ON COMPLEX ROOTS OF MEROMORPHIC FUNCTIONS

1) HOW TO FIND ONE ZERO

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O. DE BARBIERI and D. MOREAU

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RESUME

On compare d'une façon critique et en faisant apparaître leurs limites, les différentes méthodes numériques qui permettent de trouver les racines complexes d'une fonction $f(z)$ analytique ou, plus généralement, méromorphe. On montre ensuite qu'à l'aide d'un algorithme judicieusement construit à partir d'une méthode de gradient appliquée à $|f(z)|^2$ les difficultés propres à cette méthode peuvent être surmontées.
La stratégie que nous proposons permet de trouver un zéro (ainsi que sa multiplicité) avec une précision prédéterminée si l'on connaît l'erreur avec laquelle $f(z)$ est calculée. On illustre enfin par des exemples numériques non triviaux quelques caractéristiques particulières de cet algorithme qui ne demande pas obligatoirement la connaissance de l'expression analytique de $f'(z)$. 
ON COMPLEX ROOTS OF MEROMORPHIC FUNCTIONS

1) HOW TO FIND ONE ZERO

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ABSTRACT

A number of numerical methods proposed to compute complex roots of analytic, or more generally meromorphic, functions $f(z)$ are compared critically and their basic limitations are pointed out. It is shown that these can be overcome via a suitable steepest descent algorithm for $|f(z)|^2$. The strategy proposed allows to find a zero (as well as its multiplicity) with a predetermined precision provided that the precision with which $f(z)$ is computed is known. Some peculiar features of the algorithm (which does not require an explicit knowledge of $f'(z)$) are illustrated in a number of non trivial examples.
I - INTRODUCTION

A vast literature exists (see Refs. 1/ to 31/ for a small sampling and Refs. 19/, 20/ for a larger one) and a number of codes have been documented (Refs. 15/, 28/, 29/, 30/), on the problem of finding the roots of a complex analytic function f(z). Some of these methods are specific to particular classes of functions f, like for instance polynomials (see Refs. 1/ to 7/) or functions such that the real and imaginary parts of f(z)=u(x,y)+iv(x,y), z=x+iy can be written down explicitly. In this case the problem posed by the solution of f(z)=0 is equivalent to that of finding the minima (=0) of g(x,y)=u^2(x,y)+v^2(x,y) and again many papers exist on this classical problem (see Refs. 12/, 15/, 18/, to 13/ as well as Refs. 19/, 20/).

Our interest in the problem of finding the roots of f(z)=0 is motivated by its importance for plasma physics. In this case f=0 is usually a dispersion relation which, apart from z, depends on a number of parameters that we call {a}. The problem is to find z=z({a}) such that f(z({a}),{a})=0. The analytic form of f is known and all its derivatives \( \frac{\partial^{m+n}f}{\partial z^m a^n} \); m,n = 0,1... can be computed at least in principle. Quite frequently f is transcendental in z and has an infinity of roots for any given set {a}. However one is not interested in computing either many or all of them, but very few, in many cases only one. This can be, for instance that for which Im(z) is maximum (i.e. the most unstable mode when z is a "frequency") or the one such that Im(z)=0 (in order to determine the conditions for the onset of an instability the other branches having Im(z)<0).
In the majority of the physical cases one has
\[ |\text{Re}(z(a))| \gg |\text{Im}(z(a))| \]
so that the problem of the precision on the location of the branch is of major importance. An added difficulty is that in many cases the root is just due to the balancing of two or more terms of \( f \) (usually \( f = \sum f_i(z,\{a\}) \)) whose order of magnitude may vary greatly when the parameters \( \{a\} \) change. This shows at once that all the iterative methods for solving \( f(z,\{a\}) = 0 \) in which the test to stop the algorithm is based on satisfying \( |f(z,\{a\})| < \varepsilon \) for some \( \varepsilon \) independent of \( \{a\} \), can give rise either to an unnecessary precision or to a very poor one on the location of \( z = z(\{a\}) \). And indeed it is this problem of unambiguously defining the precision which has largely motivated our work.

There are two other important points that we would like to emphasize and that have guided us.

The function \( f(z,\{a\}) \), being usually very complicated, is poorly known. As we shall see the (relative) precision on \( z = z(\{a\}) \) is fixed to a great extent by that of the algorithm which computes \( f(z,\{a\}) \) (hence it is inexpedient to go to higher precision to increase the knowledge of the root). This point is well known and sometimes very clearly stated /6/, but in the codes it seems somewhat forgotten. The second point is the following: the complicated structure of \( f(z,\{a\}) \) is such that this function is usually long to compute. This is a drawback since in order to have a fair knowledge of one of the branches as a function of \( \{a\} \) we need a method that can provide us with a mass production of zeros. Clearly we put a premium on acceleration techniques and on the minimum number of calls of \( f(z,\{a\}) \).

The plan of the paper is the following: next section is devoted
to a very brief and sketchy review of existing methods. We point out the interest of those iterative methods that do converge to a root even from a bad initial guess and explain the reasons that have led us to choose a steepest descent algorithm. In Section 3 we concentrate on the problem of just finding one root. We discuss the strategy for our algorithm and show how a steepest descent method can be adapted to find a triangular region in the complex z plane which contains the root (see Section 3-7) of a poorly known f(z). An apparent drawback of our algorithm is that it requires the knowledge of f'(z). In Section 3-8 we show how to extend our method in order to avoid the explicit knowledge of f'(z). Section 4 is then devoted to the discussion of a number of examples. In Section 5 we discuss how to find more than one zero by supplementing our algorithm via a deflation technique.

2.- SURVEY OF EXISTING METHODS.

Rather arbitrarily we divide the methods used to find the zeros of f(z) in the two classes of non-iterative and iterative ones.

As far as we know the only algorithm of the first kind is based on Cauchy's integral formula

$$\frac{1}{2\pi i} \oint_C g(z) d\ln f(z) = \sum_{n} \frac{1}{n} g(z_n) - \sum_{m} \frac{1}{m} g(z_m)$$

(1)

z_n, n=1,2..N, (z_m, m=1,2..M) being the zeros (poles) of the meromorphic function f inside a simply connected open set A bounded by the closed curve C. Each zero or pole of f in A must be counted with its order of multiplicity, f is meromorphic in A and C does not contain any poles or zeros of f.

Taking in succession g=1, z, z^2; and if f has no poles in A, one obtains in succession the number of zeros, their sum, the sum of their squares,
cubes and so on. This method has recently been documented /14/ and a FORTRAN program has been written /15/. The idea is not new, however, since according to Ref./16/ Cauchy himself realized in the year 1819 that it is possible to compute numerically the zeros of \( f \) with the aid of Eq.(1). This algorithm seems splendid but, as is pointed out in /6/ it may bring us into a "blind alley". This is due to the fact that it may be quite difficult to ascertain that the conditions under which the theorem is demonstrated are valid (e.g. that no zero of \( f \) is located on \( C \)) and because of the very well known difficulty of performing an integral over the arbitrary function \( z^k f'(z)/f(z), k=0,1,2... \) with a predetermined relative precision. To illustrate the last point consider for instance the case \( k=0 \) when the contour passes close to a zero. The integrand can become locally very large, but the integral must be \( O(1) \).

We think that this algorithm can, at most, lead the user to suspect that in a given region of the complex plane some zeros (or/and poles) of \( f \) are located. This piece of information can then be used as the first guess of some iterative methods (see below).

The knowledge of a suitable initial guess for \( z \) is provided by graphical methods also (see Refs./17/, /18/, /31/ for instance). These combined graphical-iterative algorithms are quite interesting, but rather slow. Aside from human intervention of some kind, they necessitate the construction of the function \( f \) over a number of mesh points. Then, having chosen the initial guess, all but one (or a few) of these function values are thrown away, and this is as waste.
We now come to the iterative methods whose number is as great as human fantasy is. To a certain extent the problem of finding the zeros of a complex \( f(z) = u(x,y) + iv(x,y) \) can be thought to be equivalent to that of finding the common zeros \( \{x_n, y_n\} \) of the two real functions \( u(x,y), v(x,y) \). This is a well known problem on which a large body of specialized literature exists (see for instance Refs. /19/ and /20/ and the references cited therein as well as /8/ to /13/). The simple requirement of analyticity on \( f \) imposes rather stringent conditions on the combined behavior of \( u \) and \( v \). Moreover, in a number of cases, it is almost impossible to write down \( u \) and \( v \) explicitly as functions of \( x \) and \( y \). This immediately shows that many of the methods discussed in Refs. /19/, /20/ which cover the case of \( n \) real functions in \( n \) real variables are either too cumbersome or not applicable at all (like e.g. Brown's algorithm, see Ref. /13/ and Ref. /20/ p. 281).

Iterative methods are based on an initial guess about the location of the required root of \( f(z) \) and on a refinement of this guess, via an appropriate algorithm, which is repeated until some test is satisfied. A great deal of work has been done about the definition of the iterated guesses, their convergence properties and the economy of the entire operation. For a number of algorithms (as e.g. Newton's one) it is very difficult, if not impossible at all, to know a priori the set of all initial guesses for which the iterates are well defined. However methods exist which are guaranteed to converge to a root from an arbitrary guess \( z_0 \) at which \( f(z) \) is regular. These are, for instance, Schechter's nonlinear successive overrelaxation process /21/, combinational search algorithms /22/, /23/ based on a constructive proof of Brouwer's fixed point theorem, Davidenko's imbedding technique /24/ to /27/, /7/, /10/, and gradient methods /5/, /11/, /12/, /19/ chapter 8, /28/, /32/.
We do not discuss in detail the first two kinds of algorithms. Suffice it to say here that Schechter's method is adapted (like Brown's method) to the case in which \( u(x,y) \), \( v(x,y) \) are known separately and that combinatorial search algorithms even though able to get into a region of local convergence from a poor initial guess, are such that it may take as much time to obtain a second significant digit of accuracy in the solution as it did to arrive at the first significant digit.

Davidenko's technique, on the other hand, can be used with an entangled \( f(z) \) and may be sufficiently rapid. In its simplest and original form it consists in defining the imbedding

\[
f(z(t)) = (1-t)f(z_0), \quad z_0 = x_0 + iy_0
\]  

(2)

from which

\[
\frac{dz}{dt} = -f(z_0)(\frac{df}{dz})^{-1} = F(z_0, z)
\]

(3)

This is a couple of differential equations for \( x(t), y(t) \) with initial conditions \( x_0, y_0 \) at \( t=0 \) whose solution at \( t=1 \) gives a root \( z_1 = z(t=1) \) of \( f(z)=0 \). To justify this method it has to be proved that Cauchy's problem has a solution which is moreover a single solution for the whole range \( 0 \leq t \leq 1 \). This has been done in Ref. /27/ for the imbedding defined by Eq.(2) as well as for:

\[
z(t) - z_0 + t[f(z(t))-z(t)+z_0] = 0
\]

(4)

which gives:

\[
\frac{dz}{dt} = \frac{z-z_0-f(z)}{1+t(\frac{df}{dz} - 1)} = F(z_0, z, t)
\]

These imbeddings are such that overflow problems can be encountered in the case of multiple roots. These problems may be overcome by skillfully programming \( F \) in the region \( t=1 \) where \( \frac{df}{dz} = 0 \). However it is quite annoying to realize that these imbeddings may break down at the end of the iteration
process. Notice also that a proof that Cauchy's problem has a solution for the case of multiple roots is apparently lacking. Multiple roots may occur in practice since in many physical applications the function $f$ depends on the parameters $\{a\}$ in such a way that two or more simple roots may coalesce to a multiple one for a certain choice of $\{a\}$ and, in general, these critical cases are the most interesting ones. Now, the main defect of Davidenko's method is that in going from $z(t)$ to $z(t+\delta t)$ one must compute the R.H.S. of the equation for $\frac{dz}{dt}$ a number of times. These are fixed by the method of integration adopted, but are at least two per step (as in a predictor-corrector method). Moreover the problem of the number of significant digits of the root is not solved in a way as simple and clear cut as it is with our method.

The algorithm we propose belongs to the large class of gradient-descent methods (also in this case their originator seems to be Cauchy /32/). We think that the strategy we have adopted is new so that many of the problems usually encountered with the steepest descent /5/, /28/ are overcome.

The choice of the first iterated guess $z_1 = z_0 + pe^{i\theta}$ is one of them since we want to avoid a too small step size $p$. We shall show that, knowing $f(z_0), f'(z_0), f''(z_0), z_0$ being the first guess, it is possible to give an order of magnitude estimate for the step size $p$ such that (the angle $\theta$ being given in terms of $f(z_0), f'(z_0)$) it is possible to be reasonably sure that $|f(z_1)| < |f(z_0)|$. Otherwise $p$ is multiplied by a reduction factor $K_p$ (more on this later) a number of times until this inequality is satisfied. The large number of tests we have made with increasingly complicated functions (see Sect.4) has shown that in general it is not necessary to change the estimate for $p$ to $p K_p^n, n>1$. The knowledge of $f''(z_0)$ seems disturbing. However this second derivative is needed
only at the initial step and can be obtained numerically. Notice that 
the problem of the initial step size exists also when using an imbedding 
technique. In the case of ordinary differential equation the step size \( \Delta t \) 
is usually chosen intuitively and then doubled or halved until some 
precision test is satisfied.

We shall discuss how to define the subsequent iterations until 
\[ |f(z_{n+1})| > |f(z_n)| \] 
with \( |z_{n+1} - z_n| < \rho_{\text{crit}} \) where \( \rho_{\text{crit}} \) depends on the 
precision requested. In this case three possibilities may occur: we are 
either close to a zero (simple or multiple) or to a saddle point or in the 
bottom of a long valley. We shall show how to discriminate between these 
situations. The only problem which can be difficult to treat with our 
gradient algorithm is that of a function whose modulus exhibits a very narrow valley 
(a "canyon"). This may cause the iterated point \( z_n \) to move in a zig zag 
line over the bottom line making the convergence very poor. In the next 
section we show that the surface described by the function \( \phi(x,y) = |f(x+iy)|^2 \) 
cannot have such a shape when \( z \) is sufficiently close to a zero.

We want now to point out a decisive advantage of a steepest 
descent technique over an imbedding technique. Both make use of the 
knowledge of \( f' \), but while at each step \( z_n \rightarrow z_{n+1} \) the former uses as 
a criterion the rather mild condition \( |f(z_{n+1})| < |f(z_n)| \) the second 
uses the much more stringent conditions concerning the integration of 
ordinary differential equations. In our opinion it is a waste of time 
to test the precision all along the trajectory when in reality it is 
needed at the final step only. This suggests that the number of times 
that \( f \) and \( f' \) are computed with an imbedding technique may be far greater 
than that needed by a well conceived steepest descent algorithm. On the
other hand it might seem that a great disadvantage of this method is the necessity to compute both $f$ and $f'$. However we point out that in many practical cases a number of the elementary building blocks that are used to compute $f(z)$ can be stored in a memory and used again to construct $f'(z)$. In those cases in which $f(z)$ is so complicated that it is out of question to programme $f'(z)$ (e.g. when $f(z)$ is the determinant of a $N \times N$ matrix) we show that our method can still be used by computing $f'(z)$ numerically.

3. - DESCRIPTION OF THE METHOD

3.1) Direction of Steepest Descent and Initial Step Size.

Let us associate to a meromorphic function $f(z), z=x+iy$, the non negative real function $\phi(x,y)$:

$$\phi(x,y)=|f(x+iy)|^2$$  \hspace{1cm} (5)

It is easy to check that

$$\frac{\partial \phi}{\partial x} = 2 \text{Re}(f'(f')^*)$$

$$\frac{\partial \phi}{\partial y} = 2 \text{Im}(f'(f')^*)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2|f'|^2 + 2\text{Re}(f(f'')^*)$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 2\text{Im}(f(f'')^*)$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2|f'|^2 - 2\text{Re}(f(f'')^*)$$  \hspace{1cm} (6)
and so on for the higher derivatives. Putting now:

\[
\begin{align*}
X(x,y) &= \frac{\partial \phi}{\partial x} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right]^{-1} \\
Y(x,y) &= \frac{\partial \phi}{\partial y} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right]^{-1}
\end{align*}
\]

(7)

the steepest descent paths \( x(\tau), y(\tau) \) satisfy:

\[
\begin{align*}
\frac{dx}{d\tau} &= X(x,y) \\
\frac{dy}{d\tau} &= Y(x,y)
\end{align*}
\]

(8)

\( \tau \) being a parameter. The singular points \( x_s, y_s \) of this system, at which \( X(x_s, y_s)=Y(x_s, y_s)=0 \), are located at the zeros, poles and saddle points of \( f(z) \) as it is easy to see from Eqs. (6), (7). The converse is also true: a regular point of the system (8) is such that both \( f(z)\neq0 \) and \( f'(z)\neq0 \).

A direct application to Eq. (8) of the well known perturbation analysis of two dimensional real autonomous systems (see Ref [33/ p. 370] near any singular point \( z_s=x_s+i y_s \) shows that \( z_s \) is either a proper node or a saddle point. \( z_s \) is never a spiral point. This statement has very important consequences for us. It can be established most easily by considering the level curves \( \phi(x,y) = \text{const.} \) In fact consider first \( f(z) \) with a zero of order \( n \) at the origin, we can write:

\[
f(z) = \frac{f^{(n)}(0)}{n!} z^n + \frac{f^{(n+1)}(0)}{(n+1)!} z^{n+1} + O(|z|^{n+2}).
\]

Then, with \( z=\rho e^{i \theta} : \)

\[
\phi(x,y) = \rho^{2n} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \left( 1 + \frac{2}{n+1} \frac{f^{(n+1)}(0)}{f^{(n)}(0)} \rho \cos(\theta-\phi) + O(\rho^2) \right)
\]

where \( \theta = \text{arg} \{ f^{(n)}(0)/f^{(n+1)}(0) \} \). Hence if:

\[
\rho < \frac{n+1}{2} \left| \frac{f^{(n)}(0)}{f^{(n+1)}(0)} \right|
\]

(9)
the level curves are (almost) circles. The same argument applies near a pole with \( f(z) \sim \frac{1}{f(z)} \). Hence the steepest descent lines, which are orthogonal to the \( \phi(x,y) \) const. curves, are directed along the rays stemming from a pole (or zero) of \( f(z) \). A zero of \( f(z) \) is a sink and a pole is a source for the integral curves of Eq. (8).

Near a saddle point we can write:

\[
f(z+\delta z) = f(z) + \frac{f^{(n)}(z)}{n!}(\delta z)^n + O(|\delta z|^{n+1})
\]

where by \( f^{(n)}(z) \) we indicate the first non zero derivative of \( f \) at (the saddle point) \( z \). Hence with \( \delta z = pe^{i\theta} \) and \( \beta = \arg \{f^n f^{(n)}\} \):

\[
\phi = |f(z+\delta z)|^2 = |f(z)|^2 + 2\text{Re}[f^n f^{(n)}] \cos(n\theta + \beta) + O(p^{n+1})
\]

which shows the well known property of meromorphic functions that around a saddle point of order \( n-1 \) there is a starlike pattern made up of \( n \) sectors of decrease separated by \( n \) sectors of increase for \( \phi(x,y) \). Thus we have shown that no singular point of the system (8) is of the spiral type.

From Eq. (11) it follows also the well known theorem on meromorphic functions which states that at a point \( z \) at which \( f \) is regular, its modulus cannot have either a maximum or a minimum (\( \neq 0 \)) but is at most stationary (\( f' = 0 \)). This is basic for our method since all the steepest descent lines (except the "exceptional" ones leading to a saddle point) must end to a zero of \( f(z) \).

At a regular point the direction of steepest descent is given by Eq. (7) and by \( \cos(n\theta + \beta) = -1 \) in Eq. (11) for a singular point. Hence for \( n > 1 \) we take

\[
\theta(z) = \arg \left[ -f(z)/f^{(n)}(z) \right]^{1/n}
\]
from which:
\[ \delta z = R(z) (-f(z)/f^{(n)}(z))^{1/n}. \] (13)

where \( R(z) \) is real, positive and "small". This result immediately suggests the following iterative procedure: given a point \( z\_m \), the subsequent point is
\[ z\_{m+1} = z\_m + R\_m (-f(z\_m)/f^{(n)}(z\_m))^{1/n}. \] (14)

Of course this does not ensure that \( |f(z\_{m+1})| < |f(z\_m)| \) since \( R\_m \) can be too large. An estimate of \( R\_m \) can be obtained in the following way:
calling \( \delta z\_m = z\_{m+1} - z\_m \)
we have from Eq. (14)
\[ (\delta z\_m)^n f^{(n)}(z\_m) + R\_m^n f(z\_m) = 0. \]

Compute now:
\[ f(z\_{m+1}) - f(z\_m) (1 - \frac{R\_m^n}{n!}) = f(z\_{m+1}) - f(z\_m) - (\delta z\_m)^n f^{(n)}(z\_m) \]
\[ = \sum_{j=0}^{n+1} \frac{f^{(j)}(z\_m)}{j!} \left( \frac{R\_m}{f^{(n)}(z\_m)} \right)^{1/n} \].

Putting
\[ A\_n(z\_m) = \sum_{j=0}^{n+1} \frac{f^{(j)}(z\_m)}{j!} \frac{|f^{(n)}(z\_m)|}{|f^{(j)}(z\_m)|^{1/n}} \]
we can write
\[ \frac{|f(z\_{m+1})|}{|f(z\_m)|} \leq \left| 1 - \frac{R\_m^n}{n!} \right| + R\_m^n A\_n(z\_m) \]
and to ensure that \( |f(z\_{m+1})| < |f(z\_m)| \) it is sufficient that
\[ \left| 1 - \frac{R\_m^n}{n!} \right| + R\_m^n A\_n(z\_m) \leq 1. \] (15)
To discuss this inequality let us take \( n = 1 \), i.e., the point \( z_m \) is not a saddle point, and let us take for \( A_1 \) the first term only of the series above. From \( \delta z = - Rf/f' = \rho e^{i\theta} \) we get the following condition for the step length \( \rho \) at \( z_m \):

\[
\rho = \rho(z_m) \leq \min \left\{ 2 \left| \frac{f'}{f''} \right|, \left| \frac{f''}{f''} \right| \left| -1 + \sqrt{1 + 4 \left| \frac{f''}{f'} \right|^2} \right| \right\}
\]  

(16)

Notice that this condition must be understood as only giving the order of magnitude estimate for \( \rho \). Therefore we shall use it only to compute the length of the first step and make sure that the resulting \( \rho_1 \) lies between two tolerable limits \( \rho_{\text{max}}, \rho_{\text{min}} \) imposed a priori, in order to avoid too large or too small steps at the beginning of the iterative procedure.

3.2) The Minimum Significant Step Size \( \bar{\rho}(z) \)

Suppose now that we have reached the point \( z_{m+1} \) from \( z_m \) with a step size \( \rho_m \) and an angle of steepest descent \( \theta_m \):

\[
z_{m+1} = z_m + \rho_m e^{i\theta_m}
\]  

(17)

\( z_m \) can be the initial point \( z_0 \). In this case, if \( f'(z_0) = 0 \), we start from a new \( z_0 \) located in a random direction at a distance \( \rho_{\text{min}} \) apart, for instance:

\[
z_{\text{new}} = z_0, \text{old} + \rho_{\text{min}}.\)

The strategy we shall adopt to make the following iteration will depend basically on the magnitude of \( |f(z_{m+1})| \) with respect to \( |f(z_m)| \). However, in order that it be successful, there is a fundamental point we want to stress. On a short scale length, we must think the value of \( f(z) \) as comprised within some error bars. These are due in part to the random fluctuations caused by truncation in a finite length word computer and in part due to the intrinsic uncertainty characterizing the algorithm used in computing \( f(z) \).
The scale length of these fluctuations will determine a lower bound $\bar{\rho}(z)$ for our step length and it will be illusory to isolate a root of $f(z)$ inside a domain smaller than $\bar{\rho}(z)$. In other words, it is of the greatest practical importance that the step length $\rho$ be large enough so that the difference between $f(z_m)$ and $f(z_{m+1})$ be significant i.e.

$$|f(z_{m+1}) - f(z_m)| > K_1 \left( |\Delta f(z_{m+1})| + |\Delta f(z_m)| \right)$$

Here $|\Delta f|$ is an estimate of the error on $f$ and $K_1$ is a constant larger than unity and whose necessity will be made clear in the following (in section 4 we shall give a convenient choice for this constant as well as for the other $K_n$ we shall define in the sequel).

We shall consider that $|f(z_{m+1})|=|f(z_m)|$ when condition (18) is not satisfied and $|f(z_{m+1})|\neq|f(z_m)|$ in the opposite case. Therefore it is important to have an accurate estimate of $|\Delta f|$.

It will be shown in Sect.3.7 that the fact that we have put $K_1>1$ has important consequences also regarding the possibility of obtaining an accurate numerical value of the first derivative of $f(z)$.

3.3) Predetermined Uncertainty on the Root

Before describing our method in detail, we must point out that in a number of cases the value of $\bar{\rho}(z)$ obtained from (18) can be much smaller than the precision requested on the zero. Indeed there is a critical minimum value $\rho_{\text{crit}}(z)$ of the step size which is fixed by the wanted precision. Clearly $\rho_{\text{crit}}$ is of the order of the given uncertainty of the root.
In general one is interested in finding a root $z_0$ of $f(z) = 0$ with a given number $n$ of significant digits. In this case one may take:

$$\rho_{\text{crit}}(z) = \varepsilon |z|, \quad \varepsilon = 10^{-n} \quad (19)$$

if it is known, a priori, that $|\Re{z_0}|$ and $|\Im{z_0}|$ are, almost, of the same order. As already said it is possible to have $|\Im{z_0}| \ll |\Re{z_0}|$ and in this case it would be more convenient to take:

$$\rho_{\text{crit}}(z) = \varepsilon |\Im{z}|, \quad \varepsilon = 10^{-n} \quad (20)$$

if the imaginary part of $z$ is of interest. Of course this $\rho_{\text{crit}}(z)$ may become tremendously small if some root approaches too much the real axis ($\Im{z_0} = 0$), but in this case (see the next subsection) the iteration will be automatically stopped when the minimum significant step size $\rho(z)$ is reached. In this case we shall obtain the maximum precision which is compatible with the uncertainty in the algorithm which computes $f(z)$. In other words, at each step, we ascertain that $\rho(z)$ satisfies the following inequality:

$$\rho(z) > \rho_0(z) = \max\left\{\rho(z), \rho_{\text{crit}}(z)\right\}. \quad (21)$$

3.4 ) Test of the Zero : The Criterion of the Triangle.

The purpose of the iterative procedure that we shall describe in Sect. 3-7 is to isolate, if possible, a particular region of dimensions $\rho$ of the order of $|\rho_0(z)|$ in which there is either a zero or a saddle point of $|f(z)|$ or the bottom of a "canyon" of $|f(z)|^2$. Another problem is that of giving simple criteria which allow the computer to discriminate between these three cases.
We begin to explain how to decide whether the point we have reached is either close to a zero of \( f(z) \) or not. Let us consider \( F(z) \) defined as:

\[
F(z) = \frac{f(z)}{f'(z)}
\]

so that if \( f(z) \) has a zero of order \( n \) at \( z_0 \):

\[
f(z) = A_n(z-z_0)^n + O(|z-z_0|^{n+1})
\]

\( F(z) \) has only a simple zero at \( z_0 \) and is linear in some vicinity of \( z_0 \):

\[
F(z) = \frac{1}{n}(z-z_0) + O(|z-z_0|^2)
\]

Let us now call \( z_1 \) and \( z'_1 \) (see Fig.1) the last two points, at a distance \( \rho(z'_1) = \rho_0(z'_1) \) apart, that we have obtained and which are such that \( |f(z_1)| > |f(z'_1)| \) although \( z_1 \) is in the direction of steepest descent from \( z'_1 \). If in this region of the complex plane, there is a (multiple) zero of \( f(z) \), (and hence a simple one of \( F(z) \)) it must be situated near the segment joining \( z_1 \) and \( z'_1 \).

Let us construct a triangle whose vertices are \( z_1, z_2, z_3 \) with

\[
z_2 = z_1 + \alpha p(z'_1) e^{i(\theta - \pi/6)} , \quad z_3 = z_1 + \alpha p(z'_1) e^{i(\theta + \pi/6)}
\]

where \( \theta \) is the direction of steepest descent from \( z_1 \) (notice that this direction is, in principle, different from that computed at \( z'_1 , \theta(z'_1) \neq \theta(z_1) \)) and where \( \alpha \) is slightly greater than unity. For instance \( \alpha = 1.6 \). The reason why it is better to choose \( z_1 \) instead of \( z'_1 \) as the first vertex of the triangle will be made clear in Sect. 3-8.
We adopt then the following line of action. First we make sure that in the triangular region $z_1$, $z_2$, $z_3$ the function $F$ is "roughly linear". This is ascertained by testing whether:

$$J_{12} = \frac{F(z_1) - F(z_2)}{z_1 - z_2}, \quad J_{13} = \frac{F(z_1) - F(z_3)}{z_1 - z_3} \quad (26)$$

agree with an accuracy of some percent (say), i.e. if:

$$C = |1 - J_{12}/J_{13}| < K_2 < 1 \quad (27)$$

If the precision demanded on the root is not very large, $\rho_{cr}(z_1)$ may be some orders of magnitude larger than $\bar{\rho}(z_1)$ so that $\rho(z_1)\rho_{cr}(z_1) > \bar{\rho}(z_1)$ because of Eq (21). In such a case it may occur that it is impossible to satisfy the inequality (27) since $\rho(z_1)$ is too large. For this reason we are obliged, when necessary, to forget about Eq (21) and to let $\rho(z_1)$ become so small as to satisfy the inequality (27) provided that it be always larger than $\bar{\rho}(z_1)$, i.e. provided that the fundamental inequality (18) be satisfied. This point will be discussed in more detail in sect. 3-7.

In practice (see Sect. 4) this condition is always satisfied if there is a zero nearby. If $C > K_2$ and if $\rho = \bar{\rho}(z)$ we shall try to leave this region by what we call the "heavy flak scenario" which will be described later.

We now come back to the normal case where the linearity is satisfied and recalling that $F(z)$ has simple zeros only, we can assert that $F(z)$ has a root at a point $z_0$ located inside the triangle $z_1$, $z_2$, $z_3$ if and only if the sign of $\text{Re}(F)$ and of $\text{Im}(F)$ changes twice and alternatively when making a complete tour $z_1 \to z_2 \to z_3 \to z_1$ (see Fig. 1). Notice that this criterion can be seen
as the simplest and most natural extension in the complex domain of the well known property of real continuous functions that if \( f(x_1)f(x_2) < 0 \) then \( f(x_0) = 0 \) at some \( x_0 \) satisfying \( x_1 < x_0 < x_2 \).

In practice (and this is one of the reasons why we need \( F \) to be "roughly linear") we shall test whether the above statement is valid by linear interpolation. Represent each side of the triangle via a parameter \( t \) running from 0 to 1. Taking for instance the side \( z_1, z_2 \) of Fig. 1 we have \( \text{Re}\{F\} = 0 \) for:

\[
t_R = -\frac{\text{Re}\{F(z_1)\}}{\left| \text{Re}\{F(z_2)\} - \text{Re}\{F(z_1)\} \right|}
\]  

(28)

and we know whether \( \text{Re}\{F\} \) vanishes "before" or "after" \( \text{Im}\{F\} \) when going from \( z_1 \) to \( z_2 \) by comparing \( t_R \) and \( t_I \) with:

\[
t_I = -\frac{\text{Im}\{F(z_1)\}}{\left| \text{Im}\{F(z_2)\} - \text{Im}\{F(z_1)\} \right|}
\]  

(29)

at which \( \text{Im}\{F\} = 0 \).

It is now easy to understand the necessity of the parameter \( K_1 > 1 \) introduced in Eq.(18). The lines \( \text{Re}\{F\} = 0 \) and \( \text{Im}\{F\} = 0 \) have a finite thickness due to the imprecise calculation of \( F \); hence the triangle must be large compared to this thickness. However it must not be too large since, on one hand, its dimensions represent the uncertainty on the zero \( z_0 \) and, on the other hand, \( F \) must be approximately linear in order to make Eqs.(28) and (29) meaningful.

At last we must observe that it does not make sense to apply the criterion of the triangle if the function \( f \) does not satisfy the inequalities:

\[
|f(z_i)| > |\Delta f(z_i)| \quad i = 1, 2, 3
\]  

(30)
3.5) Better Approximation of the Zero and its Multiplicity

If the criterion introduced above is satisfied, $F(z)$ has a simple zero inside the triangle and is sufficiently well represented in this region by the linear function

$$F(z) = \frac{1}{n'}(z-z_0)$$

where $n'$ and $z_0$ are unknown. These can be determined by imposing that this form of $F$ be exact at two vertices $z_i, z_j$. We obtain:

$$n' = \frac{z_i - z_j}{F(z_i) - F(z_j)}$$

(32)

for the multiplicity and for the root $z_0$:

$$z_0 = z_i - n'F(z_i) = z_j - n'F(z_j).$$

(33)

In order to avoid the loss of significant figures at this final step it is convenient to choose as vertices $z_i, z_j$ those enclosing that side of triangle at which both Re{$F$} and Im{$F$} change sign.

Of course the multiplicity $n'$ determined via Eq.(32) is a complex number and it must be transformed into the integer $n$ closest to it.

The refined zero $z_0$ and its multiplicity $n$ are taken to be acceptable if the following two consistency conditions are satisfied, namely: i) $n'$ has a positive real part and differs in modulus from an integer $n$ by less than some constant $K_3$:

$$\text{Re}(n') > 0 \quad |n' - n| < K_3 < 1,$$

(34)
and: ii) if calling $z_c$ (see Fig. 1) the centre of the triangle $z_1, z_2, z_3$ we have:

$$|z_0 - z_c| < |z_1 - z_c| = \rho_{\text{fin}}$$

(35)

If both requests are met we can assert that the absolute error on the modulus of $z_0$ is given by $\rho_{\text{fin}}$. This can be pessimistic since the final precision on the root obtained can be much greater than that requested (in many cases, see Sect. 4, we get at least one more significant digit on $z_0$). In practice it occurs that $|f(z_0)|$ is some orders of magnitude smaller than $|f(z_1)|$ suggesting that $z_0$ is indeed quite close to the root.

It is easy to understand why the knowledge of the root can be improved via a single step iteration. It is possible to look at Eqs. (32), (33) either as a secant method applied to $F(z)$ or as a "damped" Newton method applied to $f(z)$. The curious feature of this Newton-type method is that the (modulus of the) "damping" factor $n'$ is larger than (or equal to) unity and is automatically determined, whereas in a normal scheme $|n'| < 1$ and arbitrary.

At last we point out a simple geometrical interpretation of the point $z_0$. Let us call $z'_0$ the point of intersection of the steepest descent directions at $z_i, z_j$. Since $\arg(z_0^i - z_k) = \arg(F_k^i), k = i, j$, while $\arg(z_0 - z_k) = \arg(-F_k) + \arg(n'), k = 1, j$, from Eq. (33), it is clear that $z_0$ and $z'_0$ practically coincide if $n'$ is close to a positive integer.

3.6) "Heavy Flak" Scenario

We now deal with the case in which the criterion of the triangle shows that the point $z_1^i$ at which $|f(z_1^i)|$ is seen as a minimum is not situated close to a zero. The point $z_1^i$ can be close either to a saddle point or to the bottom of a canyon for $\phi(x,y) = |f(x+iy)|^2$. The first case is rather unusual since it is very hard to approach a saddle point. One must start from and
move right on the divide of $\phi(x,y)$ which is an
unstable line as far as a steepest descent method is concerned. Rounding
off in $z$ due to the finite length of the computer word suffices to drift
away from this line (which is in general a curve in the $z$-plane) after a
few iterations. The opposite situation occurs at the bottom of a canyon. In
a sufficiently small region, of dimension $R$ (say), the steepest descent curves
are almost parallel and provided that the step size $\rho(z)$ is such that
$\bar{\rho}(z)<\rho(z)\approx R$ the polygonal of the steepest descent points will first reach and
then follow the bundle of slightly converging descent curves embracing the
bottom line. When this is the case there is no problem in sliding down along
the bottom of the valleys of $\phi(x,y)$ to reach the zero (notice that the
problems encountered in Ref. /28/ stem from the fact that for the functional
$\phi(x,y)=|\text{Ref}|+|\text{Imf}|$ the steepest descent curves have a cusp at the bottom
line). The problem is that it is not easy to evaluate $R(z)$. Because of this
and in order to move away from a possible saddle point we use what we have
called the "heavy flak scenario". This consists in computing $f(z)$ at a
number of points (in our code 20) distributed randomly and uniformly on the
circumference of a circle centered at $z_1$ with radius $\rho=|z_1-z_1|$. Out of
these points we choose that for which $|f(z)|$ is minimum and smaller (in the
sense of Eq. (18)) than $f(z_1)$. If such a point exists we take it as the new
"initial guess" and start the whole procedure again (i.e. estimate of $\rho_0$ via
the second derivative of $f(z)$ etc ...).

If $z_1^*$ is close to a saddle point we succeed in moving off from $z_1$
via this simple method, although we do not have any control on which particular
root of $f(z)$ we will, perhaps, obtain.
If the error $|\Delta f|$ on $f(z)$ has been grossly underestimated the step size $\rho$ can become extremely small. In this case it is very easy to run into the troubles of the heavy flak scenario. We must point out that, after having correctly estimated $|\Delta f|$, we have never encountered problems despite our efforts to construct "difficult" functions. At first sight this may seem strange. For the case of non negative real functions of two or more variables examples are known /34/ of deep curved valleys leading to a zero. A simple one is: $\phi(x,y) = 100(y-x^2)^2 + (1-x)^2$.

For an arbitrary real function $\phi(x,y)$ of two variables all the derivatives $\frac{\partial^{m+n}\phi}{\partial x^m \partial y^n}$ are independent. So a subclass of these functions can display canyons of arbitrary monstruosity. If, as in our case, $\phi(x,y)$ is given by Eq.(5), the derivatives of $\phi$ are no more free as it is shown by Eq.(6).

Consider now, for simplicity's sake, a function having simple zeros only and think of a (curved) valley oriented, at least locally, along the $y$-axis. A canyon would correspond to very small values of $\frac{\partial \phi}{\partial y}$, $\frac{\partial^2 \phi}{\partial y^2}$ and to very high values of $\frac{\partial \phi}{\partial x}$, $\frac{\partial^2 \phi}{\partial x^2}$ (for instance $f(z,a) = \text{Cosh } z - a$ with a real and $0 < a \ll 1$ gives rise to a $\phi(x,y)$ which has these properties). However when we approach the zero we have:

$\frac{\partial \phi}{\partial x}$, $\frac{\partial^2 \phi}{\partial x \partial y}$, $\frac{\partial^2 \phi}{\partial x^2} \rightarrow 0$ and $\frac{\partial^2 \phi}{\partial y^2} \neq 0$ (and, of course, we did not encounter any problem when finding numerically the zeros of $f = \text{Cosh } z - a$ with $10^{-6} < a < 1$).

### 3.7) The Strategy of the Iterative Process

The purpose of the iterative procedure, that we are going to describe, is to isolate a region of dimension $\rho_0(z)$, Eq.(21), in which there is either a zero of $f(z)$ or a saddle point i.e. a zero of $f'(z)$. We have already discussed how to discriminate between these possibilities.
After having jumped from $z_m$ to $z_{m+1}$ according to Eq.(17), three cases may happen when we compare $|f(z_m)|$ with $|f(z_{m+1})|$. 

A) $|f(z_{m+1})| < |f(z_m)|$ i.e. we are "going down". In this case we use $z_{m+1}$ as a new point, compute here a new angle of descent $\theta_{m+1}$, lengthen the step size by putting $\rho_{m+1} = K_4 \rho_m$, $K_4 > 1$, in order to approach faster the region of the root, and make a new step.

B) $|f(z_{m+1})| = |f(z_m)|$. We lengthen the step size $\rho_m \rho_m + K_5 \rho_m - K_5 > 1$, if at the previous iteration we had $|f(z_{m-1})| > |f(z_m)|$ (the cases $|f(z_{m+1})| < |f(z_m)|$ are discussed below) and starting again from $z_m$, we compute another point $z'_{m+1}$ along the old direction $\theta_m$. The process is repeated until condition (18) is satisfied.

C) $|f(z_{m+1})| > |f(z_m)|$ i.e. we are "going up". We compute $\rho_{crit}(z_m)$, Eq.(19) for instance. Two possibilities arise whether $\rho_m |z_{m+1} - z_m| > \rho_{crit}(z_m)$.

i) $\rho_m < \rho_{crit}(z_m)$. We have attained the required precision and we construct the triangle as indicated in Sect. 3-4. If the linearity of $F$ is satisfied, Eq. (27), in this region then: either the triangle contains a zero (so that we can find a better approximation to it as well as its multiplicity, Sect. 3-5, and we stop the search), or not. In this case we use the "heavy flak" scenario to leave this region. If $F$ is not "linear" we adopt the same strategy as that of the case:

ii) $\rho_m > \rho_{crit}(z_m)$, we shorten the step size $\rho_m \rho_m + K_6 \rho_m - K_6 > 1$, and from $z_m$ we compute another $z'_{m+1}$ always in the direction $\theta_m$.

If $|f(z'_{m+1})| < |f(z_m)|$, we have case A) again and, in principle, we might continue the search as stated there taking $z'_{m+1} + z_{m+1}$ as a new point of descent. However, since a "going up" step has just been made, we can suspect to be close to a zero. In this case, lengthening the step size as in A) could give rise to a going up step at the next iteration.
In order to avoid this unpleasant situation we make a certain number of iterations (3 in our case) with $K_4=1$. If for all these steps we are "going down", then we put $K_4>1$ again. With this simple recipe, the number of useless iterations is substantially reduced.

If $|f(z_{m+1})|=|f(z_m)|$ we cannot go back to case B) above since, in general, we would make a "going up" step again. Two situations may occur: either $z_m$ and $z_{m+1}$ are so close that $|z_{m+1}-z_m|<\delta(z_m)$ or they are on the same level curve. In this case (see Fig. 3) they are, roughly, symmetrically placed with respect to the root. To remove this ambiguity we make, from $z_m$, a new step with a smaller step size ($\rho_m=\rho_m/2$). If we find $|f(z_{m+1}^*)|=|f(z_m)|$ again it means (with the exception noted below) that we have reached the lower bound of the significant step size $\bar{\rho}(z)$, (see Fig. 2). Therefore it is impossible to reach the predetermined precision $p_{\text{crit}}(z_m)$ in this case. We must stop the search at this point and ascertain (see Sect. 3.4) if there is a root of $f(z)$ within a distance equal to the length of the last significant ($|f(z_{m+1}^*)|\neq|f(z_m)|$) step we have made. Otherwise we continue the iteration (see Fig. 3). The exception here is a situation like that illustrated in Fig. 4. It may occur, for instance, when two iterated points $z_{m+1}$ (the old one, point N°2 at a distance $\rho_m/K_6$ from $z_m$ and the new one, point N°3, at a distance $\rho_m/2K_6$) lie in two distinct valleys of $|f(z)|$. This may occur for instance (see Sect. 4, case 3 for an example) when two or more roots are located at a distance comparable to or less than the smallest significant step $\bar{\rho}(z)$ allowable. In this case our strategy may fail and the only remedy is to have a more precise algorithm to compute $f(z)$ in this critical region.
If \(|f(z_{m+1})|>|f(z_m)|\) we are in the same situation as at the beginning of this case C). We repeat the whole procedure by putting \(z_{m+1}^l = z_m^l + \rho_{m+1}^l + \rho_m^l\) and so on. There is no danger of iterating ad infinitum since if \(\rho_m^l \to \delta(z_m)\) we stop the search and check in the usual way (Sect. 3.4) if there is a zero nearby. If no zero is found we let loose the "heavy flak" (Sect. 3-6).

3.8) **The Numerical Calculation of the First Derivative**

We consider now the case in which the function \(f(z)\) is so involved that it is out of question to compute \(f'(z)\) analytically. In this case, we must compute the first derivative numerically.

According to the strategy already discussed we need \(f'(z)\) in two cases only. The first one occurs when we are "going down" i.e. when \(|f(z_{m+1})| < |f(z_m)|\) since the angle of descent at \(z_{m+1}\) is expressed in terms of \(f'(z_m+1)\), see Eq. (12). Notice that we do not need a very large precision on \(f'\) in this case. A small error in the angle \(\theta\) simply means that we will not approach the zero along the optimum polygonal.

The first derivative may also be necessary when we are ready to perform the test of the triangle. However if we know, a priori, that \(f\) has simple zeros only we can identify \(F\) with \(f\) in Eq. (22) and we can get rid of \(f'\) at this stage. But either if we lack this piece of information or if \(f\) has multiple roots we need a quite accurate computation of the first derivative. In fact if we compute \(f'(z)\) with a too large imprecision, the function \(F\) is not accurately represented by a linear form and the test of the triangle will not work even though we are close to a zero.
In order to see how to compute a reliable $f'(z)$, at least in most cases, we use the usual Taylor expansion of $f(z+h)$ to write:

$$f'(z) = D_p(h,f(z)) + \frac{h^p}{(p+1)!} f^{(p+1)}(z) + O(|h|^{p+1})$$  \hspace{1cm} (37)

where $D_p$ is a $p$-points finite difference expression constructed with $f(z)$ and where $h$ is the step size of the mesh points (for instance for two points we have $D_2(h,f(z)) = \frac{f(z+h) - f(z-h)}{2h}$, and so on). $D_p$ will represent the numerical derivative of $f(z)$ and $h^p f^{(p+1)}/(p+1)!$ an error term ($\Delta f'$) due to the finite step size $h$. Our problem is to make this error term negligible.

Intuitively it is obvious that in the case of a multiple root $h$ should be smaller than the distance between $z$ and the zero $z_0$. In fact we show that if the multiplicity $n$ of the zero at $z=z_0$ is greater than the number $p$ of mesh points, we have $|\Delta f'(z)| = |f'(z)|$ if $|h| = |z-z_0|$. Indeed let us take:

$$f(z) = (z-z_0)^n g(z) = (z-z_0)^n \sum_{k} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k.$$

From

$$\frac{d^n}{dz^n} (z-z_0)^n g(z) = \sum_{k} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k$$

with $K = \max(0, s-n)$, it follows that if $n \geq p+1$:

$$|\frac{\Delta f'}{f'}| = \frac{(n-1)!}{(p+1)!(n-p-1)!} \frac{h}{|z-z_0|} \left| \frac{1}{1+O(|z-z_0|)} \right|$$  \hspace{1cm} (38)
while if $n < p + 1$

$$\left| \frac{\Delta f}{f} \right| = \frac{1}{n(p+1-n)!} \left| \frac{g(p+1-n)(z_0)}{g(z_0)} \right| \frac{h}{|z-z_0|} \left| z-z_0 \right|^{p+1-n} \left(1+O(|z-z_0|) \right) \quad (39)$$

Hence, apart from inessential numerical factors, we can write:

$$\left| \frac{\Delta f}{f} \right| = \frac{h}{|z-z_0|} \left| z-z_0 \right|^{p+1-n} \left\{ \begin{array}{ll} O(1) & \text{if } n > p + 1 \\ O(|z-z_0|^{p+1-n}) & \text{if } n < p + 1 \end{array} \right. \quad (40)$$

When we are going to perform the test of the triangle we expect that $z_0$ is located near the segment joining the points $z_1$ and $z_2$ where $z_1 = z_1^* + p(z_1^*) e^{i\theta(z_1^*)}$, (see Sect.3.4). Hence $|z_1 - z_0|^{\kappa p(z_1)}$ and $|z_1 - z_0|^{\kappa p(z_1^*)}$. Eq.(40) then shows that if we take $h = p$ the error term $|\Delta f'|$ can be as large as $|f'|$ if the unknown multiplicity $n$ is larger than $p$. Even if we take:

$$h = p/K_7 \quad (41)$$

with $K_7 > 1$ we cannot circumvent this difficulty since we do not know where $z_0$ is. It may occur that the distance $|z_1^* - z_0|$ between the root $z_0$ and the last point of descent $z_1^*$ is smaller than $p/K_7$ whatever $K_7$ is. This means that, in principle, the derivative at $z_1^*$ is not reliable. However we can show that this does not occur at the point $z_1^*$ if the function $F$ is (almost) linear in a region of dimension $\rho(z_1)$ centered around $z_1^*$. In this case the level curves of $|f|^2$ are (almost) circular and $|f(z_1^*)|^2 > |f(z_1)|^2$ implies that $z_0$ is closer to $z_1^*$ than to $z_1$. Hence $|z_1 - z_0| > \frac{1}{2}|z_1 - z_1^*| = \frac{1}{2}p$. Then by taking $K_7 > 2$ in Eq.(41) we can compute a reliable $f'$, and hence a reliable $F$, at $z_1$. It is precisely for this reason that we use the point $z_1^*$ as the first vertex of the triangle, see Eq.(25), and that we put $\alpha > 1$ in the same equation in order to move away from $z_0$ the other two vertices $z_2$ and $z_3$. 
4.- NUMERICAL RESULTS

We now present a series of numerical results obtained with the aid of a program based on the strategy illustrated previously. Test runs have allowed us to choose a set of values for the constants $K_1, \ldots, K_7$.

We have taken $K_1=5$. A greater value (for instance $K_1=10$ or larger) is equivalent to an unnecessary overestimate of the error $|\Delta f|$, which, in turn, gives an overestimate of $\rho(z)$. On the other side a value of $K_1=1$ or smaller might lead us to construct a triangle of sides comparable with the "width" of the lines $\text{Re}(F)=\text{Im}(F)=0$.

We have taken $K_2=0.1, K_3=0.2$. These are certainly very large overestimates of the values of $C$, Eq.(27), and of $|n-n'|$, Eq.(34), that we obtain (see the following examples).

Moreover we have taken $K_4=2, K_5=3, K_6=4$. It has seemed to us that with this choice the number of the iterative steps necessary to go from the initial guess up to the first vertex $z_1$ of the triangle is somewhat minimized. However no systematic work in this sense has been made by us even for some simple classes of $f(z)$. May be there is not a "best set" for an arbitrary $f$.

The first derivative of $f(z)$ has been calculated numerically by using the 2-points formula:

$$D_2(h, f) = \frac{1}{2h} \{f(z+h)-f(z-h)\} \quad (42,a)$$
during the phase of approach to the zero. When the triangle must be constructed a more precise evaluation of \( f' \) is needed and in this case we use the 4-points formula:

\[
D_4(h,f) = \frac{1}{4h} \left[ f(z+h) - f(z-h) - i\left(f(z+ih) - f(z-ih)\right) \right]
\]  

(42, b)

to compute the function \( F \), see Eq.(22), at each vertex of the triangle as well as the angle \( \theta(z_1) \) needed in Eq.(25). At last we have taken \( K_\gamma = 10 \) in Eq.(41).

The second derivative of \( f(z) \) which is needed to evaluate the order of magnitude estimate of the initial step size (see Eq.(16)) is obtained via the following formula:

\[
f''(z) = \frac{1}{h^2} \left[ f(z+h) - 2f(z) + f(z-h) \right]
\]  

(43)

and we have taken

\[
h = 10^{-1} (|z| + 10^{-1})
\]

in order to avoid a too small increment \( h \) if \( |z| = 0 \) initially.

All the numerical tests which follow have been performed on a CDC 7600 computer.

CASE 1

We begin with the simple function:

\[ f(z) = (z^m - 1)^n \]  

(44)

with:

\[ m = 1, 2, \ldots 10 \quad , \quad n = 1, 2, \ldots 10 \]  

(45)
whose roots (of multiplicity \( n \)) are the \( m \) roots of unity:

\[
z_{\lambda}(m,n) = \exp\left\{ i\left( \frac{2\pi}{m} \lambda \right) \right\} \quad \lambda = 0,1,\ldots,m-1 \tag{46}
\]

Notice also that this \( f \) exhibits a saddle point of order \( m-1 \) at the origin and that \( |f(z)| \) is quite flat for large \( m \) and \( n \) if \( |z|<1 \). Outside this region \( |f(z)| \) increases abruptly.

We have taken \( \rho_{\text{crit}}(z) \) as given by Eq.(19) with \( \varepsilon = 10^{-6} \). For an \( f(z) \) as simple as this, the error \( |\Delta f| \) is fixed by machine precision in the various multiplications, subtractions and exponentiations and depends on the way in which \( f(z) \) is built up. We have somewhat overestimated \( |\Delta f| \) by imposing that in any elementary operation the relative error made is equal to \( 10^{-14} \). At last the initial step size has been allowed to vary between \( \rho_{\text{min}} = 10^{-6} \) and \( \rho_{\text{max}} = 1 \) and we take \( \rho_{\text{crit}}(z) \) as given by Eq.(19).

We begin to illustrate the results for the case in which \( f'(z) \) is computed via its analytic expression.

If we start the search from:

\[
z_0 = 2 \times 10^{-2}(1+i/10) \tag{47}
\]

i.e. if \( m>1 \) from the vicinity of the saddle point at \( z=0 \), we always find one of the zeros (46). The first iterated point moves away from the origin and in the great majority of the cases the estimate of the initial step size afforded by Eq.(16) is correct (one could also say that for this \( f(z) \) and for this \( z_0 \) the choice \( \rho_{\text{max}} = 1 \) is correct). After about 10 iterations for the first few values of \( m \) and \( n \) to about 20-25 for \( m,n = 8 \times 10 \) the minimum step size \( \rho_0 \), Eq.(21), is reached. A triangle is then constructed, the parameter \( C \) in Eq.(27) is quite small, \( C = 10^{-4} \) at most but usually smaller, the criterion of the triangle is always satisfied and the final iteration described in Sect. 3-5 always gives a correct value for the multiplicity \( n \).
The same description can be made for the cases in which we start the search either from

\[ z_0 = 0.2(1+i/10) \]  \hspace{1cm} (48.a)

or from

\[ z_0 = 2(1+i/10) \]  \hspace{1cm} (48.b)

or from

\[ z_0 = 20(1+i/10). \]  \hspace{1cm} (48.c)

We stress that in all these 400 cases the radius \( \rho_{\text{fin}} \), Eq.(35), of the circle circumscribing the triangle, is a few \( 10^{-7} \) while the difference between the root computed via Eq.(33) and the exact root is a few \( 10^{-12} \) at most. This confirms the fact that \( \rho_{\text{fin}} \) gives a pessimistic evaluation of the error.

Now, if we compute numerically the first derivative, the same story is repeated in all the 400 cases. We always obtain an approximation, within the predetermined limits, of one of the zeros of \( f(z) \), \( \rho_{\text{fin}} \) as again a very pessimistic evaluation of the precision attained and the multiplicity \( n \) is always correctly determined.

It is very interesting to remark that less exciting results are obtained if, instead of using the point \( z_1 \) as the first vertex of the triangle, the final point of descent \( z_f \) (recall Sect. 3-4 and Fig.1) is used for the same purpose. Indeed if we repeat the whole search of the 400 zeros by using at any step of descent the 4-points formula given by Eq.(42b) to compute \( f'(z) \) we find that in 8 cases over 400 the root is not reliable (i.e. the criterion of the triangle fails and hence we cannot compute the multiplicity as described in Sect.3.5), while in 61 cases over 400 a root is found inside the triangle but the multiplicity is not reliable. These
deceiving results occur for high values of \( n \) only, \( n > 5 \), in agreement with the estimate of \(|\Delta f'|\) given by Eqs.(38) to (40). In these cases the final point of descent \( z_1^* \) is just too good i.e. its distance from the zeros is either very slightly larger or positively smaller than the step size \( h = p(z_1^*)/k_7, k_7 = 10 \), used to compute \( f'(z_1^*) \).

**CASE 2**

We consider the more complicated case of

\[
f(z) = \left(\sin(z^m - 1)\right)^n
\]

with \( m, n \) varying according to Eq.(45). The roots are now given by the solutions of

\[
(z_j^0)^m = 1 + j\pi \quad j = 0, \pm 1, \pm 2, \ldots
\]

i.e.

\[
z_j^0 = \left\{\left[1 + j\pi \right]\right\}^{1/m} \exp\left\{i \frac{\pi}{m} \left\{ x + 2\lambda \right\} \right\} \quad , \lambda = 0, 1, 2 \ldots m - 1
\]

where

\[
\chi = 0 \quad \text{if} \quad j > 0 \quad , \quad \chi = 1 \quad \text{if} \quad j < 0
\]

Clearly there is an infinity of roots (each of multiplicity \( n \)) situated along the directions indicated in Eq.(51). These roots are separated by (simple) saddle points located on the same directions at:

\[
z_j^{(s)} = \left\{\left[1 + \frac{\pi}{2} + j\pi \right]\right\}^{1/m} \exp\left\{i \frac{\pi}{m} \left\{ x + 2\lambda \right\} \right\}
\]

Moreover all these points are joined by the level curve \(|f(z)| = 1\) as is easily seen. At last there is (for \( n > 2 \)) an isolated saddle point of order \( n-1 \) at the origin.
As shown by Eq. (51) the "density" of zeros increases with \( m \). This corresponds to the fact that our \( f(z) \) is an entire function of order \( m \) (see Ref. /35/ p. 248). Denoting in this case by \( N(r) \) the number of zeros such that \( |z^{(0)}_{j,\lambda}| < r \) we have \( N(r) = O(r^m) \), ( /35/ p. 249), this can be seen directly from Eq. (51). Moreover for high values of \( j \) the zeros and the saddle points become closer and closer until, for high values of \( |z| \), they coalesce into a common magma and cannot be anymore distinguished by the computer. Notice also that while \( |f(z)| = o(1) \) for small \( n \) and any \( m \) if \( z^m \) is on or near the real axis, \( |f(z)| \) increases very rapidly with \( |z| \) if \( z^m \) is on or near the imaginary axis.

Let us now come to the numerical results. We have made runs with the same values of \( \rho_{\text{min}}, \rho_{\text{max}} \) as for Case 1. \( \rho_{\text{crit}}(z) \) is given by Eq. (19) with \( \epsilon = 10^{-6} \) (as before) and \( \epsilon = 10^{-10} \) and we have evaluated \( |\Delta f| \) as in the previous case. We have used \( z_0 \) as given by Eqs. (47), (48) as starting guess and have taken \( 1 < m, n < 10 \), see Eq. (45).

In all the cases treated i.e. if the first derivative \( f'(z) \) is computed either via its analytic expression or via Eqs. (42, a,b) we have found one of the zeros given by Eq. (51) with the correct multiplicity and with a precision much larger than that requested.

By way of an example we illustrate the results obtained for the case:

\[ m = n = 10 \]

with \( \rho_{\text{crit}}(z) = \epsilon |z|, \epsilon = 10^{-10}, z_0 = 0.2(1+i/10) \) and with \( f' \) computed numerically. At the initial point \( |f(z_0)| = 0.1779 \ldots \) and the initial step size \( \rho(z_0) = 4.04 \times 10^{-2} \) is such that \( |f(z_1)| < |f(z_0)| \) at the first iterate \( z_1 \).
In 36 iterations we arrive at (see Fig. 1)

\[ z_1 = 1 + 8.2 \times 10^{-12} + i \times 3.23 \times 10^{-11} \]

at which \(|f(z_1)| = 1.71 \times 10^{-35}\). With \(\rho(z_1) = 7.5 \times 10^{-11}\) the limit step size fixed by \(\rho_{\text{crit}}\) is reached and at the next point

\[ z_1 = 1 + 2.3 \times 10^{-11} - i \times 4.15 \times 10^{-11} \]

the modulus is \(|f(z_1)| = 5.83 \times 10^{-9}\) i.e. we are "going up". A triangle is then constructed with first vertex at \(z_1\), the parameter \(C\) of linearity is \(C = 6.8 \times 10^{-3}\) and the test of the triangle shows that it contains a zero i.e. that the points \(z_1\) and \(z_2\) are indeed close to a zero within the prescribed limits. The final iteration gives for the multiplicity

\[ n' = 10.024 + i \times 9.17 \times 10^{-2} \Rightarrow n = 10 \]

and the final iterated point:

\[ z_0 = 1 - 1.2 \times 10^{-12} - i \times 1.74 \times 10^{-13} \]

at which \(|f(z)| = 7.9 \times 10^{-110}\) is contained in a circle centered at

\[ z_c = 1 + 4.55 \times 10^{-12} + i \times 1.070 \times 10^{-11} \] with radius \(\rho_{\text{fin}} = 5.03 \times 10^{-11}\). We have found the zero corresponding to \(j = \lambda = 0\) in Eq. (51).

**CASE 3.**

We consider now a situation in which two or more roots coalesce when some parameter approaches a limit value. To this aim we have considered two cases represented via two simple polynomials:

\[
\begin{align*}
  f_2(z,a) &= (z - z_1(a)) (z - z_2(a)) \\
  f_3(z,a) &= (z - z_3(a)).f_2(z,a)
\end{align*}
\]

with

\[
\begin{align*}
  z_1(a) &= 1 + i a \\
  z_2(a) &= 1 - i a/10 \\
  z_3(a) &= 1 + (1-i)a/10
\end{align*}
\]
and let $a \to 0$ by taking
\[ a = 10^{-u}; \quad u = 0,15(1). \] (56)

We have made runs by always starting the search from the point $z_0 = -1 + 2i$. All the results we discuss do not depend on the way in which $f'(z)$ is computed.

We begin to illustrate what we obtain by using $\rho_{\text{crit}}(z)$ as given by Eq. (19) with:
\[ \epsilon = 10^{-v}, \quad v = 3,13(1). \] (57)

We plot the results for $f_2(z,a)$ in Fig. 5. We see that a polygonal ABCDE and a line FG divide the $(u,v)$ plane in four sectors plus one half strip.

Above FG we are unable to localize a root with the requested precision. This comes about because here $\epsilon$ is so small that $\rho_{\text{crit}}$ becomes smaller than the smallest significant step $\bar{\rho}(z)$. Recall that in our strategy we do not let the step size $\rho(z)$ become smaller than the $\bar{\rho}(z)$ over which $f(z)$ begins to behave wildly. Here $\bar{\rho} = 10^{-12}$. Below the line FG, $\rho_{\text{crit}}(z) > \bar{\rho}(z)$ and the precision can be (and indeed is) attained.

Let us now come to the multiplicity of the roots. To the right of, and below, the polygonal ABCDEG the code finds a root of multiplicity 2. In this region the distance $\rho_{12}(a) \equiv |Z_1(a) - Z_2(a)|$ between the roots is either comparable to or smaller than the last significant step made. We may be unable to establish the multiplicity if $\rho_{12}(a)$ is of the order of the size of the triangle. Moreover in this case the function $F = f/f'$ may be far from being linear. This explains why for $v > 12$ and $u = 12,13$ our strategy is unable to
decide about the existence of roots. We are in a situation similar to that illustrated in Fig. 4 and already indicated at the end of Sec. 3.7 as a possible failure of our strategy. Apart from these cases we always find a simple root in the region to the left of AB (but below FG). Clearly these lines do not limitate sharply defined regions, but indicate only the location of transition zones. This phenomenon of fuzziness occurs also, of course, for \( f = f_3(z,a) \). In this case (see Fig. 6) for \( \nu = 4 \) we find in succession simple, double and triple roots for \( \mu = 3, 4, 5 \) respectively.

Now we take \( \rho_{\text{crit}}(z) \) according to Eq.(20), keep all the other quantities unchanged and repeat the entire run always starting from \( z_0 = -1 + 2i \). In this case, as already observed in Sect. 3-3, if we succeed in finding a root with the predetermined relative precision and if \( |\text{Im}(z)| \ll |\text{Re}(z)| \) the precision attained on \( \text{Re}(z) \) at the last step can be very high. For instance for \( f = f_2(z,a) \), \( \epsilon = 10^{-3}, a = 10^{-3} \) we find \( z_{\text{root}} = Z_{\text{f}}^{\text{a}} \). More precisely \( \text{Re}(z) = 1 - 5.2 \times 10^{-11}, \text{Im}(z) = 10^{-3} + 2.1 \times 10^{-11} \) (with \( \rho_{\text{crit}} \) given by Eq.(19) we find \( \text{Re}(z) = 1 - 2.6 \times 10^{-6}, \text{Im}(z) = 10^{-3} + 1.7 \times 10^{-5} \)). The results we obtain for \( f = f_2, f = f_3 \) are displayed in Figs. 7, 8 respectively. Below and to the left of the zig-zag line AB we always find a simple root with the predetermined precision. In the region above and to the right of AB the precision requested is not attained. These findings are easily understood. We always let \( \rho(z) \) become as small as \( \rho_0(z) \), Eq.(21). For the functions \( f_2, f_3 \) we have \( \rho(z) = 10^{-12} \) for \( |z| = 1 \) so that \( \rho_0 > \rho \) provided that \( \mu + \nu \leq 12 \) and indeed \( \mu + \nu = 11 \leq 12 \) is, roughly, the equation of the line AB. The fact that the program is unable to decide about the existence of roots if \( \mu = 11, 12 \) and that either a double root (Fig.7) or a triple root (Fig.8) is found for \( \mu > 12 \) is easily understood recalling that for our roots \( |z| = 1 \) and \( \rho = 10^{-12} \).
CASE 4

We consider now a case in which the algorithm that computes \( f(z) \) is not very precise. As an example we compute Landau's pole i.e. the solution closest to the real axis of Landau's dispersion relation:

\[
f(z,a) = 1 + 2a^2 \left[ 1 + XZ(X) \right], \quad X = za
\]

where \( Z(X) \) is the plasma dispersion function which can be defined as:

\[
Z(X) = 2i \int_0^\infty e^{-Xt} \frac{e^{-t^2}}{t} dt
\]

for any complex \( X \). We have numerically computed \( Z \) by using the method described in Ref. /36/. This is based on an algorithm given by Gautschi /37/ for \( X < 6 \), together with a careful matching for \( |X| > 6 \) with the asymptotic series for \( Z'' \). The absolute error \( |\Delta f|_a \), as estimated in /36/ is:

\[
|\Delta f|_a = \begin{cases} 
3.0 \times 10^{-9} & \text{if } |X| \leq 6 \\
7.7 \times 10^{-8} |X|^{-2} & \text{if } |X| > 6
\end{cases}
\]

and is always much greater than the truncation error \( |\Delta f|_t \); consequently we take \( |\Delta f| = |\Delta f|_a \).

Eq.(58) has an infinity of branches \( z = z(a) \), but we are interested in that having the smallest \( |\text{Im}(z)| \). In this case an (asymptotic) analytic solution has been given by Landau /38/ for \( a \gg 1 \):

\[
\Omega = \text{Re}[z(a)] = (1 + 3/2a^2)^{1/2} \\
\gamma = \text{Im}[z(a)] = -\sqrt{\pi} a^3 e^{-a^2/2}
\]

We take \( \rho_{\text{crit}}(z) \) according to Eq.(20) with \( \epsilon = 10^{-4} \) and \( a = 1,10(0.25) \) and use Landau's solution as a first guess \( z_0 \).
For a > 3.5 for which $\gamma < 10^{-n}$ it is impossible to compute the root with the precision requested (see Fig.9) because of the too large error $|\Delta f_a|$. In this case the uncertainty $\rho_{\text{fin}} = |z_1 - z_c|$ is larger than $\gamma$. Anyway we see that our results show a tendency to go into those of Landau.

For small $a$ we can check our results against those given by Canosa /39/ who determined $z = z(K), K = \frac{\sqrt{2}}{a}$ and $K = 0.25, 2(0.05)$ via Muller's algorithm /1/. The comparison is interesting since Canosa pretends to give $z(K)$ with five significant figures. However he did not consider (as we do) the effect of the error $|\Delta f_a|$ on the precision attainable. In general the results agree except for the last figures on $|\text{Im}(z)|$. For instance at $K = 0.25$ we find $\gamma = -2.1641 \times 10^{-3}$, $\rho_{\text{fin}} = 1.12 \times 10^{-7}$ (i.e. we have four figures) while Canosa gives $\gamma = -2.1693$ (i.e. he has three figures). If we take $\epsilon = 10^{-6}$ we always have five figures for $K \geq 0.40$. For instance if $K = 0.40$ we have $\gamma = -6.6128 \times 10^{-2}$, $\rho_{\text{fin}} = 5.1 \times 10^{-8}$ while in /39/ we find $\gamma = -6.6133 \times 10^{-2}$. For higher values of $K$, $\gamma$ increases in modulus and the roots are correctly given in Ref./39/. For $K \geq 1.1$ both methods yield five significant figures.

5. - FURTHER SOLUTIONS OF $f(z) = 0$

Once a number of roots $z_1, z_2, ..., z_N$ with (integer) multiplicities $n_1, n_2, ..., n_N$ is known of $f(z)$ it can be of interest to compute further roots of $f(z) = 0$. Leaving aside the possibility offered by the application of circular arithmetics /4/ we examine how to combine our algorithm with a deflation technique. As is well known this consists in defining a new function $f_N$ which retains those roots of the original function which remain to be computed but no longer tends to zero values as one of the known roots is approached.
The function $f_N(z)$ is given by:

$$f_N(z) = A_N f(z) \left\{ \prod_{j=1}^{l-N} \frac{1}{(z-z_j)^{n_j}} \right\}^{-1}$$

so that

$$f'_N(z) = A_N f'(z) \left\{ \prod_{j=1}^{l-N} \frac{1}{(z-z_j)^{n_j}} \right\}^{-1} - f_N(z) \sum_{j=1}^{l-N} \frac{n_j}{z-z_j}$$

(61)

Here $A_N$ is a normalization factor whose necessity will be made clear in a moment.

This approach would be perfect if we knew the roots $z_j$ with infinite precision. In reality we know them with the error $\rho_{\text{fin},j} \neq 0$ (in general) and by doing the deflation as indicated in Eq.(60) we will automatically change any holomorphic function $f$ into a meromorphic function $f_N$ which has poles located near the zeros. This may be disturbing if after having found first a root of $f$ via a steepest descent technique we want to find those of $f_N$ by applying the same technique. Having found a root the problem is of how to give the initial guess for the subsequent one in order not to converge again to that (or those) previously found.

In order to make this point clear let us take $f(z) = z^n g(z)$ with $g(0) = o(1)$. By deflating we have $f_1 = g(z) \left( \frac{z}{z-z_0} \right)^n$ where $z_0 \neq 0$ is the root found and $|z_0| = \rho_{\text{fin},0} \ll 1$. It appears (see Fig.10) that we can find a new root of $f$ (i.e. a root of $g(z)$) if we start from an initial guess $Z$ such that $|Z-z_0| \gg \rho_{\text{fin},0}$. 
In order to test this idea we have looked for the zeros of

\[ f(z) = \text{Det} |A_{iK}(z)| \]

with

\[
\begin{vmatrix}
  z & \cos(z+1) & z+4 \\
  \sin(z-1) & \sin(z+5.23) & z \sin z \\
  z-3 & z-1+i \times 10^{-2} & 1
\end{vmatrix}
\]

this \( f(z) \) is an entire function of 2nd order, hence has an infinity of zeros. We do not know where any of the roots is located and we ignore their multiplicities. We have used our steepest descent program, with \( f'(z) \) computed numerically, we have estimated the truncation error by overestimating it somewhat heavily, we have taken \( \rho_{\text{crit}}(z) \) as given by Eq. (19) with \( n = 8 \) and we have used \( z = 0.5(1+i) \) as first initial guess. The subsequent initial guesses are defined by moving away from the last (approximate) zero in a random direction with a step much greater than the radius \( \rho_{\text{fin}} \) of the last error circle found. We set \( N = 50 \) as upper limit on the number of zeros to find and we took other values for the first initial guess. Each time the program found 50 roots of multiplicity equal to 1, the test of the triangle is satisfied, the test of the linearity for \( F \equiv f'/f \) (Eq. (27)) gives for the constant \( C \sim 10^{-6} \) at most and we have never converged to a previously known root. The only problem we have encountered, which is well known when \( f \) is an entire function, is that there is a tendency at a certain moment to find roots having larger and larger moduli. This may cause underflow problems in \( f_N \) and \( f'_N \) if we take \( A_N = 1 \). A simple remedy exists, it consists in taking, for instance:

\[
A_N = \frac{1}{\prod_j (Z_N - z_j)} {_j}^n
\]

where \( Z_N \) is the \( N \)'th initial guess.
Other functions have been tested by us like \( f(z) = \sin^5(\pi z) \) and again all the (SO) roots demanded were always obtained with the correct multiplicity. In a typical run we find, for instance zeros located at \( z = 1, 6, 7, 9, 11, 13, 16, \ldots 34, 41, 49, \ldots 434, 521, 625 \ldots \) and so on (rounding off small errors) up to \( z = 2239 \).

All this seems good enough, but, in reality, is unsatisfactory. The true problem is that of finding all the roots contained in a given region and not that of finding, somewhere, as many roots as desired.

For polynomial \( f \) splendid ad hoc algorithms exist /4/ which solve this problem. When \( f \) is either an entire or a meromorphic function, the problem is more difficult. Some work has been done in this case by Delves and Lyness /40/ by studying numerical methods particularly adapted to compute the Cauchy integral, Eq.(1) with \( g(z) = 1 \), around either circular or square contours.

We think a better approach may be based on Jensen's formula (see Ref./34/ p.125):

\[
\log \frac{r^n |f(a)|}{r_1 r_2 \cdots r_N} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \tag{63}
\]

Here \( f(z) \) is holomorphic for \( |z| < R \), \( f(0) \) is not zero, \( r_1, r_2 \ldots r_N \) are the moduli of the zeros of \( f \) in the circle \( |z| < R \) arranged as a nondecreasing sequence (a zero of order \( n \) is counted \( n \) times) and \( r_N < r < r_{N+1} \). Notice that Jensen's formula (in contrast with Cauchy's integral) does not need knowing \( f'(z) \), that it involves integration of a real function and that the integrand has always the same (negative) sign if \( f(re^{i\theta}) \) is
normalized to its maximum modulus on the circle $z=r$. These points make Eq. (63) very attractive for numerical applications. One can be sure that all the zeros have been found in a given circular region by comparing the two sides of Eq. (63). We are now working along these lines with some first encouraging results.
CONCLUSIONS.

A steepest descent technique has been proposed to solve \( f(z) = 0 \) where \( f \) is, at most, meromorphic and poorly known. This is the case most frequently occurring in practice where, because of the complexity of \( f \), only a few significant figures of this function can be computed.

The basic points of the method consist in 1) having shown how it is possible to construct a triangular region situated close to an unknown root, and in 2) having given a simple test to ascertain that the triangle does indeed contain a root of \( f(z) = 0 \). The dimensions of the triangle are fixed by the precision requested on the root and (or) by that of the algorithm which computes \( f(z) \).

The efficiency and applicability of a code based on the strategy proposed have been illustrated in a number of examples (a preliminary and less performing version of the code has already been used by one of us, see Ref/41/). From these examples two limitations of the method apparently emerge. The first one occurs when the distance between two or more roots is comparable with the minimum step size \( \hat{\rho} \) (see Case 3), the second one occurs when the root cannot be obtained with the precision requested because of errors in the computation of \( f(z) \) (see Case 4 with \( a > 3 \div 3.5 \)). The first limitation is similar to that occurring in microscopy wherein it is impossible to distinguish two points situated at a distance \( \lambda \) apart if illuminated by light with wave-length \( O(\lambda) \). The remedy parallels that used in optics: one must increase the precision on the root and go (if necessary) to a longer word length provided that no restrictions are introduced by the computation of \( f(z) \). As far as our method is concerned the second limitation is only apparent; it can be overcome only by increasing the precision in the calculation of \( f(z) \).
At last we point out that in spite of the fact that our steepest descent algorithm has always performed successfully, we believe in the following statement that has a strong Picard-like flavour: given a method to solve \( f(z) = 0 \), it works for all functions but one.

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Fig. 1 - Triangle $z_1, z_2, z_3$ and lines $\text{Re}(F)=\text{Im}(F)=0$ in a "small" region where $F$ is roughly linear. The dotted lines indicate the region of uncertainty in the knowledge of $\text{Re}(F)=0$, $\text{Im}(F)=0$. Since $F(z)$ has a simple zero, the lines $\text{Re}(F)=0$, $\text{Im}(F)=0$ cross at $\pi/2$.

Fig. 2 - Graph of $|f(z)|$ along a polygonal (indicated by $\bar{z}$) in the complex plane. Step 1 is a "going up" step, but condition (18) is not satisfied for the reduced steps 2 and 3. Point 1 is the last at which Eq.(18) is satisfied. Therefore $\rho_1$ is taken as the minimum step size $\bar{\rho}$ and a triangle is constructed with point 1 as first vertex $z_1$ (see Sect. 3-4).

Fig. 3 - Same type of graph as for Fig. 2. Step 1 is a "going up" step. Condition (18) is not satisfied between $z_m$ and point 2 after reduction of the step size $\rho_m$ by $K_6$, but is satisfied between $z_m$ and point 3 after reduction of the step size by $2K_6$. Point 3 will be taken as the new $z_{m+1}$.

Fig. 4 - Same type of graph as for Fig. 2. In this very unusual case our strategy would fail since $\rho_m$ would be considered as the minimum significant step size $\bar{\rho}$ and the function $F$ (see Sect. 3-4) is certainly far from being linear in this region.

Fig. 5 - Multiplicity of the roots as a function of the parameters $\nu$ and $\mu$, Eqs. (56) and (57), for $f(z)=f_2(z)$, Eq. (54), and for $\rho_{\text{crit}}(z)$ given by Eq. (19).

Fig. 6 - Same graph as that of Fig. 5, but for $f(z)=f_3(z)$, Eq. (54).

Fig. 7 - Same graph as that of Fig. 5 for $f(z)=f_2(z)$ and for $\rho_{\text{crit}}(z)$ given by Eq. (20).

Fig. 8 - Same graph as that of Fig. 7, but for $f(z)=f_3(z)$. 
Fig. 9 - Comparison between the numerical and analytical solutions of Landau's dispersion relation.

Fig. 10 - Showing the introduction of a pole close to a zero when deflating with an imprecise root.
precision unattainable

simple root

double root

Fig. 5
Fig. 5

Precision unattainable

Simple root

Triple root
precision unattainable

simple root

simple root

root not found
double root

Fig. 7
precision unattainable

Fig. 8
Fig. 9

- Numerical
- Landau