare eddy current heating in plates with different but related shapes. For comparison theorems to be useful, we must know the eddy current heating exactly for certain shapes. Exact results are quoted for the circle, ellipse, and equilateral triangle; the series solution is used for the rectangle.

Geometrization theorems relate eddy current heating to certain geometric quantities dependent on the size and shape of the plate. The most powerful theorem of this class follows from application of Pólya and Szegö's[5] method of assigned level lines, in which the level lines of the stream function are taken to be a family of curves geometrically similar to the boundary curve of the plate. This theorem gives a variational estimate of the eddy current heating in all plates having convex shapes and in many (though not all) plates having reentrant shapes. Evaluation of the required geometric quantities is extremely simple for polygons, for which results can often be written by inspection. Another geometrization theorem relates the eddy current heating to the moments of inertia of the plate. Experience shows that this theorem is less useful than the first geometrization theorem. Attempts to improve it lead to the variational equations of Trefftz.[6]

The plan of the paper is as follows: (1) exhibit the complementary variational principles; (2) quote the exact results for circle, ellipse, equilateral triangle, and rectangle; (3) state, but not prove, the theorems and various corollaries; (4) rework some of Sikora's examples (rectangles, isosceles triangles, sectors of circular annuli) in order to compare his methods with ours; and (5) work some examples for which the methods of this paper are more suitable than Sikora's methods (rhombus, L-shaped plate). Proofs of the theorems and other assertions cannot be given in this paper because of space limitations but can be found in a recent ORNL report.[7]

Basic Equations

Let us choose the axes so that the Z axis is perpendicular to the plate with the positive Z axis pointing in the direction of the applied external field. Let C denote the boundary curve of the plate and A its interior. In Cartesian coordinates, the electrodynamic equations we need in this problem are the following:

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\dot{B} , \tag{1a}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 , \tag{1b}$$

$$P = \frac{1}{A} \iint_A \sigma(E_x^2 + E_y^2)\, dxdy , \tag{1c}$$

where $\vec{B}$ is the externally applied magnetic field, $\vec{E}$ is the electric field induced by it, σ is the conductivity of the plate, A is the area inside C, and P is the average eddy current dissipation per unit volume of the plate, and $\dot{B}$ is the spatially uniform value of $(\partial\vec{B}/\partial t)_z$.

To solve these equations, we begin by introducing a stream function ψ defined by

$$E_x = -\dot{B}\frac{\partial\psi}{\partial y} , \tag{2a}$$

$$E_y = \dot{B}\frac{\partial\psi}{\partial x} . \tag{2b}$$

These equations satisfy (1b) identically. Equations (1a) and (1c) become

$$\nabla^2\psi = -1 , \tag{3a}$$

$$P = \frac{\sigma\dot{B}^2}{A} \iint_A (\nabla\psi)^2 dA . \tag{3b}$$

The boundary condition ψ must obey on C expresses the fact that the current density $\sigma\vec{E}$ at C must be parallel to C. Thus, the normal component of $\vec{E}$

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vanishes on C. If $\vec{n} = (n_x, n_y)$ is the normal vector to C, then (n_x, n_y) is perpendicular to (E_x, E_y) on C or, what is the same thing, (n_x, n_y) is perpendicular to $(-\partial\psi/\partial y,\ \partial\psi/\partial x)$. Because $(-\partial\psi/\partial y,\ \partial\psi/\partial x)$ is perpendicular to $(\partial\psi/\partial x,\ \partial\psi/\partial y)$, the latter is parallel to (n_x, n_y). In other words, $\nabla\psi$ is parallel to $\vec{n}$. However, this means that ψ is constant on C; and because ψ is unique only up to an arbitrary constant, we can choose

$$\psi = 0 \text{ on C.} \tag{4}$$

We solve Eqs. (3a) and (4) and obtain estimates of P using a pair of complementary variational principles.[2] This means we will find two functionals $K[\psi]$ and $J[\psi]$ such that: (1) K is a minimum and J a maximum for the solution of (3a) and (4); and (2) the minimum of K equals the maximum of J, and both are equal to the value of $PA/2\sigma\dot{B}^2$ calculated for the solution of (3a) and (4).

We have

$$K[\phi_2] = \frac{1}{2}\int_A (\nabla\phi_2)^2 dA \geq \frac{PA}{2\sigma\dot{B}^2} \geq \int_A \left[\phi_1 - \frac{1}{2}(\nabla\phi_1)^2\right] dA \equiv J[\phi_1] \ , \tag{5a}$$

where

$$\nabla^2\phi_2 = -1 \text{ but no value of } \phi_2 \text{ is specified on C,} \tag{5b}$$

$$\phi_1 = 0 \text{ on C but is otherwise arbitrary ,} \tag{5c}$$

and the left- and right-hand terms are the same when $\phi_1 = \phi_2 = \psi$, i.e., $K[\psi] = PA/2\sigma\dot{B}^2 = J[\psi]$.

Exact Results

An elementary calculation shows that for a circle of radius R,

$$P = \frac{\sigma\dot{B}^2R^2}{8} = \frac{\sigma\dot{B}^2A}{8\pi} \ . \tag{6}$$

As we shall see later on, for all plates of a given area, the circle has the largest dissipation P. Hence, if we express the dissipation as a fraction f of the dissipation for a circle of equal area, i.e., if we form the fraction $f = 8\pi P/\sigma\dot{B}^2A$, then

$$f \leq 1 \ , \tag{7}$$

with equality if, and only if, the plate is a circle.

For ellipses with semimajor axis b and semiminor axis c, $f = 2\left(\frac{b}{c} + \frac{c}{b}\right)^{-1}$. For equilateral triangles, $f = \frac{2\pi\sqrt{3}}{15} = 0.7255$. For rectangles of dimensions $2b \times 2c$,

$$f = \frac{2}{3}\pi\frac{c}{b}\left[1 - \frac{192}{\pi^5}\frac{c}{b}\sum_{\ell=1,3,5,..} \ell^{-5} \tanh\left(\frac{\pi b\ell}{2c}\right)\right] \tag{8}$$

a result given by Sikora et al.[8]

Inclusion Theorem

Using the right-hand part of the variational principle (6a), we can prove the following theorem. If a plate A is divided into two plates A_1 and A_2 such that $A = A_1 + A_2$, then $PA > P_1A_1 + P_2A_2$ or equivalently $fA^2 > f_1A_1^2 + f_2A_2^2$. This theorem says simply that the total eddy current loss PA in a body is greater than the sum of the eddy current losses in its parts, the latter being reckoned separately. It is only of marginal utility in estimating eddy current dissipation.

Isoperimetric Theorem

The isoperimetric theorem states: Of all plates of a given area A, the circular plate has the largest P. It is a special case of the more general theorem: f and P increase under Steiner symmetrization.[5] When a solid is Steiner-symmetrized, a plane T, called the plane of symmetrization, is selected. The solid is then broken into a bundle of paraxial, infinitesimal, cylindrical elements, all perpendicular to the plane T. These elements are slid parallel to one another until their midpoints all lie in the plane T. Repeated Steiner symmetrization of a plate with respect to an infinitude of planes all containing the normal to the plate will reduce it to a circular plate of the same area. Therefore, any quantity which is increased by Steiner symmetrization will be a maximum for the circular plate.

The increase of f and P under Steiner symmetrization can be used to prove a number of theorems other than the isoperimetric theorem. First, if a semicircle is symmetrized along its diameter, the result is an ellipse with a ratio of major to minor axes equal to 2. Thus, we find that for a semicircle, $f < 4/5$. Second, if we symmetrize a triangle repeatedly about its altitudes, we get in the limit an equilateral triangle. Hence, of all triangles of a given area, the equilateral triangle has the largest f. Third, if we symmetrize a rhombus with diagonals 2b and 2c about an axis perpendicular to a pair of its sides, we get a rectangle, the sides of which are in the ratio $(1/2)(b/c + c/b)$. The value of f for the rectangle, which must exceed that for the rhombus, can be calculated from the series (8). Other theorems can be derived from the increase of f under Steiner symmetrization, but those noted above are the ones we use in this paper.

Method of Assigned Level Lines

Using Pólya and Szegö's method of assigned level lines,[5] we can prove the following theorem. If $r = R(\Theta)$ is the polar representation of the curve C and the quantity a is defined by

$$a = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{R}\frac{dR}{d\theta}\right)^2 d\theta \ , \tag{9a}$$

then

$$P \geq \frac{\sigma\dot{B}^2A}{8\pi(1+a)} \ , \text{ i.e., } f \geq (1+a)^{-1} \ . \tag{9b}$$

Equality holds for circles.

The quantity a is simple to evaluate when the plate has the form of a convex polygon

$$a = \frac{1}{2\pi}\sum_i \frac{s_i}{h_i} - 1 \ , \tag{10a}$$

so that

$$f \geq (1+a)^{-1} = 2\pi\left(\sum_i \frac{s_i}{h_i}\right)^{-1} \ , \tag{10b}$$

where s_i is the length of the ith side of the polygon, and h_i is the length of the altitude on it from the center 0.

The right-hand side of (10b) depends on which interior point of the polygon is chosen as the origin 0. The best estimate of f is obtained for that point 0 which minimizes the sum $\sum_i s_i/h_i$.

Connection with Moments of Inertia

Suppose we choose the origin at the centroid of A and set $I_{xx} = \int_A x^2 dA$, etc. Then

$$P \le \sigma \dot{B}^2 A \frac{I_{xx}I_{yy} - I_{xy}^2}{A^2(I_{xx} + I_{yy})}, \quad \text{i.e.,}$$

$$f \le \frac{8\pi(I_{xx}I_{yy} - I_{xy}^2)}{A^2(I_{xx} + I_{yy})}. \tag{11}$$

There is equality for circular plates.

Because the numerators and denominators of the fractions expressing P and f involve the determinant and trace of the moment of inertia tensor, respectively, these fractions are invariant to rotation of the coordinate axes.

If the origin of coordinates is chosen at a point other than the centroid, then the moments of inertia in the new coordinates are related to those around the centroid by the parallel axes theorem. A straightforward but tedious calculation shows that the fraction $(I_{xx}I_{yy} - I_{xy}^2)/(I_{xx} + I_{yy})$ has its smallest value when calculated with the origin at the centroid, and this gives the lowest upper bound attainable with this theorem.

Rectangular Plate

Shown in Fig. 1 is the fraction f for rectangles of dimension 2b × 2c. The curve marked "series" has been calculated using (8). The curve marked "assigned level lines" is the curve $f = (\pi/2)\ (b/c + c/b)^{-1}$, while the curve marked "moments of inertia" is the curve $f = (2\pi/3) \times (b/c + c/b)^{-1}$. These two curves are lower and upper bounds, respectively, to f, which must lie in the shaded band. Because the estimates differ by a constant factor of 4/3, their geometric mean gives an estimate of f which is never in error by more than 15%.

Triangles

In order to apply (10) to triangles most efficiently, we must choose as origin 0 that point which makes theorem $\sum_i s_i/h_i$ smallest. It turns out that 0 is the common intersection of the angle bisectors. The altitudes h_i from the intersection of the angle bisectors are all equal to the common value 2A/P, where P is the perimeter of the triangle. It follows then from (10) that

$$f \ge \frac{4\pi A}{P^2} \tag{12}$$

The moment of inertia theorem gives

$$f \le \frac{8\pi}{3} \frac{A}{a^2 + b^2 + c^2} \tag{13}$$

for a triangle with sides a, b, c, and area A.

We can test how good these bounds are for equilateral triangles, for which we know f exactly. We find

$$\frac{\pi}{3\sqrt{3}} = 0.6046 \le f = \frac{2\pi}{5\sqrt{3}} = 0.7255$$

$$\le \frac{2\pi}{3\sqrt{3}} = 1.2092. \tag{14}$$

The lower limit, calculated by assigned level lines, is fairly close to the correct value (17% low), but the upper limit, from the moment of inertia theorem, is much too high (67% high).

Sikora et al.[1] have used the Ritz method with a carefully chosen trial function for isosceles triangles with sides 2m, $\sqrt{1+m^2}$ and $\sqrt{1+m^2}$. Their result, obtained from functional J, reads

$$f \ge \frac{4\pi}{5} \frac{m}{1 + 3m^2}. \tag{15}$$

From assigned level lines and the moment of inertia theorem, we have

$$\frac{\pi m}{(m + \sqrt{1+m^2})^2} < f \le \frac{4\pi}{3} \frac{m}{1 + 3m^2}. \tag{16}$$

Sikora's result is exact for equilateral triangles ($m = 1/\sqrt{3}$), a fact which seems to have escaped notice.

The bounds (15) and (16) are plotted in Fig. 2 together with the exact value for the equilateral triangle, which, by the isoperimetric theorem, is also an upper bound. The correct value of f must lie in the shaded area.

Sectors of Circular Annuli

Sikora et al. have used the method of Kantorovich[4] to find a lower limit for f of sectors of circular annuli. Their result is

$$f \ge \frac{2\pi}{3\theta_0} \frac{1-k}{1+k} \left[1 - \frac{\tanh(\beta\theta_0)}{\beta\theta_0}\right], \tag{17a}$$

$$\beta = \left[\frac{2(1-k)^2(1-k^2)}{(1-k^2)(1-8k+k^2) - 12k^2 \ln k}\right]^{1/2}, \tag{17b}$$

where θ_0 is the half-angle of the sector, and k is the ratio of the inner to the outer radius. When k = 0 and $\theta_0 = \pi/2$, the plate is a semicircle. Sikora's formula gives $f \ge 0.7471$, and the symmetrization theorem gives $f \le 0.8000$, as noted earlier. The moment of inertia theorem gives the upper bound $f \le 0.8738$.

For assigned level lines, the origin for the semicircle should be chosen on the normal bisector of the diametral edge 0.545 radii from the diametral edge. Then $f \ge 0.7149$. The geometric mean of the closest estimates, 0.7471 and 0.8000, is 0.7731 and has a maximum percentage error of 3.5%.

Next we turn our attention to very thin circular annuli, i.e., annuli for which $1 - k \ll 1$. We intuitively feel that f for such annuli should be very close to f for rectangles of the "same" dimensions. To express this feeling rigorously, we choose as our trial function the exact stream function for a rectangle of dimensions $2R\theta_0 \times 2\delta$, where R is the mean radius and δ the thickness of the annulus. The error we commit by doing so is $\sim(\delta/R)^2$ because of the variational nature of the functional J (the error in the stream function itself is of order δ/R). Therefore, if we plot f as a function of δ/R, the slope at $\delta/R = 0$ should be the same as calculated from the "equivalent" rectangle. Comparing with (8), we see that this slope is $2\pi/3\theta_0$. Translated to the variable k, this means that $(df/dk)_{k=1} = -\pi/3\theta_0$.

Shown in Fig. 3 are Sikora's curves (17) of f vs k as well as the exact slopes at k = 1. It is easy to see from the figure that Sikora's formula (17) has the correct slope at k = 1 [as can be shown rigorously by expanding (17) in powers of 1 - k]. Some of Sikora's curves (e.g., that for $\theta_0 = \pi/8$) show maxima. For $\theta_0 = \pi/8$, the maximum comes about because as k changes from 0 to 1, the shape of the sector changes from circular (k = 0) to roughly a square ($k \sim 0.5$) and then to a long, thin rectangular strip [$(1 - k) \ll 1$]. From our experience with the symmetrization theorem, we know that equilateral shapes, like the square, have the highest f.

Rhombus

The rhombus is a simple shape not dealt with by Sikora et al. that can be treated by our theorems. Shown in Fig. 4 are three bounding curves for f of the rhombus whose diagonals are 2c and 2b, respectively. The lower bound given by assigned level lines is $f \geq (\pi/2)\ (b/c + c/b)^{-1}$. The upper bound given by the moment of inertia theorem is $f \leq (2\pi/3)\ (b/c + c/b)^{-1}$. If we Steiner-symmetrize a rhombus about an axis perpendicular to a pair of sides, we get a rectangle with a ratio of sides of $(1/2)\ (b/c + c/b)$. The value of f for this rectangle, which is an upper bound to f of the rhombus, can be calculated from Sikora's series (8). The exact value of f must lie in the shaded band.

The L-Shaped Plate

Shown as an insert in Fig. 5 is an L-shaped plate that can be dealt with by the methods of this paper. In applying assigned level lines, we have used the best origin so as to obtain the highest lower bound. We get an upper limit to f useful near a = 1 by using the inclusion theorem and treating the L-shaped plate as a square with a smaller square cut out of it. If we symmetrize the plate about its axis of reflection symmetry, we get a rectangle with isosceles right triangles capping each end. This figure can be handled reasonably well with the moment of inertia method. We can find the exact slope of the curve of f vs a at a = 0 just as we did for the annular sectors. The details are given in Ref. 7. We find $(df/da)_{a=0} = \pi/3$; this slope is shown in Fig. 5. Because f must equal f for a square when a = 1 and have the slope $\pi/3$ when $a \ll 1$, we can draw the dashed curve as a practical estimate for the L-shaped plate. In drawing the dashed curve, we have been guided by the shape of the bound from the method of assigned level lines.

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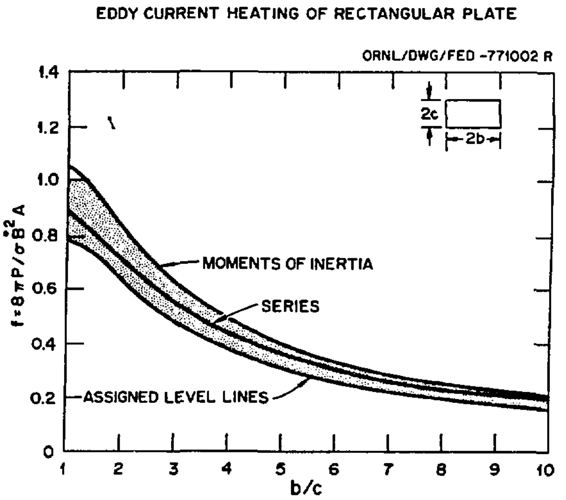

Fig. 1. Various bounds of f for rectangles.

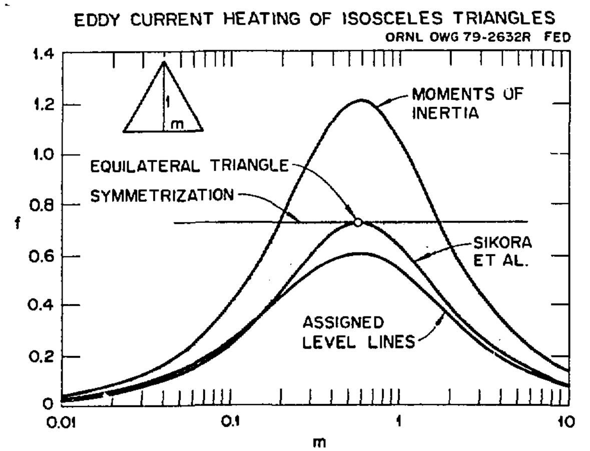

Fig. 2. Various bounds of f for isosceles triangles.

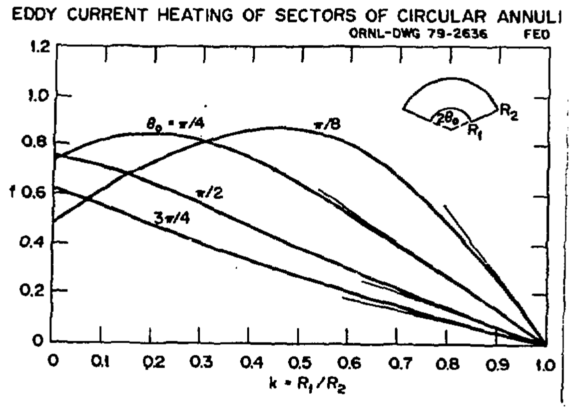

Fig. 3. Sikora's curves of f for sectors of circular annuli together with the exact slopes at k = 1.

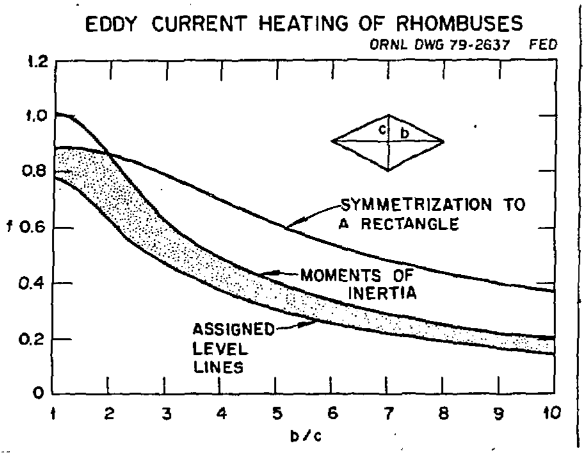

Fig. 4. Various bounds of f for rhombuses.

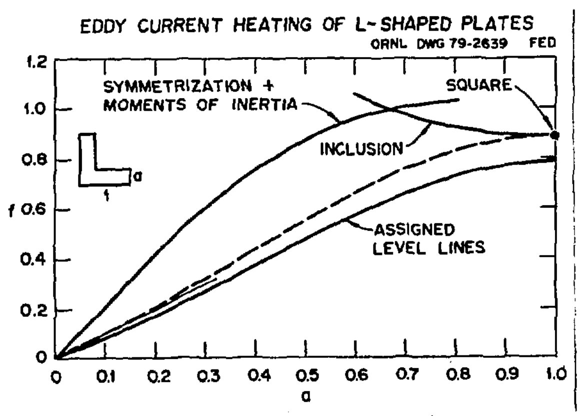

Fig. 5. Various bounds and estimates of f for the L-shaped plate.