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Zero-Point Energy of Confined Fermions

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Abstract

A closed form for the reduced Green's function of massless fermions in the interior of a spherical bag is obtained. In terms of this Green's function, the corresponding zero-point or Casimir energy is computed. It is proposed that a resulting quadratic divergence can be absorbed by renormalizing a suitable parameter in the bag model (that is, absorbed by a contact term). The residual Casimir stress is attractive, but smaller than the repulsive Casimir stress of gluons in the model. The result for the total zero-point energy is in substantial disagreement with bag model phenomenological values.

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I. Introduction

It is widely accepted today that hadrons consist of confined fermion and vector fields, the fields of quarks and gluons. The precise mechanism by which this occurs remains unknown. However, the mere existence of confinement gives rise to an interesting quantum phenomenon which can be calculated in principle, and which has significant bearing on the confinement issue itself. This phenomenon is the zero-point, or Casimir energy, due to quantum fluctuations in the fields.[1]

The simplest way of approaching an understanding of the role of zero-point energy in the internal structure of hadrons is in the bag model,[2,3] which may be a crude approximation to the true situation existing in QCD. In that model the quarks and gluons are free (or weakly coupled) inside the bag, but are absolutely confined to the interior of the bag. (This is a rough representation of asymptotic freedom.) For such a simple situation it would seem to be a straightforward matter to compute the Casimir energy. Unfortunately, it is well-known that there are difficulties associated with quantum field fluctuations in the presence of curved boundaries.[4-6] There is one known conspicuous exception: that of a perfectly conducting spherical shell in electrodynamics, where because of delicate cancellations between TE and TM modes, and between interior and exterior modes, a finite <u>repulsive</u> Casimir energy results.[7-9] Since there are no exterior modes in the bag model, it is immediately apparent that divergences may be expected there. Indeed, Bender and Hays[6] explicitly computed the zero-point energy due to fermion and massless vector fields confined by a spherical shell, and found the divergences expected from general considerations.[4,5]

As a consequence, model builders[2,3] have treated the zero-point energy term as a phenomenological parameter, of the form $-Z/a$, a being the bag radius, where

Z is determined from mass fits to be ~ 1.[10] However, it is recognized that this is a stop-gap procedure since the underlying theory (QCD) should determine Z. In fact, recently Johnson has returned to the fundamental picture in his model of the vacuum[11] by adopting as the gluonic zero-point energy of an "empty" bag surrounded by other such bags eight times the QED value. This is incorrect, however, since there are no QCD modes exterior to an ideal bag: the Casimir energy of a world filled with contiguous empty bags in simply the sum of the energies due to the modes <u>within</u> each bag.

Elsewhere[12,13] I have recomputed the zero-point energy due to fluctuations of free gluon fields confined by a spherical bag. I have suggested that the quadratic divergence should be absorbed by a suitable renormalization of a phenomenologically determined parameter, leaving a finite, calculable result. The latter is found to be, approximately,

$$E_g = +0.51/a, \tag{1.1}$$

which includes the effects of eight gluons. It is worth noting: (a) this zero-point energy is of opposite sign to, and of about half the magnitude of, that used in bag model fits,[2,3] and (b) E_g is substantially greater than eight times the QED value, which Johnson[11] used. However, what remains to be included is the fermionic contribution. Could that change the result significantly?

Bender and Hays[6] did compute the zero-point energy of confined vector and fermion fields some years ago. But a reconsideration seems worthwhile because there are some errors in their paper, and more significantly because they did not proceed beyond isolating the divergent terms. The calculations which I have presented elsewhere,[12,13] and which I will present here, are more complete, since I retain the finite terms that remain if the divergent terms are absorbed by a suitable renormalization. Of course, the latter procedure remains arguable, but at least it forms a hypothesis for extracting physics from what is ultimately a poorly understood phenomenon.

II. Green's Function

The fermion (quark) Green's function satisfies

$$\left(\gamma \frac{1}{i} \partial\right) \mathcal{G}(x,x') = \delta(x-x') \tag{2.1}$$

provided the quark masses are negligible, a good approximation, perhaps, for u, d, and s quarks. Equation (2.1) is to be solved subject to the linear boundary condition[2]

$$(1 + i\,\vec{n}\cdot\vec{\gamma})\mathcal{G}(x,x') \Big|_s = 0, \tag{2.2}$$

where $\vec{n}$ is the outward normal to the static bag. (The bag model also possesses a non-linear boundary condition, which we will ignore for the present.) To solve (2.1) we introduce a time Fourier transform:

$$\mathcal{G}(x,x') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} G(\vec{r},\vec{r}\,',\omega) \tag{2.3}$$

where, since

$$\vec{\gamma} = i\gamma^0 \gamma_5 \vec{\sigma}, \tag{2.4}$$

G satisfies

$$(-\omega + \gamma_5 \vec{\sigma}\cdot\vec{\nabla}) G(\vec{r},\vec{r}\,') = \gamma^0 \delta(\vec{r}-\vec{r}\,'). \tag{2.5}$$

We adopt a representation in which γ_5 is diagonal

$$i\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.6}$$

so that (2.5) reads

$$(-\omega \pm i\vec{\sigma}\cdot\vec{\nabla}) G_{\mp\mp}(\vec{r},\vec{r}\,') = 0, \tag{2.7a}$$

$$(-\omega \pm i\vec{\sigma}\cdot\vec{\nabla}) G_{\mp\pm}(\vec{r},\vec{r}\,') = \pm i\delta(\vec{r}-\vec{r}\,'), \tag{2.7b}$$

where the subscripts denote the eigenvalues of $i\gamma_5$.

To proceed, we make an angular momentum decomposition. The eigenstates of $\vec{J} = \vec{L} + \frac{1}{2}\vec{\sigma}$ are

$$Z_{JM}^{\ell=J\pm\frac{1}{2}}(\Omega) = \left(\frac{\ell+\frac{1}{2}\mp M}{2\ell+1}\right)^{\frac{1}{2}} Y_{\ell M-\frac{1}{2}}(\Omega)\,|+\rangle$$

$$\mp\left(\frac{\ell+\frac{1}{2}\pm M}{2\ell+1}\right)^{\frac{1}{2}} Y_{\ell M+\frac{1}{2}}(\Omega)\,|-\rangle\,, \tag{2.8}$$

which have the property

$$\vec{\sigma}\cdot\hat{r}\; Z_{JM}^{\ell=J\pm\frac{1}{2}}(\Omega) = Z_{JM}^{\ell=J\mp\frac{1}{2}}(\Omega)\,. \tag{2.9}$$

Then, in the two-dimensional spin space spanned by $Z_{JM}^{\ell=J\pm\frac{1}{2}}$, the operator in (2.7) becomes

$$(-\omega\pm i\vec{\sigma}\cdot\vec{\nabla}) = \begin{pmatrix} -\omega & \pm\frac{i}{r}\left[\frac{\partial}{\partial r}r - J - \frac{1}{2}\right] \\ \pm\frac{i}{r}\left[\frac{\partial}{\partial r}r + J + \frac{1}{2}\right] & -\omega \end{pmatrix}\,. \tag{2.10}$$

Expanding $G_{ab}(\vec{r},\vec{r}\,')$ $[a,b=\pm=(i\gamma_5)']$ in terms of these angular momentum eigenstates

$$G_{ab}(\vec{r},\vec{r}\,') = \sum_{JM}\left\{f_J^{ab}(r,\vec{r}\,')Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega) + g_J^{ab}(r,\vec{r}\,')Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega)\right\}\,, \tag{2.11}$$

and using the orthonormality property

$$\int d\Omega\; Z_{JM}^{\ell}(\Omega)Z_{J'M'}^{\ell'}(\Omega)^{*} = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}, \tag{2.12}$$

we find the following component equations

$$-\omega f_J^{\pm\mp} \mp \frac{i}{r}\left(\frac{\partial}{\partial r}r - (J+\tfrac{1}{2})\right)g_J^{\pm\mp} = \mp\frac{1}{r^2}\,\delta(r-r')Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega')^{*}\,, \tag{2.13a}$$

$$\mp\frac{i}{r}\left(\frac{\partial}{\partial r}r + J + \tfrac{1}{2}\right)f_J^{\pm\mp} - \omega g_J^{\pm\mp} = \mp\frac{1}{r^2}\,\delta(r-r')Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega')^{*}, \tag{2.13b}$$

and

$$-\omega f_J^{\pm\pm} \mp \frac{i}{r}\left(\frac{\partial}{\partial r} r - J - \tfrac{1}{2}\right) g_J^{\pm\pm} = 0, \tag{2.14a}$$

$$\mp \frac{i}{r}\left(\frac{\partial}{\partial r} r + J + \tfrac{1}{2}\right) f_J^{\pm\pm} - \omega g_J^{\pm\pm} = 0. \tag{2.14b}$$

The system (2.13) can be solved in terms of scalar Green's functions satisfying

$$\left(\frac{1}{r}\frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2} + \omega^2\right)\Delta_\ell(r,r') = -\frac{1}{r^2}\,\delta(r-r'), \tag{2.15}$$

which have the form, for $r, r' < a$, of

$$\Delta_\ell = i k j_\ell(kr_<)[h_\ell(kr_>) - A_\ell j_\ell(kr_>)]. \tag{2.16}$$

The scalar Green's functions corresponding to f and g in (2.11) will be denoted by F_ℓ and G_ℓ, respectively. Then

$$\begin{aligned} G_{\pm\mp}(\vec{r},\vec{r}\,') = \sum_{JM} \Big\{ & \frac{1}{r'}\left(\frac{\partial}{\partial r'} r' + J + \tfrac{1}{2}\right) F_{J+\frac{1}{2}}(r,r') Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega')^* \\ & + \frac{1}{r'}\left(\frac{\partial}{\partial r'} r' - J - \tfrac{1}{2}\right) G_{J-\frac{1}{2}}(r,r') Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega')^* \\ & \mp i\omega F_{J+\frac{1}{2}}(r,r') Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega')^* \\ & \mp i\omega G_{J-\frac{1}{2}}(r,r') Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega')^* \Big\}, \end{aligned} \tag{2.17}$$

where G_ℓ and F_ℓ are related by

$$G_{J-\frac{1}{2}}(r,r') = \frac{1}{\omega^2}\,\frac{1}{r}\left(\frac{\partial}{\partial r} r + J + \tfrac{1}{2}\right)\frac{1}{r'}\left(\frac{\partial}{\partial r'} r' + J + \tfrac{1}{2}\right) F_{J+\frac{1}{2}}(r,r'), \tag{2.18}$$

which means that, in the notation of (2.16),

$$A^G_{J-\frac{1}{2}} = A^F_{J+\frac{1}{2}}, \tag{2.19}$$

since

$$\left(\frac{d}{dr}\, r \pm \ell\right) j_{\ell}(kr) = \pm kr\, j_{\ell \mp 1}(kr). \tag{2.20}$$

The homogeneous system (2.14) is subject to a further constraint, that of the antisymmetry of the Green's function: [(2.17) manifestly satisfies this requirement]

$$[\gamma^0 G(x,x')]^T = -\gamma^0 G(x',x), \tag{2.21}$$

which implies the form

$$\begin{aligned} G_{\pm\pm}(\vec{r},\vec{r}\,') = \sum_{JM} \Big\{ & a_J j_{J+\frac{1}{2}}(kr) j_{J+\frac{1}{2}}(kr') Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega')^* \\ & + b_J j_{J-\frac{1}{2}}(kr) j_{J-\frac{1}{2}}(kr') Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega')^* \\ & \pm c_J j_{J-\frac{1}{2}}(kr) j_{J+\frac{1}{2}}(kr') Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega')^* \\ & \pm c_J j_{J+\frac{1}{2}}(kr) j_{J-\frac{1}{2}}(kr') Z_{JM}^{\ell=J+\frac{1}{2}}(\Omega) Z_{JM}^{\ell=J-\frac{1}{2}}(\Omega')^* , \end{aligned} \tag{2.22}$$

where we have anticipated that a_J, b_J are even functions of ω, while c_J is odd.

Now we are ready to impose the boundary condition (2.2). In my basis it reads

$$\left.\begin{pmatrix} 1 & -\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r} & 1 \end{pmatrix} \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}\right|_{|\vec{r}|=a} = 0 . \tag{2.23}$$

The implications are

$$a_J = -b_J = -i\frac{k}{\omega} c_J = \frac{1/a^2}{[j_{J+\frac{1}{2}}(ka)]^2 - [j_{J-\frac{1}{2}}(ka)]^2} \tag{2.24a}$$

and

$$A^F_{J+\frac{1}{2}} = \frac{h_{J+\frac{1}{2}}(ka) j_{J+\frac{1}{2}}(ka) - h_{J-\frac{1}{2}}(ka) j_{J-\frac{1}{2}}(ka)}{[j_{J+\frac{1}{2}}(ka)]^2 - [j_{J-\frac{1}{2}}(ka)]^2} . \tag{2.24b}$$

In simplifying (2.24a) we used the identity

$$h_{\ell-1}(x)j_{\ell}(x) - h_{\ell}(x)j_{\ell-1}(x) = \frac{i}{x^2}. \qquad (2.25)$$

Equations (2.17), (2.22), and (2.24) supply a closed form for the reduced Green's function. It differs from the free Dirac Green's function $G^{(o)}$ by

$$G = G^{(o)} + \tilde{G} \qquad (2.26)$$

where, using a matrix notation for the two-dimensional spin space spanned by $Z_{JM}^{\ell=J\pm\frac{1}{2}}$,

$$\tilde{G}_{\pm\mp} = -ik \sum_J \frac{h_{J+\frac{1}{2}}(ka)j_{J+\frac{1}{2}}(ka) - h_{J-\frac{1}{2}}(ka)j_{J-\frac{1}{2}}(ka)}{[j_{J+\frac{1}{2}}(ka)]^2 - [j_{J-\frac{1}{2}}(ka)]^2}$$

$$\times \begin{pmatrix} \mp i\omega j_{J+\frac{1}{2}}(kr)j_{J+\frac{1}{2}}(kr') & kj_{J+\frac{1}{2}}(kr)j_{J-\frac{1}{2}}(kr') \\ -kj_{J-\frac{1}{2}}(kr)j_{J+\frac{1}{2}}(kr') & \mp i\omega j_{J-\frac{1}{2}}(kr)j_{J-\frac{1}{2}}(kr') \end{pmatrix} \qquad (2.27a)$$

and

$$\tilde{G}_{\pm\pm} = -ik \sum_J \frac{1/k^2a^2}{[j_{J+\frac{1}{2}}(ka)]^2 - [j_{J-\frac{1}{2}}(ka)]^2}$$

$$\times \begin{pmatrix} -ikj_{J+\frac{1}{2}}(kr)j_{J+\frac{1}{2}}(kr') & \mp\omega j_{J+\frac{1}{2}}(kr)j_{J-\frac{1}{2}}(kr') \\ \mp\omega j_{J-\frac{1}{2}}(kr)j_{J+\frac{1}{2}}(kr') & ikj_{J-\frac{1}{2}}(kr)j_{J-\frac{1}{2}}(kr') \end{pmatrix} . \qquad (2.27b)$$

We will verify in Appendix A that this Green's function implies the usual bag wavefunctions for the quarks. Here we directly proceed to compute the corresponding zero-point energy.

III. Casimir Stress

There are quite a number of equivalent approaches to calculating the zero-point energy.[5-9,14-17] Perhaps the most direct means is to compute the stress on the surface due to the fluctuating fields. The fermionic stress tensor is

$$T^{\mu\nu} = \tfrac{1}{2}\psi\gamma^0\tfrac{1}{2}\left(\gamma^\mu\frac{1}{i}\partial^\nu + \gamma^\nu\frac{1}{i}\partial^\mu\right)\psi + g^{\mu\nu}\mathcal{L}, \tag{3.1}$$

$\mathcal{L}$ being the fermionic Lagrange function. The contribution to $T^{\mu\nu}$ arising from quantum fluctuations is obtained by making the replacement

$$i\psi(x)\psi(x')\gamma^0 \rightarrow G(x,x'), \tag{3.2}$$

which implies for the radial stress

$$T_{rr} = \tfrac{1}{2}\frac{\partial}{\partial r}\,\mathrm{tr}\,\vec{\gamma}\cdot\hat{r}\,G(x,x')\Big|_{x'\rightarrow x} \; . \tag{3.3}$$

Here we ignored a δ-function contribution coming from the Lagrange function, since we adopt the point-splitting attitude in which $x'\rightarrow x$ only at the end of the calculation. Using the representation (2.6) we find ($\tau = t-t'$)

$$T_{rr} = \frac{i}{2}\int\frac{d\omega}{2\pi}\,e^{-i\omega\tau}\,\frac{\partial}{\partial r}\,\mathrm{tr}\,\vec{\sigma}\cdot\hat{r}\,(G_{-+}+G_{+-})$$

$$= i\int\frac{d\omega}{2\pi}\,e^{-i\omega\tau}\frac{\partial}{\partial r}\sum_{JM}\mathrm{tr}\left\{\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \tfrac{1}{2}\right)F_{J+\frac{1}{2}}(r,r')Z_{JM}^{J-\frac{1}{2}}(\Omega)Z_{JM}^{J-\frac{1}{2}}(\Omega')^*\right.$$

$$\left. + \frac{1}{r'}\left(\frac{\partial}{\partial r'}r' - J - \tfrac{1}{2}\right)G_{J-\frac{1}{2}}(r,r')Z_{JM}^{J+\frac{1}{2}}(\Omega)Z_{JM}^{J+\frac{1}{2}}(\Omega')^*\right\}, \tag{3.4}$$

making use of (2.17) and (2.9). The completeness relation for the Z's reads

$$\sum_{M=-J}^{J}\mathrm{tr}\,Z_{JM}^{J\pm\frac{1}{2}}(\Omega)Z_{JM}^{J\pm\frac{1}{2}}(\Omega')^* = \frac{2J+1}{4\pi} \quad , \tag{3.5}$$

so when the points are allowed to approach each other through a temporal direction, we have

$$T_{rr} = i\frac{\partial}{\partial r}\sum_{J=\frac{1}{2}}^{\infty} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \frac{2J+1}{4\pi}\left\{\frac{1}{r'}\left(\frac{\partial}{\partial r'} r' + J + \tfrac{1}{2}\right) F_{J+\frac{1}{2}}(r,r')\right.$$

$$\left.\left. + \frac{1}{r'}\left(\frac{\partial}{\partial r'} r' - J - \tfrac{1}{2}\right) G_{J-\frac{1}{2}}(r,r')\right\}\right|_{r'=r} . \tag{3.6}$$

It is now necessary to remove from this expression the "vacuum" or volume energy which this formalism would supply even if no bag were present (that is, if $a \to \infty$). That means we must subtract from each Green's function in (3.6) its vacuum part,

$$\Delta_\ell^{(o)} = i\, k\, j_\ell(kr_<) h_\ell(kr_>) , \tag{3.7}$$

which means we use $\tilde{G}_{\pm\mp}$ defined by (2.27a), leaving us with

$$T_{rr}\Big|_a = \sum_{J=\frac{1}{2}}^{\infty} \frac{2J+1}{4\pi} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} k^3$$

$$\times \frac{h_{J+\frac{1}{2}}(ka) j_{J+\frac{1}{2}}(ka) - h_{J-\frac{1}{2}}(ka) j_{J-\frac{1}{2}}(ka)}{[j_{J+\frac{1}{2}}(ka)]^2 - [j_{J-\frac{1}{2}}(ka)]^2}$$

$$\times \left\{ j'_{J+\frac{1}{2}}(ka) j_{J-\frac{1}{2}}(ka) - j'_{J-\frac{1}{2}}(ka) j_{J+\frac{1}{2}}(ka) \right\} . \tag{3.8}$$

This gives the force/area on the surface; to obtain the free energy we multiply by $4\pi a^3$. If in addition we make a Euclidean rotation[9]

$$\omega, k \to i\omega, ik,$$

$$\tau \to i\tau, \tag{3.9}$$

we have ($\delta \to 0$), with $x = ka$, $y = \omega a$,

$$E = \frac{1}{2\pi a}\sum_{J=\frac{1}{2}}^{\infty} (2J+1) \int_{-\infty}^{\infty} dy\, e^{iy\delta} x^2 \frac{I_{J+1}(x)K_{J+1}(x) - I_J(x)K_J(x)}{[I_{J+1}(x)]^2 + [I_J(x)]^2}$$

$$\times \{ I'_{J+1}(x) I_J(x) - I'_J(x) I_{J+1}(x) \} , \tag{3.10}$$

which coincides with the formula given by Bender and Hays.[6] We prefer, instead, to introduce the functions

$$s_\ell(x) = \sqrt{\frac{\pi x}{2}}\, I_{\ell+\frac{1}{2}}(x)$$

$$e_\ell(x) = \sqrt{\frac{2x}{\pi}}\, K_{\ell+\frac{1}{2}}(x) \tag{3.11}$$

in which case the Casimir energy expression takes the form ($\delta \to 0$)

$$E = \frac{1}{2\pi a} \sum_{J=\frac{1}{2}}^{\infty} (2J+1) \int_0^\infty dx\, x \cos x\delta \left\{ \frac{d}{dx} \ln\left(s^2_{J+\frac{1}{2}} + s^2_{J-\frac{1}{2}} \right) \right.$$

$$\left. - 2\left(s'_{J+\frac{1}{2}} e_{J-\frac{1}{2}} + s'_{J-\frac{1}{2}} e_{J+\frac{1}{2}} \right) \right\} . \tag{3.12}$$

We now turn to the evaluation of this expression.

IV. Numerical Evaluation

A. $J=\frac{1}{2}$ Contribution.

As mentioned in Sec. II, in addition to the linear boundary condition (2.2), the bag model is usually subjected to an additional non-linear boundary condition,[2]

$$-\frac{\partial}{\partial r}\sum_i \bar{\psi}_i\gamma^0\psi_i\Big|_a = 2B, \tag{4.1}$$

where the sum ranges over the various quarks, and B is the bag constant. This restricts valence quark states to only those characterized by $J=\frac{1}{2}$. However, it imposes no condition on $\mathcal{G}$ (how could it, since $\mathcal{G}$ is already uniquely determined?) but rather expresses the zero-point, quantum fluctuation contribution to the bag constant. This is discussed in Appendix B.

Nevertheless, it is interesting to compute the $J=\frac{1}{2}$ contribution to (3.12), since the lowest mode might be thought to be the most important, and because it provides a check on the accuracy of the approximations to be made subsequently. Since

$$s_0(x) = \sinh x, \quad s_1(x) = \cosh x - \frac{1}{x}\sinh x,$$

$$e_0(x) = e^{-x}, \quad e_1(x) = e^{-x}\left(1+\frac{1}{x}\right), \tag{4.2}$$

we have for the $J=\frac{1}{2}$ contribution to the zero-point energy

$$E^{\frac{1}{2}} = \frac{2}{\pi a}\int_0^\infty dx\, x f(x), \tag{4.3}$$

with

$$f(x) = \left[\left(1+\frac{1}{2x^2}\right)\coth 2x - \frac{1}{2x^2}\,\operatorname{csch} 2x - \frac{1}{x}\right]^{-1} - \frac{1}{x} - e^{-x}\left[\left(1+\frac{1}{x^2}\right)\sinh x + \cosh x\right]. \tag{4.4}$$

Numerical integration yields

$$E^{\frac{1}{2}} = -\frac{1}{\pi a}\,(0.18) = -\frac{1}{a}\,(0.057)\ . \tag{4.5}$$

Note that this contribution is attractive. It is instructive to compare this with the lowest mode contribution of a confined vector field. The latter is[13]

$$E_V^1 = -\frac{3}{2\pi a}\int_0^\infty dx\, x\left\{\frac{s_1'}{s_1}+\frac{s_1''}{s_1'}+2(e_1's_1'-e_1s_1'')\right\}$$

$$= \frac{3}{2\pi a}\int_0^\infty dx\, x\, f(x) + E^1_{V,\text{shell}}, \qquad (4.6)$$

where $E^1_{V,\text{shell}}$ is result of interior and exterior mode contributions,[7-9]

$$E^1_{V,\text{shell}} = -\frac{1}{a}\,(.047), \qquad (4.7)$$

and the integral cancels off the exterior modes. Explicitly

$$f(x) = -\left[\frac{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{1+\frac{1}{x}} + \frac{\left(1+\frac{2}{x^2}\right)\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}\right] + 2 + \frac{2}{x^2} - \frac{1}{x^4} + e^{-2x}\left(\frac{2}{x^3}+\frac{1}{x^4}\right), \qquad (4.8)$$

and numerical integration yields

$$\int_0^\infty dx\, x\, f(x) = -0.28\ . \qquad (4.9)$$

As a result, the lowest mode vector contribution is

$$E_V^1 = -\frac{1}{a}\,(0.18) \qquad (4.10)$$

which is also attractive, but more than 3 times larger than the lowest mode spinor contribution, (4.5).

B. Sum Over All Modes.

What is most remarkable about the classic spherical shell calculation is that although the low mode contributions are all attractive, when the sum over all modes is preformed, the sign of the zero-point energy reverses. A similar phenomenon happens here. The simplest way to see this is to use the uniform asymptotic expansions for the Bessel functions:[12]

$$s_\ell(x) \sim \tfrac{1}{2} z^{\frac{1}{2}} t^{\frac{1}{2}} e^{\nu\eta}\left[1 + \sum_{k=1}^{\infty} \nu^{-k} u_k\right]$$

$$e_\ell(x) \sim z^{\frac{1}{2}} t^{\frac{1}{2}} e^{-\nu\eta}\left[1 + \sum_{k=1}^{\infty} (-1)^k \nu^{-k} u_k\right]$$

$$s_\ell'(x) \sim \tfrac{1}{2} z^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{\nu\eta}\left[1 + \sum_{k=1}^{\infty} \nu^{-k} v_k\right]$$

$$e_\ell'(x) \sim z^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\nu\eta}\left[1 + \sum_{k=1}^{\infty} (-1)^k \nu^{-k} v_k\right] \qquad (4.11)$$

as $\ell \to \infty$. Here

$$\nu = \ell + \tfrac{1}{2}, \quad x = \nu z, \quad t = (1 + z^2)^{-\frac{1}{2}}$$

$$\eta = t^{-\frac{1}{2}} + \ell n\,(z/[1 + t^{-1}]) \qquad (4.12)$$

and

$$u_1(t) = \frac{1}{24}\,(3t - 5t^3)$$

$$v_1(t) = \frac{1}{24}\,(3t + 7t^3)$$

$$u_2(t) = \frac{1}{1152}\,(81t^2 - 462t^4 + 385t^6)$$

$$v_2(t) = \frac{1}{1152}\,(-63t^2 + 474t^4 - 455t^6)\ . \qquad (4.13)$$

Using these it is straightforward to approximate (3.12) by ($\delta \to 0$)

$$E \sim -\frac{1}{4\pi a}\sum_{J=\frac{1}{2}}^{\infty}(2J+1)\int_0^{\infty} dz\, \cos\nu z\delta\, z^2 t^5, \tag{4.14}$$

where $\nu = J+1$. [Note that the $J=\frac{1}{2}$ contribution is approximated by

$$E^{\frac{1}{2}} \sim -\frac{1}{\pi a}\frac{1}{6} = -\frac{1}{a}\,(.053)$$

nearly the correct value given in Eq.(4.5). The similar approximation to the $\ell = 1$ contribution for the vector field is[12,13]

$$E_V^1 \sim -\frac{1}{\pi a}\,\tfrac{1}{2} = -\frac{1}{a}\,(.159)$$

not far from the correct value given in Eq.(4.12). Including the next-to-leading approximation here changes this to

$$E_V^1 \sim -\frac{1}{\pi a}\left(\tfrac{1}{2}+\frac{3\pi}{128}\right) = -\frac{1}{a}\,(.183)$$

which yields nearly perfect agreement.] The z integral in (4.14) is

$$\nu\delta K_1(\nu\delta) - \tfrac{1}{3}(\nu\delta)^2 K_2(\nu\delta), \tag{4.15}$$

making it simple to evaluate the J sum using

$$\int_0^{\infty} dx\, x^n K_m(x) = 2^{n-1}\,\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n-m+1}{2}\right) \quad . \tag{4.16}$$

The result is

$$E \sim \frac{1}{\pi a}\left\{\frac{1}{3}\frac{1}{\delta^2} - \frac{1}{48}\right\} \quad . \tag{4.17}$$

Compare this to the corresponding result for the confined vector field[13]

$$E_V \sim \frac{1}{\pi a}\left\{-\frac{4}{3}\frac{1}{\delta^2} + \frac{1}{8}\right\} \quad . \tag{4.18a}$$

Here we have included only the leading asymptotic approximation; inclusion of the next to leading term yields[13]

$$E_V \sim \frac{1}{\pi a}\left\{-\frac{4}{3}\,\frac{1}{\delta^2}+\frac{1}{8}+\frac{3\pi}{128}\right\} \quad , \tag{4.18b}$$

a roughly 50% modification of the finite term. Presumably a similar modification of the finite term would occur in (4.17) if next to leading terms were included.

V. Conclusions

Since at this stage in our understanding numerical precision is not the issue we confine our remarks to the leading approximations (4.17) and (4.18a). Restating these results, for 8 gluons and N quarks,

$$E_q \sim \frac{N}{\pi a}\left\{\frac{1}{3}\frac{1}{\delta^2} - \frac{1}{48}\right\} \tag{5.1}$$

$$E_g \sim \frac{1}{\pi a}\left\{-\frac{32}{3}\frac{1}{\delta^2} + 1\right\} , \tag{5.2}$$

we first must address again the question, how to deal with the quadratically divergent terms, as $\delta \to 0$. Now, since (5.1) and (5.2) are of opposite sign, the possibility is opened that the divergence could cancel between the quark and gluon contributions, which would occur provided

$$N = 32. \tag{5.3}$$

This is impossible, however, in a colored theory, where N must be a multiple of 3.

The solution to this problem appears to be that in the bag model we should introduce phenomenological terms in the energy as follows

$$H' = BV + \sigma A + Fa \tag{5.4}$$

where B is the bag constant, σ a constant surface tension, and F a constant force. Here V is the volume, A the area, and a the radius of a spherical bag. The necessity of introducing B and σ has been previously recognized;[2,3] however, equally well there is no reason to exclude F. In fact, all such terms may be thought of as contact terms, since they are polynomials in the bag radius. Since $\delta = i\tau/a$, we appreciate that the divergent parts of (5.1) and (5.2) are of the form of Fa, and so merely renormalize that phenomenological parameter. (Note that the volume energy subtraction can be thought of as a renormalization of B, but no term of the form σA occurs in zero-point calculations.[4]) Equation (5.4) would appear to be the correct phenomenology, not the usual

$$H'' = BV + \sigma A - Z/a. \tag{5.4'}$$

Let us therefore suppose that this conception is correct, and only the finite parts of (5.1) and (5.2) remain after renormalization. The corresponding zero-point energy is

$$E_{ZP} = E_g^{ren} + E_q^{ren} \sim \frac{1}{\pi a}\left(1 - \frac{N}{48}\right) . \tag{5.5}$$

The quarks most nearly massless are u, d, and s, so $N \cong 3 \times 3 = 9$, and

$$E_{ZP} \sim \frac{1}{\pi a}\ (1 - 0.19) = \frac{.26}{a}. \tag{5.6}$$

It is of interest to contrast this with Johnson's recent speculation,[11] that

$$E_g \cong \frac{0.37}{a}\ , \quad E_q \cong \frac{0.28}{a}, \quad E_{ZP} \cong \frac{0.65}{a}\ ; \tag{5.7}$$

before drawing conclusions in this comparison, however, note that E_g^{ren} and E_q^{ren} are subject to perhaps 50% corrections from higher order asymptotic terms. But it is undoubtedly impossible to reconcile (5.6) with the value found in bag model fits,[10]

$$E_{ZP} \cong -1/a. \tag{5.8}$$

This problem must be squarely faced by the model builders.

Appendix A

Here my purpose is to show that the Green's function (2.27) yields the correct bag wavefunctions. Since the latter are subject to the boundary condition (4.1), we will extract only the $J = \frac{1}{2}$ wavefunctions. For this purpose it is most convenient to change to the basis where

$$\gamma^o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\gamma_5 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{A1}$$

which is effected by the unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} . \tag{A2}$$

The Green's function becomes

$$G \to UGU^{-1} = \frac{1}{2} \begin{pmatrix} G_{++} + G_{--} + i(G_{+-} - G_{-+}) & G_{+-} + G_{-+} - i(G_{--} - G_{++}) \\ G_{+-} + G_{-+} + i(G_{--} - G_{++}) & G_{++} + G_{--} - i(G_{+-} - G_{-+}) \end{pmatrix} = \hat{G} . \tag{A3}$$

Using (2.27), we identify the wavefunction from the residue of the poles: for example, the positive frequency poles of $\hat{G}_{++}$ are

$$\hat{G}^{J=\frac{1}{2}}_{++} = -\Bigg\{ \frac{1}{ka - \xi_+} \frac{1}{2a^2 j_o^2(\xi_+)} \frac{1}{1+\xi_+} j_1(k_+ r)\vec{\sigma}\cdot\hat{r}\Big[|+\rangle\langle+| + |-\rangle\langle-|\Big]\vec{\sigma}\cdot\hat{r}' j_1(k_+ r')$$
$$- \frac{1}{ka - \xi_-} \frac{1}{2a^2 j_o^2(\xi_-)} \frac{1}{1-\xi_-} j_o(k_- r)\Big[|+\rangle\langle+| + |-\rangle\langle-|\Big] j_o(k_- r')\Bigg\} \tag{A4}$$

where $\xi_\pm$ is a root of the equation

$$j_o(\xi_\pm) = \mp j_1(\xi_\pm), \tag{A5}$$

that is

$$\tan \xi_\pm = \frac{\xi_\pm}{1 \pm \xi_\pm} , \tag{A6}$$

and

$$k_\pm a = \xi_\pm . \tag{A7}$$

[In (A4) the sum over the different roots of (A5) is implicit.] In this way we find

$$\hat{G}^{J=\frac{1}{2}} = -\sum_{\pm} \frac{1}{k - k_{\pm}} \psi^{(\pm)}(\vec{r})\psi^{(\pm)}(\vec{r}\,')^{*}\gamma^{o} \tag{A8}$$

where, with χ an arbitrary Pauli spinor,

$$\psi_n^{(+)}(\vec{r}) = \frac{1}{N_+}\begin{pmatrix} j_1(k_+r)\vec{\sigma}\cdot\hat{r}\,\chi \\ -j_o(k_+r)\chi \end{pmatrix} \tag{A9a}$$

$$\psi_n^{(-)}(\vec{r}) = \frac{1}{N_-}\begin{pmatrix} j_o(k_-r)\chi \\ j_1(k_-r)\vec{\sigma}\cdot\hat{r}\,\chi \end{pmatrix}, \tag{A9b}$$

$$N_{\pm}^2 = 2a^3 j_o^2(\xi_{\pm})(\xi_{\pm} \pm 1). \tag{A10}$$

These wavefunctions are equivalent to the usual ones and evidently satisfy the boundary condition (2.2) since they are, by (A5), annihilated by

$$\begin{pmatrix} 1 & -\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r} & 1 \end{pmatrix} \tag{A11}$$

on the surface.

Appendix B

When we make the replacement (3.2) in the non-linear boundary condition (4.1), we find

$$i \frac{\partial}{\partial r} \operatorname{tr} \mathcal{G}(x,x) = \frac{2B}{N} \tag{B1}$$

where N is the number of quarks. From (2.27b) we see this is

$$\frac{2B}{N} = -i \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \sum_{JM} a_J \frac{\partial}{\partial a} \Big\{ j_{J+\frac{1}{2}}(ka) j_{J+\frac{1}{2}}(ka) \operatorname{tr} Z_{JM}^{J+\frac{1}{2}}(\Omega) Z_{JM}^{J+\frac{1}{2}}(\Omega)^* $$

$$- j_{J-\frac{1}{2}}(ka) j_{J-\frac{1}{2}}(ka) \operatorname{tr} Z_{JM}^{J-\frac{1}{2}}(\Omega) Z_{JM}^{J-\frac{1}{2}}(\Omega)^* \Big\}$$

$$= i \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \sum_{J=\frac{1}{2}}^{\infty} \frac{2J+1}{4\pi} \left(-\frac{1}{a^2}\right) \frac{\partial}{\partial a} \ln\left[j_{J+\frac{1}{2}}^2(ka) - j_{J-\frac{1}{2}}^2(ka) \right] \tag{B2}$$

or

$$B = \frac{N}{8\pi^2 a^4} \sum_{J=\frac{1}{2}}^{\infty} (2J+1) \int_{\delta}^{\infty} x \, dx \cos x\delta \frac{d}{dx} \ln\left(\frac{s_{J+\frac{1}{2}}^2(x) + s_{J-\frac{1}{2}}^2(x)}{x^2} \right) \tag{B3}$$

This formula, which is very similar to (3.12), is an expression of the quantum field fluctuation component of the bag constant. It opens up a possibility of computing B from first principles.

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