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LINEAR MODE CONVERSION IN A TOROIDAL PLASMA

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ABSTRACT

Linear mode conversion at the perpendicular ion cyclotron resonance has been treated for an axially symmetric toroidal plasma. The mode conversion appears between a fast electromagnetic wave and a slow-quasi electrostatic wave, due to finite electron inertia. The problem reduces to the Orr-Sommerfeld equation where the coefficients determining the reflection, transmission and conversion are functions of the arc length along a poloidal intersection of the resonance surface. These coefficients can be determined from eigenfunctions of an ordinary differential equation.

Introduction

Linear mode conversion at the perpendiculal ion cyclotron resonance or the shear Alfvén wave resonance is of great importance for ion cyclotron resonance heating as well as Alfvén wave heating. A number of absorption mechanisms are available in these frequency regimes. One of the crucial questions is whether the externally launched fast electromagnetic wave will in the neighbourhood of the perpendicular ion cyclotron resonance either be absorbed locally or reflected or converted into another wave. Which of the processes is the dominating one depends on the magnetic field strength, collisionality, temperature, density etc. To localize the resonance surfaces and to determine which is the dominating process is then of great importance. The localization of the resonance surfaces in the ICRF regime for a toroidal geometry was recently treated by Hellsten and Tennfors[1]. Investigations of the different processes occurring at the resonance have hitherto only been studied in the one-dimensional case. Absorption caused by resistivity or viscosity was treated by Tartaronis[2], and Hasegawa and Chen[3]. The two last mentioned authors treated also the case of linear mode conversion into a kinetic Alfvén wave caused by finite Larmor radius[4]. Linear mode conversion due to finite electron inertia was studied by Swanson[5].

In this report we investigate the linear mode conversion, due to finite electron inertia, between a fast electromagnetic wave and a slow quasi-electrostatic wave for axially symmetric toroidal equilibria. We shall here use the cold plasma equations. For an one-dimensional slab the mode conversion is described by the Orr-Sommerfeld equation

$$u^{(iv)} + \lambda^2(xu'' + \alpha u' + \beta u) = 0 \qquad (1)$$

Asymptotic solutions of this equation for large $|\lambda|$ have been derived by Rabenstein[6] from which the reflected, transmitted and converted fractions can be determined.

In cases of two dimensional equilibria the corresponding problem of mode conversion, described by partial differential equations, can be reduced to an equation of similar form as Eq. (1) where λ now is a function of the arc length along a poloidal intersection of the resonance surface. To determine λ, the eigenfunctions of an ordinary differential equation have to be solved.

The asymptotic solutions of Eq. (1) can then be used for matching the solutions outside the boundary layer obtained by two-dimensional numerical codes.

2. Basic Equations and the Perpendicular Ion Cyclotron Resonance

A cold magnetized multicomponent plasma can be described by the following set of equations where thermal corrections and damping effects have been neglected[7]

$$\frac{d\underline{V}_\alpha}{dt} = \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{V}_\alpha \times \underline{B}}{c}\right) \tag{2}$$

$$\frac{1}{c}\frac{\partial \underline{B}}{\partial t} = -\underline{\nabla} \times \underline{E} \tag{3}$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\sum_\alpha q_\alpha n_\alpha \underline{V}_\alpha \tag{4}$$

where $\underline{V}_\alpha$, n_α, m_α and q_α denote velocity, density, mass and charge for particles of type α, $\underline{E}$ denotes the electric field, $\underline{B}$ the magnetic field and c the velocity of light.

After linearizing and Fourier decomposing Eqs. (2-4) by putting them proportional to $\exp i\omega t$ we substitute Eqs. (2 and 3) into Eq. (4) yielding

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} = \underline{\underline{\varepsilon}}\,\underline{E} \tag{5}$$

The nature of the wave propagation described by Eq. (5) can be visualized by studying a plane slab geometry, in which the equations can be reduced to ordinary differential equations. Studying the case with a density gradient parallel with the x-axis and perpendicular to a uniform magnetic field in the z-direction. Eq. (5) can then be Fourier decomposed with respect to the ignorable coordinates y and z. In the quasi-homogeneous case $\partial/\partial x$ is replaced by ik_x yielding[8]

$$\varepsilon_{xx}k_x^4 + \left[(k_z^2 - \varepsilon_{xx})(\varepsilon_{xx} + \varepsilon_{zz} - 2k_y^2) + \varepsilon_{xy}^2 + 2k_z^2k_y^2\right]k_x^2 +$$

$$+ (\varepsilon_{zz} - k_y^2)\left[(k_y^2+k_z^2-\varepsilon_{xx})(k_z^2-\varepsilon_{xx})+\varepsilon_{xy}^2\right] + k_z^2k_y^2(k_y^2+k_z^2-\varepsilon_{xx})=0 \quad (6)$$

This dispersion relation, $k_x = k_x(\varepsilon_{xx})$, is outlined in Fig.1. Eq. (6) describes two waves in the ion cyclotron range of frequencies. The fast wave is fully electromagnetic, while the slow wave is essentially electrostatic. The slow wave has features similar to the slow lower hybrid wave. It is characterized by a resonance cone[9] and a resonance, Buchsbaum ion-ion hybrid resonance[10], given by $\varepsilon_{xx} = 0$. The fast wave propagates close to the perpendicular ion cyclotron resonance, $\varepsilon_{xx} - k_z^2 \simeq 0$, where it corresponds to the electromagnetic ion-cyclotron wave, and to the right of the fast wave cut-off, $\varepsilon_{xx} - k_z^2 - k_y^2 = 0$, where it corresponds to the magnetoacoustic wave. The fast wave converts into a slow wave in the neighbourhood of the perpendicular ion-cyclotron resonance.

To derive equations determining the perpendicular ion cyclotron resonance in an axially symmetric toroidal geometry it is convenient to use an orthogonal coordinate system (ψ,χ,θ), where ψ labels the magnetic surfaces, χ denotes a poloidal angle-like variable and θ is the toroidal angle. The metric tensor elements are given by

$$g_{\psi\psi} = \frac{1}{B_p^2r^2}$$

$$g_{\chi\chi} = B_p^2J^2$$

$g_{\theta\theta} = r^2$

$g_{ik} = 0$ if $i \neq k$

where the Jacobian, J , is defined by

$$J \equiv \frac{d(x,y,z)}{d(\psi,\chi,\theta)} = \sqrt{g_{\psi\psi} g_{\chi\chi} g_{\theta\theta}},$$

yielding $B^\chi = J^{-1}$ and $B^\theta = B_T/r$. B_p and B_T denote the poloidal and toroidal magnetic field strengths, respectively. In this coordinate system the dielectric tensor reads[1]

$$\varepsilon^i_j = \frac{\omega^2}{c^2}\left(\delta^i_j - \sum_\alpha \frac{\omega^2_{p\alpha} A^{\;i}_{\alpha j}}{\omega^2(\omega^2_{c\alpha} - \omega^2)}\right) \qquad (7)$$

where

$$A^{\;\psi}_{\alpha\psi} = -\omega^2$$

$$A^{\;\psi}_{\alpha\chi} = -i\omega\omega_{c\alpha} g_{\chi\chi} B_\theta / JB$$

$$A^{\;\psi}_{\alpha\theta} = +i\omega\omega_{c\alpha} r^2 B_\chi / JB$$

$$A^{\;\chi}_{\alpha\psi} = +i\omega\omega_{c\alpha} g_{\psi\psi} B_\theta / JB$$

$$A^{\;\chi}_{\alpha\chi} = \omega^2_{c\alpha} B_p^2 / B^2 - \omega^2$$

$$A^{\;\chi}_{\alpha\theta} = \omega^2_c B_\theta B^\chi / B^2$$

$$A^{\;\theta}_{\alpha\psi} = -i\omega\omega_{c\alpha} g_{\psi\psi} B_\chi / JB$$

$$A^{\;\theta}_{\alpha\chi} = \omega^2_{c\alpha} B_\chi B^\theta / B^2$$

$$A_{\alpha\theta}^{\theta} = \omega_{c\alpha}^2 B_T^2/B^2 - \omega^2$$

$$\omega_{P\alpha}^2 = 4\pi q_\alpha^2 n_\alpha/m_\alpha$$

$$\omega_{c\alpha} = q_\alpha B/m_\alpha c.$$

The operator $\underline{\nabla} \times \underline{\nabla} \times$ reads

$$(\underline{\nabla} \times \underline{\nabla} \times \underline{E})^{\psi} = \frac{1}{J} \frac{\partial}{\partial \chi}\left(\frac{r^2}{J}\left(\frac{\partial E_\chi}{\partial \psi} - \frac{\partial E_\psi}{\partial \chi}\right)\right) -$$

$$- \frac{i n g_{\chi\chi}}{J^2}\left(i n E_\psi - \frac{\partial E_\theta}{\partial \psi}\right)$$

$$(\underline{\nabla} \times \underline{\nabla} \times \underline{E})^{\chi} = \frac{i n g_{\chi\chi}}{J^2}\left(\frac{\partial E_\theta}{\partial \chi} - i n E_\chi\right) -$$

$$- \frac{1}{J} \frac{\partial}{\partial \psi}\left(\frac{r^2}{J}\left(\frac{\partial E_\chi}{\partial \psi} - \frac{\partial E_\psi}{\partial \chi}\right)\right)$$

$$(\underline{\nabla} \times \underline{\nabla} \times \underline{E})^{\theta} = \frac{1}{J} \frac{\partial}{\partial \psi}\left(\frac{g_{\chi\chi}}{J}\left(i n E_\psi - \frac{\partial E_\theta}{\partial \psi}\right)\right) -$$

$$- \frac{1}{J} \frac{\partial}{\partial \psi}\left(\frac{g_{\psi\psi}}{J}\left(\frac{\partial E_\theta}{\partial \chi} - i n E_\chi\right)\right)$$

and the ψ derivative of the Jacobian is given by

$$\frac{1}{J} \frac{\partial}{\partial \psi} J = \frac{-1}{B_p \kappa} \frac{\partial}{\partial \psi}(B_p \kappa).$$

where κ denote the poloidal curvature of the magnetic surface.

The perpendicular ion cyclotron resonance is not a resonance of the cold plasma equations, but of a system of equations consisting of the components of Eq. (5) perpendicular to the magnetic field and

$$\underline{E} \cdot \underline{B} = 0 \tag{8}$$

The resonances coincide with the magnetic surfaces and are defined for each surface as an eigenvalue problem of an ordinary differential equation[1]

$$D \frac{r^2 B_p^2}{B^2} DE_\psi + \varepsilon^{\psi\psi} E_\psi = 0 \tag{9}$$

where

$$D = \frac{1}{J} \frac{\partial}{\partial \chi} + \frac{inB_T}{r} .$$

In the limit $\omega^2/\omega_{ci}^2 \to 0$ Eq. (9) described the shear Alfvén wave resonance for a pressureless plasma.

Eq. (9) is similar to the one discussed by Budden[11] describing a resonance and a cut-off pair, here the resonance is given by the fundamental ion cyclotron resonance. To obtain periodic solutions we have to assume that the power absorbed at the fundamental ion cyclotron resonances are negligible compared to the power mode converted at the perpendicular ion cyclotron resonance.

3. Expansions near the Resonance

The wavelength perpendicular to the magnetic surfaces of the slow wave is, in general, much shorter than that of the fast wave. Near the perpendicular ion cyclotron resonance the wavelength of the fast wave decreases and becomes comparable to that of the slow wave and the two waves "couples". The mode conversion can be treated by expanding the wave functions in neighbourhood of the resonance surface.

The orthogonal coordinate system is not suitable for expanding the solutions across the magnetic surfaces. We suggest here a coordinate system (ψ,ζ,θ), in which the local wavelength parallel with $\underline{\nabla}\zeta$ is nearly constant. The coordinate system is determined by the Jacobian J_ζ

$$J_\zeta = \frac{qrc}{\omega}\sqrt{|\varepsilon^{\psi\psi}|} \text{ if } \delta < \frac{qrc}{\omega}\sqrt{|\varepsilon^{\psi\psi}|} < M$$

$$J_\zeta = \delta \text{ if } \frac{qrc}{\omega}\sqrt{|\varepsilon^{\psi\psi}|} < \delta \tag{10}$$

$$J_\zeta = M \text{ if } \frac{qrc}{\omega}\sqrt{|\varepsilon^{\psi\psi}|} > M$$

where

$$q = \frac{1}{2\pi}\oint \frac{B_T}{r} J d\chi .$$

The constants M and δ are introduced in order to obtain a well defined transformation at $\varepsilon^{\psi\psi} = 0$ and $\varepsilon^{\psi\psi} = \infty$.
In the low frequency limit $\omega_{ci}/\omega \to 0$ we have $J_y = qr/B_T$, in which coordinate system the magnetic field lines become straight[12] and

the node lines approximatively coincide with constant ζ surfaces. The orthogonal coordinate system can be chosen so that $\chi(\psi_o)=\zeta(\psi_o)$ where ψ_o denotes the resonance surface. We can then use the dielectric tensor and the operator $\underline{\nabla} \times \underline{\nabla} \times$ given in the orthogona coordinate system. When differentiating with respect to ψ we utilize that

$$\frac{\partial f}{\partial \psi}\Big|_{\chi} = \frac{\partial f}{\partial \psi}\Big|_{\zeta} + \beta \frac{\partial f}{\partial \chi} \qquad (11)$$

where $\beta = \partial\chi/\partial\psi$ defined by

$$\frac{\partial \beta}{\partial \chi} = \frac{J_\zeta}{J} \frac{\partial}{\partial \psi} \left(\frac{J}{J_\zeta}\right) \qquad (12)$$

We assume that near the resonance surface the electric field can be written as

$$E_i(\psi,\chi) = E_{i0}(\chi)e_i(\psi). \tag{13}$$

The poloidal variation of E_ψ is given by Eq. (9). The electric field parallel to the magnetic surfaces is decomposed into one component, $E_{\parallel 0}$, parallel to the magnetic field and one component, $E_{\perp 0}$, perpendicular to the magnetic field in the following way

$$E_\chi = \frac{E_{\parallel 0}B_\chi - E_{\perp 0}B^\theta}{B} \tag{14}$$

$$E_{\theta 0} = \frac{E_{\parallel 0}B_\theta + E_{\perp 0}B^\chi}{B} \tag{15}$$

The poloidal variation of $E_{\perp 0}$ is then given by

$$E_{\perp 0} = \frac{1}{B}(rB_T \frac{\partial E_{\psi 0}}{\partial \chi} - inJB_p^2E_{\psi 0}) \tag{16}$$

where the equilibrium quantities are defined at the resonance surface. Eq. (9) was derived by assuming $E_\parallel = 0$. However, the mode conversion described here occurs because $E_\parallel \neq 0$, but still being small. The poloidal variation of the parallel electric field is obtained from the parallel component of Eq. (5) by taking the leading order terms in mass ratio of the left hand side and the leading order terms of a power series expansion in $(\psi - \psi_0)$ of the right hand side yielding

$$E_{\parallel O} = -\frac{\omega_{pe}^{2}(\psi_O,\chi_O)\,r^{2}(\psi_O,\chi_O)\,r^{2}B_p^{2}}{\omega_{pe}^{2}B}\,DE_{\psi O} \tag{17}$$

where χ_O is an arbitrary surface, $E_{\parallel O}$ has been normalized to be comparable to $E_{\psi O}\sqrt{g^{\psi\psi}}$. $e_\parallel$ will then be comparable to $\omega_{pe}^2 r^2/c^2$, the equilibrium quantities are defined at the resonance surface. $E_{\parallel O}$ is then a function of χ only.

In the following we write out the zero index. We substitute the ansatz (13) into Eq. (5) and decompose the equation into one component perpendicular to the magnetic surfaces

$$\gamma_O e_\psi = \gamma_1\frac{\partial e_\perp}{\partial\psi} + \gamma_2 e_\perp + \gamma_3\frac{\partial e_\parallel}{\partial\psi} + \gamma_4 e_\parallel, \tag{18}$$

one component parallel to the magnetic field

$$h_0\frac{\partial e_\psi}{\partial\psi} + h_1 e_\psi + h_2\frac{\partial^2 e_\perp}{\partial\psi^2} + h_3\frac{\partial e_\perp}{\partial\psi} + h_4 e_\perp +$$

$$+ h_5\frac{\partial^2 e_\parallel}{\partial\psi^2} + h_6\frac{\partial e_\parallel}{\partial\psi} + h_7 e_\parallel = 0, \tag{19}$$

and one component parallel to the magnetic surfaces and perpendicular to the magnetic field

$$\ell_0 \frac{\partial e_\psi}{\partial \psi} + \ell_1 e_\psi + \ell_2 \frac{\partial^2 e_\perp}{\partial \psi^2} + \ell_3 \frac{\partial e_\perp}{\partial \psi} + \ell_4 e_\perp +$$

$$+ \ell_5 \frac{\partial^2 e_\parallel}{\partial \chi^2} + \ell_6 \frac{\partial e_\parallel}{\partial \psi} + \ell_7 e_\parallel = 0 \qquad (20)$$

where

$$\gamma_1 = -\frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{rB_T E_\perp}{JB} \right) + \frac{inB_p^2 E_\perp}{J}$$

$$\gamma_2 = -\frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{r^2}{J} L \frac{rB_T E_\perp}{B} \right) + inB_p^2 L \frac{E_\perp}{JB} - E^{\psi\perp} E_\perp$$

$$\gamma_3 = \frac{1}{J} D \frac{r^2 B_p^2}{B} E_\parallel$$

$$\gamma_4 = \frac{1}{J} \frac{\partial}{\partial \chi} \frac{r^2}{J} L \frac{rB_p^2 E_\parallel}{B} + inB_p^2 L \frac{rB_T E_\parallel}{B} - \varepsilon^{\psi\parallel}$$

$$L = \frac{\partial}{\partial \psi} + \theta \frac{\partial}{\partial \chi}$$

$$h_0 = B_p^2 r^2 D E_\psi$$

$$h_1 = B_p^2 L \frac{r^2}{J} \frac{\partial E_\parallel}{\partial \chi} + \frac{inB_T r}{J} LJB_p^2 E_\psi - \varepsilon^{\psi\psi} E_\psi$$

$$h_2 = 0$$

$$h_3 = \frac{2B_p^3 r}{JB} E_\perp L \frac{B_T}{B_p}$$

$$h_4 = B_p^2 L \frac{r^2}{J} L \frac{B_T r E_\perp}{B} - \frac{B_T r}{J} LJB_p^2 L \frac{E_\perp}{JB} +$$

$$+ \frac{in}{r^2} D \frac{E_\perp}{JB} + \frac{rB_T}{J} \frac{\partial}{\partial\chi} \left(\frac{1}{B_p^2 r^2} D \frac{E_\perp}{JB}\right) - \varepsilon^{\parallel\perp} E_\perp$$

$$h_5 = -BB_p^2 r^2 E_\parallel$$

$$h_6 = -\frac{B}{JE_\parallel} LJB_p^2 r^2 E_\parallel^2$$

$$h_7 = -\frac{rB_T}{J} LJB_p^2 L \frac{rB_T E_\parallel}{B} - B_p^2 L \frac{r^2}{J} L \frac{JB_p^2 E_\parallel}{B} +$$

$$+ \frac{in}{Jr^2}\left(\frac{\partial}{\partial\chi}\left(\frac{rB_T E_\parallel}{B}\right) - \frac{inJB_p^2 E_\parallel}{B}\right) -$$

$$- \frac{rB_T}{J} \frac{\partial}{\partial\chi}\left(\frac{1}{B_p^2 r^2 J} \frac{\partial}{\partial\chi}\left(\frac{rB_T E_\parallel}{B}\right)\right) + \frac{inB_T r}{J} \frac{\partial}{\partial\chi}\left(\frac{E_\parallel}{r^2 B}\right) -$$

$$- \varepsilon^{\parallel\parallel} E_\parallel$$

$$\ell_0 = \frac{inB_p^2 E_\psi}{J} - \frac{B_T r}{J^2} \frac{\partial E_\psi}{\partial\chi}$$

$$\ell_1 = \frac{in}{J^2} LJB_p^2 E_\psi - \frac{B_T}{rJ} L \frac{r^2}{J} \frac{\partial E_\psi}{\partial\chi} - \varepsilon^{\perp\psi} E_\psi$$

$$\ell_2 = -\frac{B}{J^2} E_\perp$$

$$\ell_3 = -\frac{B}{JE_\perp} L \frac{E_\perp^2}{J}$$

$$\ell_4 = -\frac{B_T}{rJ} L \frac{r^2}{J} L \frac{B_T E_\perp}{rB} - \frac{1}{J^2} LJB_p^2 L \frac{E_\perp}{JB}$$

$$- \frac{1}{J} D \frac{1}{r^2 B_p^2} D \frac{E_\perp}{JB} - \varepsilon^{\perp\perp} E_\perp$$

$$\ell_5 = 0$$

$$\ell_6 = -\frac{2B_p^3 r}{JB} E_\parallel L \frac{B_T}{B_p}$$

$$\ell_7 = \frac{B_T}{rJ} L \frac{r^2}{J} L \frac{JE_\parallel B_p^2}{B} - \frac{1}{J^2} LJB_p^2 L \frac{rB_T E_\parallel}{B}$$

$$- \frac{inB_T}{r^3 J^2 B_p^2} \left[\frac{\partial}{\partial\chi} \left(\frac{rB_T E_\parallel}{B}\right) - \frac{inJB_p^2 E_\parallel}{B} \right] -$$

$$- \frac{1}{J^2} \frac{\partial}{\partial\chi} \frac{1}{r^2 B_p^2 J} \left(\frac{\partial}{\partial\chi} \frac{rB_T E_\parallel}{B} - \frac{inJB_p^2 E_\parallel}{B}\right) -$$

$$- \varepsilon^{\perp\parallel} E_\parallel$$

$$\varepsilon^{\psi\perp} = \frac{\varepsilon^{\psi\theta}}{JB} - \frac{B_T}{rB}\,\varepsilon^{\psi\chi}$$

$$\varepsilon^{\psi\parallel} = JB_p^2\,\frac{\varepsilon^{\psi\chi}}{B} + \frac{B_T r}{B}\,\varepsilon^{\psi\theta}$$

$$\varepsilon^{\perp\psi} = -\,\frac{B_T}{r}\,\varepsilon^{\chi\psi} + \frac{1}{J}\,\varepsilon^{\theta\psi}$$

$$\varepsilon^{\perp\perp} = \frac{B_T^2}{r^2 B}\,\varepsilon^{\chi\chi} - \frac{B_T}{JrB}\,(\varepsilon^{\theta\chi} + \varepsilon^{\theta\chi}) + \frac{1}{J^2 B}\,\varepsilon^{\theta\theta}$$

$$\varepsilon^{\perp\parallel} = -\,\frac{B_T J B_p^2}{rB}\,\varepsilon^{\chi\chi} + \frac{B_p^2}{B}\,\varepsilon^{\theta\chi} - \frac{B_T^2}{B}\,\varepsilon^{\chi\theta} + \frac{rB_T}{JB}\,\varepsilon^{\theta\theta}$$

$$\varepsilon^{\parallel\psi} = JB_p^2\varepsilon^{\chi\psi} + rB_T\varepsilon^{\theta\psi}$$

$$\varepsilon^{\parallel\perp} = -\,\frac{JB_p^2 B_T}{rB}\,\varepsilon^{\chi\chi} - \frac{B_T^2}{B}\,\varepsilon^{\theta\chi} + \frac{B_p^2}{B}\,\varepsilon^{\chi\theta} + \frac{B_T r}{JB}\,\varepsilon^{\theta\theta}$$

$$\varepsilon^{\parallel\parallel} = \frac{J^2 B_p^4}{B}\,\varepsilon^{\chi\chi} + \frac{rB_T J B_p^2}{B}\,(\varepsilon^{\theta\chi} + \varepsilon^{\chi\theta}) +$$

$$+ \frac{r^2 B_T^2}{B}\,\varepsilon^{\theta\theta}$$

We note here that $\gamma_0 = 0$ is an apparent singularity determining the fast wave cut-off[1]. In the degenerate case where $n = 0$ and $B_T = 0$ this surface coincides with the perpendicular ion cyclotron resonance and has to be treated separately. In the following we assume that $\gamma_0 \neq 0$.

Substituting Eq. (18) into Eqs. (19 and 20) yields

$$\frac{h_0\gamma_1}{\gamma_0}\frac{\partial^2 e_\perp}{\partial\psi^2} + \left(h_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_1}{\gamma_0}\right) + \frac{h_0\gamma_2}{\gamma_0} + \frac{h_1\gamma_1}{\gamma_0} + h_3\right)\frac{\partial e_\perp}{\partial\psi} +$$

$$+ \left(h_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_2}{\gamma_0}\right) + \frac{h_1\gamma_2}{\gamma_0} + h_4\right)e_\perp +$$

$$+ \left(h_5 + \frac{h_0\gamma_3}{\gamma_0}\right)\frac{\partial^2 e_\parallel}{\partial\psi^2} + \left(h_6 + h_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_3}{\gamma_0}\right) +\right.$$

$$\left. + \frac{h_0\gamma_4}{\gamma_0} + \frac{h_1\gamma_3}{\gamma_0}\right)\frac{\partial e_\parallel}{\partial\psi} + \left(h_7 + h_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_4}{\gamma_0}\right) + \frac{h_1\gamma_4}{\gamma_0}\right)e_\parallel = 0 \qquad (21)$$

$$\left(\ell_2 + \frac{\ell_0\gamma_1}{\gamma_0}\right)\frac{\partial^2 e_\perp}{\partial\psi^2} + \left(\ell_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_1}{\gamma_0}\right) + \frac{\ell_0\gamma_2}{\gamma_0} +\right.$$

$$\left. + \frac{\ell_1\gamma_1}{\gamma_0} + \ell_3\right)\frac{\partial e_\perp}{\partial\psi} + \left(\ell_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_2}{\gamma_0}\right) + \frac{\ell_1\gamma_2}{\gamma_0} + \ell_4\right)e_\perp +$$

$$+ \frac{\ell_0\gamma_3}{\gamma_0}\frac{\partial^2 e_\parallel}{\partial\psi^2} + \left(\ell_6 + \ell_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_3}{\gamma_0}\right) +\right.$$

$$\left. + \frac{\ell_0\gamma_4}{\gamma_0} + \frac{\ell_1\gamma_3}{\gamma_0}\right)\frac{\partial e_\parallel}{\partial\psi} + \left(\ell_7 + \ell_0\frac{\partial}{\partial\psi}\left(\frac{\gamma_4}{\gamma_0}\right) + \frac{\ell_1\gamma_4}{\gamma_0}\right)e_\parallel = 0 \qquad (22)$$

Eqs. (21 and 22) describe two coupled second order differential equations. In the limit $m_i/m_e \to \infty$ h_7 is the only term approaching infinity. In this limit $e_{\parallel} \to 0$ and Eq. (22) describes the fast electromagnetic wave, which is singular when

$$\ell_2 + \frac{\ell_0 \gamma_1}{\gamma_0} = 0 \qquad (23\ a)$$

which is equivalent to

$$\frac{BE_{\perp}}{\gamma_0 J^2}\left[D \frac{B_p^2 r^2}{B^2} DE_{\psi} + \varepsilon^{\psi\psi} E_{\psi}\right] = 0 \qquad (23b)$$

Eqs. (21 and 22) can be cast into one fourth order equation

$$f_1 e_{\perp}^{(iv)} - (f_1' + a_1 f_1 + f_2) e^{(iii)} + (f_2' + a_1 f_2 + f_3 - a_2) e_{\perp}'' +$$

$$+ (f_3' + a_1 f_3 + f_4 - a_3) e_{\perp}' + (f_4' + a_1 f_4 - a_4) e_{\perp} = 0 \qquad (24)$$

where

$$f_1 = a_2/g$$

$$f_2 = (a_2' + a_3 - \frac{\ell_2 \gamma_0 + \ell_0 \gamma_1}{\ell_0 \gamma_3} - a_1 a_2 +$$

$$+ \frac{a_2 \gamma_0 (\ell_6 + \ell_0 (\frac{\gamma_3}{\gamma_0})' + \frac{\ell_0 \gamma_4}{\gamma_0} + \frac{\ell_1 \gamma_3}{\gamma_0}}{\ell_0 \gamma_3})/g$$

$$f_3 = (a_3' + a_4 - a_1 a_3 -$$

$$- \frac{\gamma_0 \ell_0 (\frac{\gamma_1}{\gamma_0}) + \ell_0 \gamma_2 + \ell_3 \gamma_0 + \ell_1 \gamma_1 - a_3 \gamma_0 \ell_6}{\ell_0 \gamma_3}$$

$$- \frac{a_3 \gamma_0 \ell_0 (\frac{\gamma_3}{\gamma_0})' + \ell_0 \gamma_4 + \ell_1 \gamma_3}{\ell_0 \gamma_3})/g$$

$$f_4 = (a_4' - a_1 a_4 -$$

$$- \frac{\gamma_0 \ell_0 (\frac{\gamma_2}{\gamma_0})' + \ell_1 \gamma_2 + \ell_4 \gamma_0 - a_4 \gamma_0 \ell_6 - a_4 \ell_0 \gamma_0 (\frac{\gamma_3}{\gamma_0})'}{\ell_0 \gamma_3} -$$

$$- \frac{a_4 \ell_0 \gamma_4 + a_4 \ell_1 \gamma_3}{\ell_0 \gamma_3})/g$$

$$g = a_1' - a_1^2 - \frac{\gamma_0 \ell_7 + \gamma_0 \ell_0 (\frac{\gamma_4}{\gamma_0})' + \ell_1 \gamma_4}{\ell_0 \gamma_3}$$

$$- \frac{a_1 \gamma_0 \ell_6 + a_1 \gamma_0 \ell_0 (\frac{\gamma_3}{\gamma_0})' - \ell_0 \gamma_4 - \ell_1 \gamma_3}{\ell_0 \gamma_3}$$

$$a_0 a_1 = \frac{\gamma_0 \ell_7 + \gamma_0 \ell_0 \left(\frac{\gamma_4}{\gamma_0}\right)' + \ell_1 \gamma_4}{\ell_0 \gamma_3} -$$

$$- \frac{\gamma_0 h_7 + \gamma_0 h_0 \left(\frac{\gamma_4}{\gamma_0}\right)' + h_1 \gamma_4}{\gamma_0 \gamma_5 + h_0 \gamma_3}$$

$$a_0 a_2 = \frac{\ell_2 \gamma_0 + \ell_0 \gamma_1}{\ell_0 \gamma_3} - \frac{h_0 \gamma_1}{h_5 \gamma_0 + h_0 \gamma_3}$$

$$a_0 a_3 = \frac{\gamma_0 \ell_0 \left(\frac{\gamma_1}{\gamma_0}\right)' + \ell_0 \gamma_2 + \ell_1 \gamma_1 + \ell_3 \gamma_0}{\ell_0 \gamma_3} -$$

$$- \frac{\gamma_0 h_0 \left(\frac{\gamma_1}{\gamma_0}\right)' + h_0 \gamma_2 + h_1 \gamma_1 + h_3 \gamma_0}{\gamma_0 h_5 + h_0 \gamma_3}$$

$$a_0 a_4 = \frac{\gamma_0 \ell_0 \left(\frac{\gamma_2}{\gamma_0}\right)' + \ell_1 \gamma_2 + \gamma_0 \ell_4}{\ell_0 \gamma_3} -$$

$$- \frac{\gamma_0 h_0 \left(\frac{\gamma_2}{\gamma_0}\right)' + h_1 \gamma_2 + \gamma_0 h_4}{\gamma_0 h_5 + h_0 \gamma_3}$$

$$a_0 = \ell_6 - h_6 + (\ell_0 - h_0)\left(\frac{\gamma_4}{\gamma_0} + \left(\frac{\gamma_3}{\gamma_0}\right)'\right) -$$

$$- (\ell_1 - h_1)\frac{\gamma_3}{\gamma_0}$$

The third order derivative can be eliminated by the substitution

$$Y = e_{\perp} \exp \int \frac{f_1' + a_1 f_1 + f_2}{4} d\psi \tag{25}$$

yielding

$$y^{(iv)} + b_2 y'' + b_1 y' + b_0 y = 0 \tag{26}$$

At the resonance $b_2 = 0$, and, in generally varies linerally with respect to ψ.In the zeroth order in m_e/m_i and $\ell_2\gamma_0 + \gamma_1\ell_0$ the functions b_o, b_1 and b_2 read

$$b_0 = \frac{\varepsilon'' E_\parallel \gamma_0}{h_0 \gamma_1 \ell_0 \gamma_3} (\gamma_0 \ell_0 (\frac{\gamma_2}{\gamma_0})' + \ell_1 \gamma_2 + \ell_4 \gamma_0)$$

$$b_1 = \frac{\varepsilon'' E_\parallel \gamma_0}{h_0 \gamma_1 \ell_0 \gamma_3} (\gamma_0 \ell_0 (\frac{\gamma_1}{\gamma_0})' + \ell_0 \gamma_2 + \ell_1 \gamma_1 + \ell_3 \gamma_0)$$

$$b_2 = \frac{\varepsilon'' E_\parallel \gamma_0}{h_0 \gamma_1 \ell_0 \gamma_3} (\ell_2 \gamma_0 + \gamma_1 \ell_0)$$

Conclusions and Discussions

The linear mode conversion occurring near the perpendicular ion cyclotron resonance and the shear Alfvén wave resonance caused by electron inertia has been treated for axially symmetric toroidal equilibria. Equations have been derived from which transmission and reflection coefficients can be calculated. To calculate these coefficients one has to solve an eigenvalue problem of an ordinary differential equation along a poloidal intersection of a magnetic surface. The solutions of the wave functions near the perpendicular ion cyclotron resonance can be used for matching with solutions outside the resonance obtained by two-dimensional numerical codes. The method outlined in this paper can be applied for studying the effect of finite Larmor radius, viscosity or resistivity. In the latter case the equation corresponding to Eq. (1) becomes of sixth order[13].

We note here that the equations describing mode conversion between the fast electromagnetic wave and the slow wave near the perpendicular ion cyclotron regime describe a similar process in another frequency regime, viz the mode conversion near the whistler resonance. However, in the case of lower hybrid heating the conditions for Rabenstein's solutions to be valid are usually not fulfilled.

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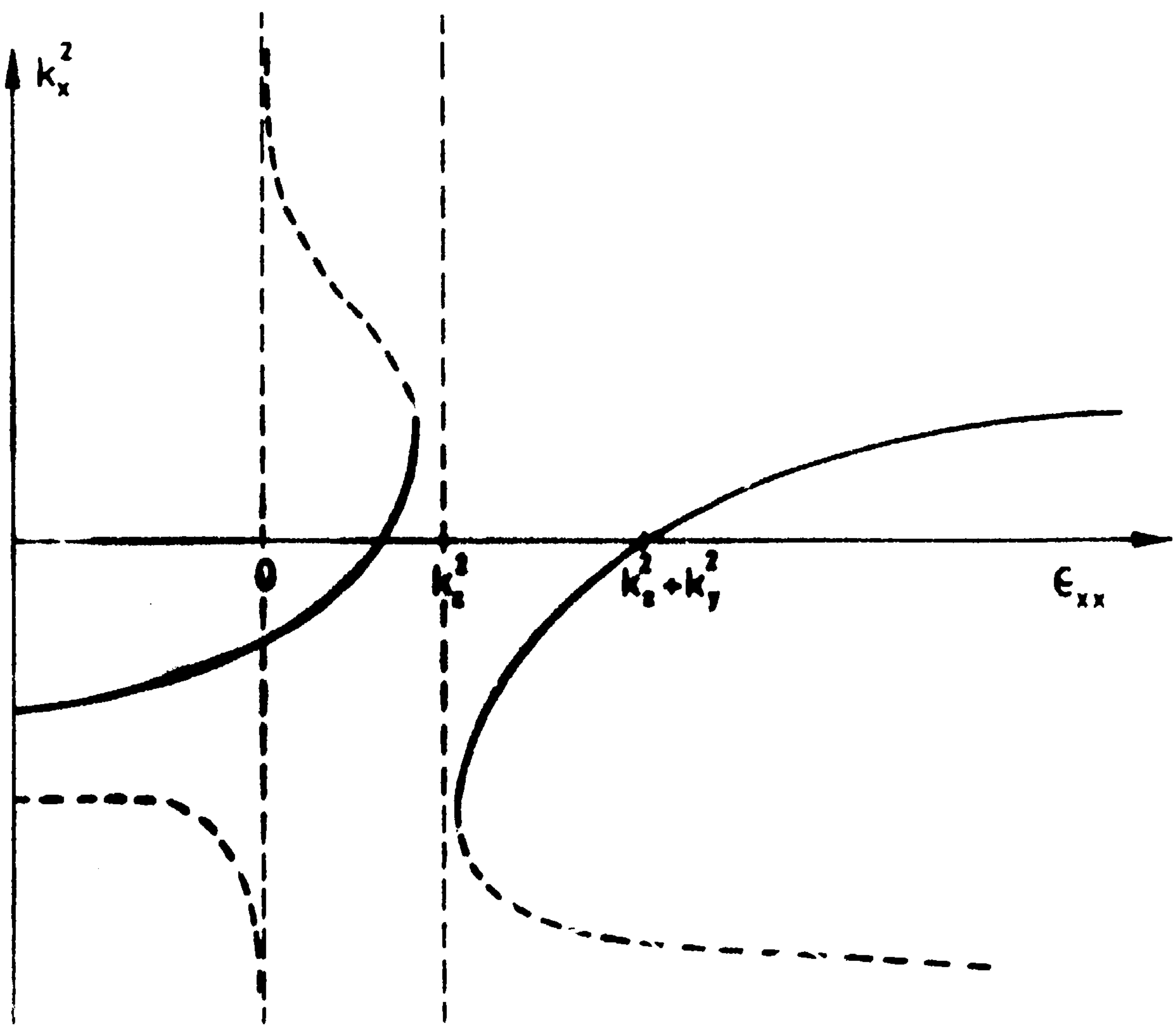
Figure Caption

Fig.1. Schematic representation of the two real roots of the cold plasma dispersion relation in a region between two ion cyclotron resonances. The slow wave is represented by the dashed line, and the fast wave by the full line. The ion-ion hybrid resonance occurs at $\varepsilon_{xx} = 0$, the perpendicular ion cyclotron resonance at $\varepsilon_{xx} - k_z^2 = 0$, and the fast wavw cut-off at $\varepsilon_{xx} - k_z^2 - k_y^2 = 0$.

Fig.1



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LINEAR MODE CONVERSION IN A TOROIDAL PLASMA

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Linear mode conversion at the perpendicular ion cyclotron resonance has been treated for an axially symmetric toroidal plasma. The mode conversion appears between a fast electromagnetic wave and a slow-quasi electrostatic wave, due to finite electron inertia. The problem reduces to the Orr-Sommerfeld equation where the coefficients determining the reflectron, transmission and conversion are functions of the arc length along a poloidal intersection of the resonance surface. These coefficients can be determined from eigenfunctions of an ordinary differential equation.

Key words: Alfvén wave heating, ion cyclotron resonance heating, linear mode conversion.