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An Electrical Analog of a Josephson Junction

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ABSTRACT

It is noted that a mathematical description of the phase-coupling of two oscillators synchronized by a phase-lock-loop under the influence of thermal white noise is analogous to that of the phase coupling of two superconductors in a Josephson junction also under the influence of noise. This analogy may be useful in studying threshold instabilities of the Josephson junction in regimes not restricted to the case of large damping. This is of interest because the behavior of the mean voltage near the threshold current can be characterized by critical exponents which resemble those exhibited by an order parameter of a continuous phase transition. As it is possible to couple a collection of oscillators together in a chain, the oscillator analogy may also be useful in exploring the dynamics and statistical mechanics of coupled junctions.

I. INTRODUCTION

Bishop and Trullinger[1] and Schneider, Stoll and Schlup[2] have recently pointed out the close analogy between the threshold for the onset of voltage in an ideal Josephson junction in the presence of noise, or the threshold properties of the identical problem, that of an overdamped Brownian particle in a periodic potential and applied field, and critical properties close to continuous phase transitions. The purpose of this note is to point out that oscillators coupled together in a phase-locked loop subject to noise are a realizable and potentially useful electrical analog of the Josephson junction in the presence of noise. In principle the study of the unlocking threshold of the oscillators should be a useful alternative to investigating the threshold instabilities of the Josephson junction. By varying circuit parameters it is possible to explore the regime for which an exact solution for the I-V characteristic exists[3,4,5] as well as the regime for which exact analytical solutions are nonexistent but which has been treated either approximately[5,6] or with a molecular dynamics simulation.[7]

In the context of the work of Refs. 1 and 2, where the dissipation threshold is viewed as a critical point, the proposed analog would permit investigations out of the overdamped limit in a way which might test their conjecture that critical rather than mean field exponents characterize the finite damping regime. Whether such critical exponents would actually be found is an open question as Bishop and Trullinger never found a mapping of the Josephson junction problem onto an equilibrium system describable by a Landau free energy functional. In effect, although their work identified an effective equation of state, it did not identify the corresponding free energy. Indeed, such a free energy may not exist.

Bishop and Trullinger further conjectured that if an appropriate mapping did exist it should be to a zero-dimensional system which would not exhibit a real phase transition. The proposed analog could be set up in a fashion that introduces a spatial degree of freedom by coupling a series of oscillators together in a chain using phase-lock-loop electronics in a way which would simulate a coupled-junction chain. It is useful to be able to do this in the light of recent studies of the statistical mechanics of coupled nonlinear pendula or similar systems subject to fluctuating forces.[8] Such problems have thus far been treated only in the overdamped limit of Brownian motion. The proposed analog might serve as a similar, although not identical, model system which could be investigated as a function of damping. As the cost of phase-locked loops in quantities of one hundred or more is relatively low, either in the form of single chips or integrated circuit chip sets, large assemblies of oscillators might be studied, thus closely approximating the behavior of a long chain.

In a related, but somewhat different context, the inductively coupled Josephson junction problem has been shown to be a discrete sine-Gordon chain for which solitary phenomena have been demonstrated.[9] The directly coupled oscillator system, although obeying a somewhat different equation, could serve as a model system for this problem in addition to serving as an analog for the statistical mechanical problems mentioned above.

II. THE JOSEPHSON JUNCTION AND PHASE-LOCKED LOOP

A small Josephson junction is one with linear dimensions small in comparison with the Josephson penetration length λ_J. Ivanchenko and Zil'berman[3] and Ambegoakar and Halperin[4] treated the problem of a current-biased small

Josephson junction in the presence of a white noise source in a way which was analogous to the problem of Brownian motion in a potential which in terms of a mechanical analogy is the problem of a particle diffusing down the potential $U(\varphi) = -\frac{1}{2}\gamma T(x\varphi + \cos\varphi)$. Here $x = I/I_1$ and $\gamma = \frac{\hbar I_1}{ek_B T}$ where I is the current flowing through the junction from an external source and I_1 is the maximum temperature-dependent Josephson current. The Langevin equation for the phase jump φ across the junction can be written as

$$\frac{d^2\varphi}{dt^2} + \frac{C}{R}\frac{d\varphi}{dt} + \frac{2e}{\hbar C} I_1 \sin\varphi = \frac{2e}{\hbar C} I + n(t) \qquad (1)$$

Here C is the capacitance of the junction, R is its shunt resistance, I_1 is the maximum Josephson current I is the current through the junction and $n(t)$ is a thermal white noise source arising from quasiparticle noise. The important physical parameter of the junction problem is the nonzero average voltage across the junction $V = \frac{\hbar}{2e}\langle \frac{d\varphi}{dt} \rangle$. $\langle \frac{d\varphi}{dt} \rangle$ is essentially the velocity of the diffusing "particle." The unusual feature of this voltage is that it can be nonzero at currents less than the maximum Josephson current. The compution of the voltage and the dc I-V characteristic involves the introduction of a time-dependent distribution function $P(\varphi,\dot{\varphi};t)$, which satisfies a two-dimensional Fokker-Planck equation which cannot be solved exactly. In the large damping limit defined by

$$\Omega = RC(2e\, I_1/\hbar C)^{1/2} \equiv \omega/\eta \ll 1 \qquad (2)$$

the Fokker-Planck equation reduces to a Smoluchowski equation for a coordinate distribution function which can be solved exactly. This limit corresponds to a Langevin equation for the phase containing only a first derivative in time. For arbitrary values of Ω the Fokker-Planck equation must be treated using either an expansion technique valid in the small Ω limit,[5,6] or by a molecular dynamics approach.[7] $\langle \frac{d\varphi}{dt} \rangle$ is then computed by averaging the Langevin equation using the distribution function.

The Langevin equation for the phase-difference of the Josephson coupling is formally identical to the equation for the phase error of two oscillators synchronized by a phase-locked loop in the presence of noise. Indeed, this analogy is so complete that all of the analytical results of calculations pertaining to the Josephson effect have actually been available in the frequency control literature for some time.[10-13]

In Fig. 1 we show a block diagram of a phase-lock loop circuit. In a circuit of this type the main oscillator (MO) and the voltaged controlled oscillator (VCO) are mixed in a phase detector which is a multiplier.[10] A voltage is produced at the output which depends on the phase difference between the MO and VCO. Both of these oscillators are taken to be sinusoidal. This voltage is passed through a low pass filter (LPF) and goes to a VCO whose frequency is controlled by the voltage in such a way that it is locked to the main oscillator. The noise which is broad-band Gaussian is introduced additively with the main oscillator signal.

The simplest case, which simulates the large damping limit of the Josephson junction, is to use a low pass filter (LPF) with an ideal cutoff (i.e., a transfer constant of unity at low frequencies and zero at high frequencies).[10] Then the Langevin equation for the phase error or phase difference between the oscillators is of the form

$$\dot{\varphi} = \Delta_0 - \Delta \sin\varphi + n(t) \tag{3}$$

Here Δ_o is the initial detuning of the oscillators, $\Delta_o = \omega_{co} - \omega_o$, and $\Delta = \frac{1}{2} \mu s A_1 A_2$ is a measure of the strength of their coupling. Also ω_{co} and ω_o are the initial frequencies of the VCO and the MO. (ω_{co} is the frequency of the VCO with no voltage applied to its input.) A_1 and A_2 are the amplitudes of the voltages supplied by the oscillators to the phase detector, μ is the gain of the mulitplier and $s = \frac{d\omega}{dV_{in}}$ gives the relationship between the input voltage to the VCO, V_{in} and its output frequency. The noise $\eta(t)$ can be assumed to be broad-band Gaussian. It is actually a sum of terms of this type associated with the random amplitude of the noise and random relative phases of the oscillators and the noise source.[10]

If the "ideal" LPF is replaced by a series RC circuit,[11] the Langevin equation for the phase error between the oscillators is second order in time and takes the form

$$\ddot{\varphi} + \alpha\dot{\varphi} + \alpha\Delta\sin\varphi = \alpha\Delta_o + \eta(t) \qquad (4)$$

where $\alpha = 1/RC$ and all of the other parameters are defined as in Eq. 3.

Equations 3 and 4 are analogs of the Langevin equation for the phase across a small Josephson junction in the overdamped and finite damping limits, respectively. Indeed, apart from difference in the physical meaning of symbols, Eq. 4 is identical in form to Eq. 1.

In the context of the oscillator problem, the counterpart of the V-I characteristic in the presence of noise is the difference between the average frequencies of the VCO and MO, $\bar{\omega}-\omega_o$ plotted against $\frac{2\Delta_o}{\Delta}$, the latter being the ratio of the initial detuning of the two oscillators to a quantity which is a measure of their coupling. In place of the parameter $\frac{\hbar I_1}{2ek_BT}$, which is the ratio of the junction coupling energy to the noise power spectral density, a parameter $D = \frac{\Lambda}{K_2}$ is introduced. The latter is a measure of the ratio of the oscillator coupling to K_2 which is an integral over time of the sum of the autocorrelation functions of the amplitude fluctuations of the noise and the phase fluctuations of the oscillators. This latter quantity is a measure of the noise intensity. In Fig. 2 we plot the I-V characteristics of a junction and the frequency shift-coupling characteristics of the coupled oscillator system in a manner which illustrates the close similarity between the two problems. It should be noted that as long as the random functions in $\eta(t)$, which have not been given here, vary rapidly, their detailed form does not affect Fig. 2b outside of determining the magnitude of K_2 and D. It should be noted that the quantity $\bar{\omega} - \omega_0$ is calculated by computing $\langle \frac{d\varphi}{dt} \rangle$ in a manner identical to that used for junctions.

If a second-order filter is used in place of a first-order filter in the phase-lock loop, the differential equation acquires a term proportional to the product of $\cos\varphi$ and $\frac{d\varphi}{dt}$ which in some sense may make the analog more realistic by incorporating the correction term to the usual Josephson current due to quasiparticle-pair interference.[14] The latter would add a term to the Langevin equation proportional to the product of $V(V = \frac{h}{2e}\frac{d\varphi}{dt})$ and $\cos\varphi$.[12,13] It should be noted that in the frequency control literature, detailed discussions of the phase-lock-loop have actually been given for arbitrary order filters in the loop.

III. DISCUSSION

With the advance of integrated circuit technology, phase-lock-loop chips having an actual circuit close to that shown in Fig. 1 are available commercially. As the applicability of the analysis depends on the validity of the circuit model for a real system, it will be necessary to use separate multiplier and VCO chips and to use carefully designed active filters. The latter are critical because the form of the Langevin equation depends in detail on the transfer characteristic of the low pass filter in the loop. All these components are available commercially at modest cost. Using frequency counter technology and sampling techniques, it should also be a simple matter to measure the difference in the frequencies of the VCO and MO for various values of the initial detuning and oscillator couplings and various intensities of Gaussian white noise which would be introduced additively with the MO signal.

A one-dimensional chain of junctions can be synthesized by using the VCO of one phase-lock-loop as the main oscillator of the second and so forth. By using separate multiplier and filter chips rather than single chip phase-lock-loop packages, a Bethe lattice of coupled oscillators could easily be constructed. It is not obvious that 2-D or 3-D lattices are actually possible.

For simulating tunnel junction phenomena the phase-lock-loop analog may be easier to deal with than real junctions as its parameters may be altered by changing the settings on potentiometers or amplifier gains rather than fabricating a new junction with a different resistance or critical Josephson current.

In the relatively new field which is the study of the statistical mechanics of coupled nonlinear mechanical systems, the phase-lock-loop coupled oscillator systems may be remarkably simple physical testing grounds for the theory capable of exploring both the heavily damped regime which can be solved exactly in many instances and the lower damping regimes for which analytical solutions do not exist.

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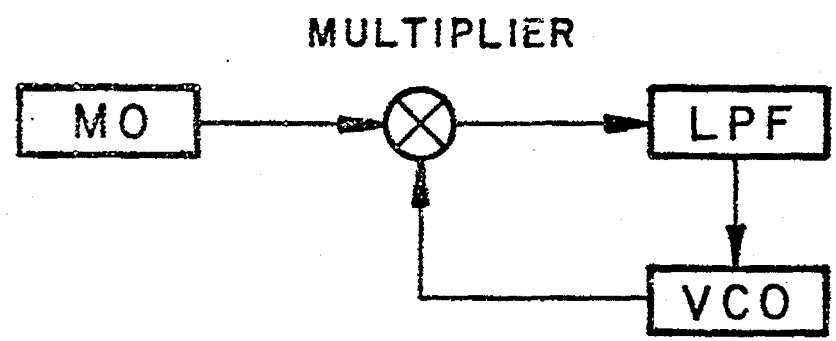
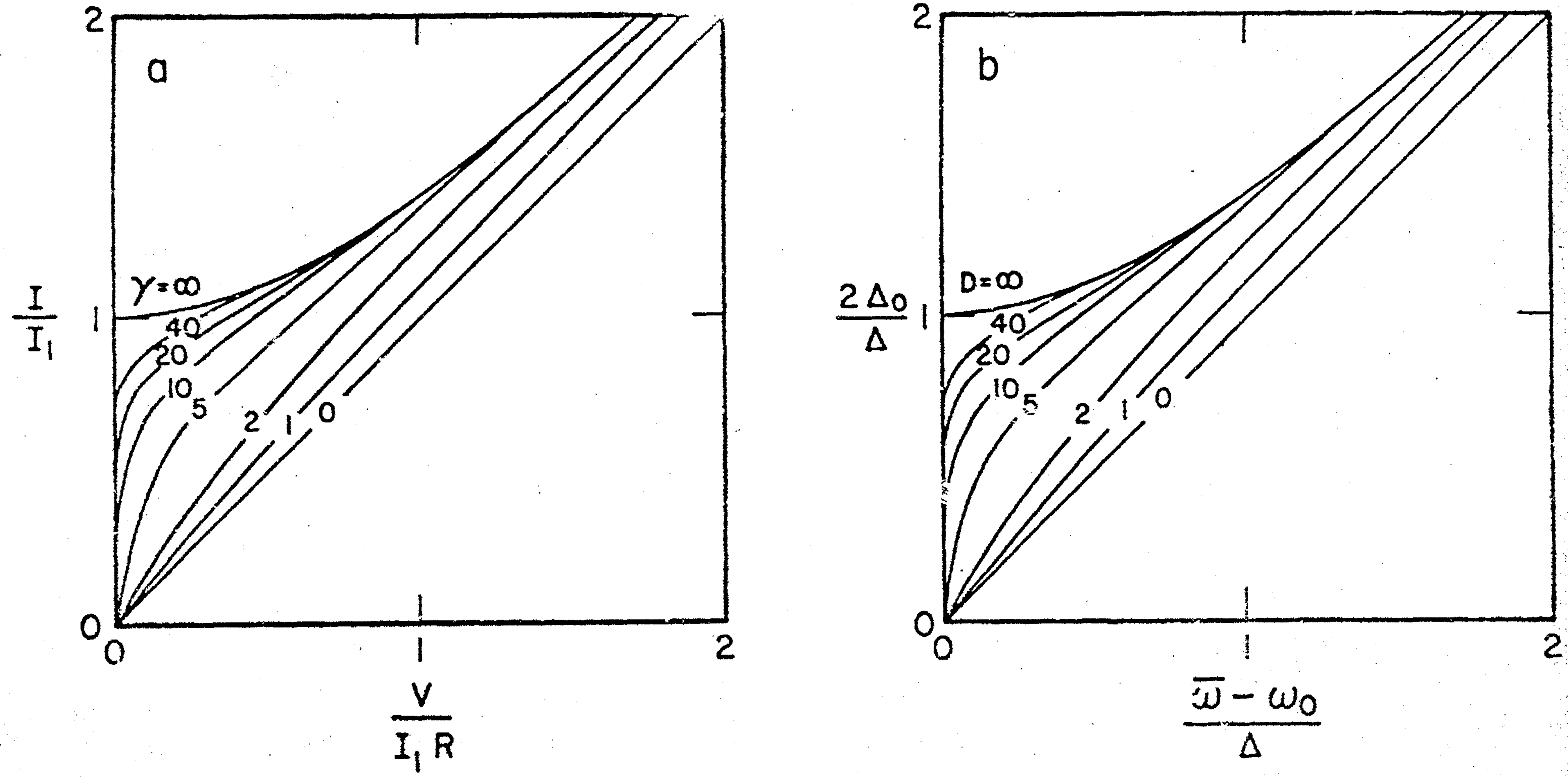
FIGURE CAPTIONS

Fig. 1. Block diagram of phase-lock loop circuit. If the two oscillators are sinusoidal, then φ is the phase difference between them.

Fig. 2a. I-V characteristic of a tunneling junction as a function of the parameter γ which is a measure of noise. These results are obtained using the heavily damped result with $\Omega = 0$.

Fig. 2b. The difference between the mean frequencies of the MO and VCO as a function of circuit characteristics and noise parameters. The curves are the same as those in Fig. 2a. This system corresponds to a phase-lock loop with an ideal low pass filter cutoff.


MULTIPLIER
MO
LPF
VCO


a
b
$\gamma = \infty$
40
20
10
5
2
1
0
$D = \infty$
$\frac{I}{I_1}$
$\frac{V}{I_1 R}$
$\frac{2\Delta_0}{\Delta}$
$\frac{\bar{\omega} - \omega_0}{\Delta}$