BUNCH-BEAM INSTABILITIES

Memorial Talk for F.J. Sacherer

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INTRODUCTION

This talk is dedicated to F.J. Sacherer whose name will always remain closely associated with bunched-beam instabilities. All the material for this lecture was drawn from the numerous reports related to the theory which he left in perfect order. Nevertheless, I have followed the natural tendency of presenting things the way I see them. One could of course enumerate a countless number of other contributions, only a few of the more classical or recent ones are referred to.

In this attempted synthesis, longitudinal and transverse instabilities will be expounded in parallel. The main results are presented here, while detailed derivations are given elsewhere. Landau damping is very briefly mentioned. Justification for this is that a companion invited paper on diagnosis and cures by A. Horman should lay more emphasis on it.

SIGNS AND COHERENT DISTRIBUTIONS

We consider a Pick Up electrode located at angular position \( \theta \) in the ring. A field particle is undergoing unperturbed linear betatron and synchrotron motions. The P.U. signal induced by its successive passings is analysed. \( \omega_s = \beta c / R \) is the angular revolution frequency of the synchronous particle. \( \tau \) is the time delay of the field particle with respect to the synchronous particle, expressed in seconds. \((\zeta, \xi)\) and \((X, X)\) are the standard rectangular coordinates in longitudinal and transverse phase spaces. \((\psi, \tilde{z})\) and \((\phi, \tilde{x})\) are the associated phase and amplitude coordinates. \( \omega_s = Q_s \omega_o \) and \( \omega_s = Q_s \omega_o \) define the focusing strength of external forces in both planes.

1.1 Longitudinal

The longitudinal signal consists of a series of nearly periodical impulses delivered at each passing.

\[
s_y(t, \theta) = \epsilon \sum_{n=-\infty}^{\infty} \delta(t - n/T) \frac{2 \pi n}{\omega_s} \sin(\omega_s t) \quad \text{(amperes)}
\]

where \( \delta \) is the Dirac function. Notice that \( s_y(t, \theta) \) is merely intensity.

An equivalent form is

\[
s_y(t, \theta) = \epsilon \sum_{p, m=-\infty}^{\infty} \int_{-\infty}^{\infty} J_m(p \omega_o \xi) e^{-j(\omega_{pm} t - p \omega_o \xi)} \frac{d\xi}{\pi}
\]

The actual spectrum of the single particle is a line spectrum at frequencies

\[
\omega_{pm} = \frac{p \omega_o + m \omega_s}{2 \pi} \quad \text{for} \quad p, m \leq \infty
\]

within envelopes represented by the \( J_m \) (Bessel function of order \( m \)) associated with the \( m \)th synchrotron satellite. This spectrum is centered at the origin of the frequency axis. Its width behaves like the reciprocal amplitude \( 1/\xi \).

As soon as two field particles are considered, one can start introducing the concept of coherency. As a matter of fact, one can arrange the initial conditions of charges in order to make the \( \epsilon e^{j m \psi} \) dependence the same for all the charges on the same synchrotron orbit \( \xi \). Then, the distribution is coherent with respect to the \( m \)th synchrotron satellite in the sense that the resulting spectral amplitude is maximum for satellite number \( m \) and...
vanishes for all other satellites.

As a net result, when dealing with longitudinal bunched-beam instabilities, one will study the behaviour of small density perturbations of the form

$$\Delta p = p_m(t) e^{\pm j m \psi} = \int p_m(t) e^{\pm j m \psi} e^{\pm j \omega_m t}$$

oscillating about an exactly circular stationary distribution

$$p_c(t)$$

Integrating over the distribution \( \Delta p \), one gets the perturbation signal

$$S_{p_m}(t, \theta) = I_0 \frac{e^{\pm j \omega_m t}}{2 \pi} \sum_{m=-\infty}^{\infty} e^{j(\Omega_m - Q)(\theta - \eta)} e^{-j \psi_m}$$

$$I_0 = \text{current in one bunch}$$

$$\sigma_{p_m}(p) = \int_{t_0}^{t_\infty} J_m(p \omega t) p_c(t) dt$$

is proportional to the spectral amplitude

$$\text{for } \omega = \omega_m + \Delta \omega_m$$

1.2 Transverse

Transverse signal of a field particle is simply the product of the longitudinal signal times the transverse position

$$s(t, \theta) = x(t) \cdot \gamma(t, \theta) \quad \text{(amperes \: m)}$$

It can be written

$$s(t, \theta) = \int e^{j \omega t} \sum_{p=-\infty}^{\infty} J_m((p \omega + Q, \omega - \omega_m) t) e^{j((\Omega_m - Q, \omega - \omega_m) t \eta + \psi_m)}$$

Again it gives a line spectrum at frequencies \((p \omega, \omega) + m \omega_m\) within the envelope \(J_m\) associated with synchrotron satellite number \(m\). This time, the spectrum is centered around \(\omega_m\) on the angular frequency axis.

$$\omega_m = \omega_m \cdot \frac{d}{d \psi}$$

where \(\omega_m = \frac{dQ}{d\psi}\) defines the chromaticity tuning

$$\psi = \frac{\omega_m}{\omega}$$

and \(\eta = \frac{\omega_m - \omega_m}{\omega_m}\) measures the proximity of the transition energy

Here also, one can arrange particles around a synchrotron orbit \(\mathcal{C}\), by making \(m \psi = \gamma\) constant, for all the particles in the bucket to be coherent with respect to satellite number \(m\). So, when dealing with transverse bunched-beam instabilities, one will consider perturbations of the form

$$\Delta P = P_m(t, t) e^{\pm j (m \psi \cdot \gamma)} = \int P_m(t) e^{\pm j (m \psi \cdot \gamma)} e^{\pm j \omega_m t}$$

The betatron amplitude \(\hat{P}(t)\) is the same for all particles on the same synchrotron orbit

The transverse perturbation signal is then

$$S_{p_m}(t, \theta) = I_0 \frac{e^{\pm j \omega_m t}}{2 \pi} \sum_{p=-\infty}^{\infty} e^{j((\Omega_p - Q, \omega_m) t - \eta) \psi_m}$$

with

$$\sigma_{p_m}(p) = \int_{t_0}^{t_\infty} J_m((p \omega - Q, \omega_m) \mathcal{C}) \hat{P}(t) p_c(t) dt$$

2. COUPLING IMPEDANCES

In order to study stability, one needs an expression for self forces induced by coherent distributions interacting with the surroundings, namely

$$F_{\text{self}}(t, \theta) = e \left[ E(t, \theta) \cdot \gamma \cdot B(t, \theta) \right]_{\text{self}}$$
where $\theta = \omega(t-x)$ when following a test particle.

The actual field expression is difficult, if not impossible, to obtain when taking into account the large number of interruptions in the vacuum chamber for step changes, P.U. electrodes, kickers and septum magnets, bellows, etc. For the problem to remain as general as possible, one brings in the notion of coupling impedances.

2.1 Longitudinal

To facilitate definitions, let us discuss this simplified case where all along the machine, the beam is completely screened by a circular pipe of radius $b$ (Fig. 1). The beam intensity $I$ at time $t$ and angular position $\theta$ is given by the longitudinal signal $S_y(t,\theta)$. There is no field outside the walls, which requires $I$ flowing downstream along the pipe, in the wall thickness.

For perfectly conducting walls, the electric field in the walls $E_w$ is null. In solving Maxwell's equations, one finds the standard longitudinal space charge electric field set up by the direct current.

For non-perfectly conducting walls, $E_w \neq 0$. The voltage drop along the wall set up by the return current is responsible for an additional uniform electric field. Both electric fields set up by direct and return currents can be expressed in terms of signal $S_y(t,\theta)$ by defining a longitudinal coupling impedance

$$Z_\parallel(\omega) = \frac{1}{2\pi R} \int_{-\infty}^{\infty} \tilde{S}_y(\omega,\theta) e^{j\omega t} d\omega$$

$Z_\parallel(\omega)$ is expressed in ohms

$S_y(t,\theta)$ is the Fourier transform of the signal already introduced in the preceding section.

2.2 Transverse

In the transverse case, the beam is off centered. As a consequence, the return current is no longer uniformly distributed around the pipe. The differential current flowing in the opposite direction on either side of the chamber pipe sets up a transverse magnetic field which can be expressed in terms of transverse signal $S_\perp(t,\theta)$ by defining a transverse coupling impedance

$$Z_\perp(\omega) = \frac{1}{2\pi R} \int_{-\infty}^{\infty} \tilde{S}_\perp(\omega,\theta) e^{j\omega t} d\omega$$

$Z_\perp(\omega)$ is expressed in ohms/m.

In the above example, the complete solution of the field equations leads to

$$Z_\perp = \frac{2c}{b^2} \frac{Z_\parallel}{\omega}$$
where \( c \) is the speed of light \((3 \times 10^8 \text{ m/sec})\).

This convenient relationship between transverse and longitudinal coupling impedances is strictly valid only for a round pipe when frequencies are sufficiently below cut-off that the fields have the simple form assumed above. There are several beam equipments often used in accelerators for which Eq. 18 does not work.

3. EQUATIONS FOR SINGLE BUNCH COHERENT MOTION, LOW INTENSITY MODES OF OSCILLATION

The procedure used to obtain the equations of motion is very similar for longitudinal and transverse planes.

3.1 Longitudinal

We proceed as follows:

- Take a realistic exactly circular distribution \( p_c(\xi) \) normalized to unity

\[
\int_{\xi=0}^{\xi=m} p_c(\xi) \, d\xi = 1 \tag{19}
\]

- Add a perturbation coherent with respect to satellite number \( m \)

\[ \Delta p_c = p_m(\xi) \, e^{-\text{i}m\omega_c} \, e^{\text{i}\Delta \omega_m t} \]

- Express the local density (normalized to unity).

\[ D(\psi_c, \xi, t) = \frac{1}{2\pi \xi} \left( p_c + \Delta p_c \right) \tag{20} \]

- Write the Vlasov equation to first order.

\[
j \frac{\partial \Delta p_m}{\partial t} = \frac{2}{\omega_c} \left( \frac{\partial}{\partial \xi} F_\psi(\psi_c, \omega_c(\xi, t)) \right) \, e^{\text{i}(m\psi - \Delta \omega_m t)} \sin(m \omega_c t + \psi_c) \tag{21} \]

- Use the coherent signal \( \frac{E_m(\psi_c, t)}{\omega_c} \) to express the coherent self force and keep only slowly varying terms.

- Finally one ends up with the following integral equation

\[
j \Delta \omega_m p_m(\xi) = \frac{m \omega_c}{\omega_c V_h \cos \phi_s} \sum_{p=-\infty}^{\infty} \frac{Z_p(p \omega, m \omega_c, \Delta \omega_m)}{p \omega} J_m(p \omega_c) J_m(p \omega) \tag{22} \]

with the convention that \( \cos \phi_s \) is positive below transition and negative above.

- Multiply both sides by \( J_m(p \omega_c) \) and integrate over \( \xi \) values in order to write this integral equation in a matrix form

\[
j \Delta \omega_m \sigma_{m,n}(t) = A^m_{m,n} \, \sigma_{n}(p) \quad m < n \leq \infty \tag{23} \]

where

\[
A^m_{m,n} = \frac{m \omega_c}{\omega_c V_h \cos \phi_s} \sum_{p=-\infty}^{\infty} \frac{Z_p(p \omega, m \omega_c, \Delta \omega_m)}{p \omega} \int_{\xi} \left( \frac{E_c}{\xi} \right) J_m(p \omega_c) J_m(p \omega) \, d\xi \tag{24} \]

For low intensity, the self force is weak when compared to the external force

\[ |\Delta \omega_{m,n}| \ll \omega_c \]

and distributions with different \( m \) oscillate at different frequencies.

With restricting to a given value of \( m \) one now has to solve the matrix equation (23). The result is an infinite number of modes \( m \) \((\omega \sim \infty)\) of oscillation. To each mode, one can associate:

- i) a coherent frequency shift \( \Delta \omega_{m,q} \) \((q^{th} \text{ eigenvalue of the matrix})\)
- ii) a coherent spectrum \( \sigma_{m,q} \)
- iii) a coherent distribution \( \sigma_{m,q} \) defined by (22).

Low order eigenvalues and eigenvectors of the infinite matrix \( A^{m}_{n} \) are found quickly by computation. Figure 2 illustrates the results for the bunch
LONGITUDINAL MODES for constant $Z_\Omega/p$

Frequency Spectrum for modes $m\nu$ when $Z_\Omega$ or $Z_\perp$ are constant

TRANSVERSE MODES for constant $Z_\perp$

Figure 2
interacting with a very broad band impedance \( Z_j^p \) constant over the mode spectrum (space charge or inductive walls).

The spectrum of mode \( m \) is peaked near \( \omega = (q+1)\pi/c_z \) and extends \( \pm 2\pi/c_z \) rad/sec. Usually, only diagonal modes \( mn \) are referred to (\( m = 1 \) dipole mode, \( m = 2 \) quadrupole, \( m = 3 \) sextupole, etc).

It should be noted that modes with the same \( q \) value, mode \( 13 \) and sextupole mode \( 33 \) for instance, lead to nearly identical spectrum envelopes and line densities although they oscillate at different frequencies with entirely different density distributions.

3.2 Transverse

Applying the same procedure to the transverse plane, one gets the transverse integral equation

\[ j \Delta \omega \frac{\hat{x}}{\hat{y}}(t) = -\epsilon \beta \frac{\pi}{4\pi R^3_0 q_0^2} \int \frac{Z_j^p(\rho, \omega) \delta \omega_m}{m} J_m(\rho \omega) \phi_m(\rho) d\hat{e} \]

which can be written in matrix form

\[ j \Delta \omega \phi_m(\rho) = A_{\hat{e} \hat{f}} \phi_m(\rho) \]

where

\[ A_{\hat{e} \hat{f}} = \epsilon \beta \frac{\pi}{4\pi R^3_0 q_0^2} \int \frac{Z_j^p(\rho, \omega) \delta \omega_m}{m} J_m(\rho \omega) \phi_m(\rho) d\hat{e} \]

Comparing transverse and longitudinal equations of coherent motion, one sees that they look nearly identical. As a matter of fact, \( Z_j^p/\rho \) and \( \frac{\delta}{\delta \hat{e}} \) in (24) are changed into \( Z_j^p(\rho) \) and \( \hat{p}(\hat{e}) \). Accordingly, for the "water-bag" bunch

\[ \hat{p}(\hat{e}) = \frac{\pi \epsilon}{c_z} \quad \frac{\epsilon \hat{e} \times R_c}{2} \]

and a broad band transverse impedance \( Z_j^p(\rho) \) constant over the mode spectrum, the results presented in figure 2 apply for both longitudinal and transverse cases.

Nevertheless, let us stress two particularities of transverse modes.

i) They start from \( m = 0 \) (dipole mode)

ii) Spectra are centered around \( \omega_0 \) on \( \omega \) axis.

4. APPROXIMATE CALCULATION OF COHERENT FREQUENCY SHIFTS

In the previous section, the procedure to obtain first order exact solutions, with realistic modes and a general interaction, has been explained in detail. It consists of finding the eigenvalues and eigenvectors of an \( A_{\hat{e} \hat{f}}^m \) complex matrix by computer, which could prove difficult in some cases. For this reason approximate formulae for eigenvalues are useful.

4.1 Longitudinal

The starting point is equation (22). Multiply both sides by \( \phi_m(\rho) \frac{\delta}{\delta \hat{e}} \) and integrate over \( \hat{e} \) values to obtain

\[ j \Delta \omega_m = \frac{m \omega_0 I_0}{\omega_0 \sqrt{1 - \cos^2 \phi}} \left[ \sum_{\hat{e}} \frac{Z_j^p(\rho \omega_m, \delta \omega_m)}{m} \phi_m(\rho) \right] \]

\[ = \frac{m \omega_0 I_0}{\omega_0 \sqrt{1 - \cos^2 \phi}} \int_{-\infty}^{\infty} \frac{\delta \omega_m}{\delta \hat{e}} \phi_m(\rho) d\hat{e} \]
Using the definition of $g_m(p)$ one can form up the following equalities

\[ \Delta \omega_{m_m} = \frac{mQ_e}{m+1} \frac{e B_{m_m}}{4 \pi \beta m_m Q_e u_{l_1}} \sum_{p=r}^{+\infty} Z_L((p+Q_e)u_{l_1}-m Q_e) A_m((p+Q_e)u_{l_1}) \]

This allows for expression of the frequency shift as a function of $Z_\phi$ and

\[ h_{m}(p) \]

provided $\frac{2}{m} (I(m))$ behaves like $\frac{1}{m}$, which implies taking $\xi_q = \frac{2 \xi q}{m} (1 - \frac{2 \xi q}{m})$, function already proposed for illustrating the result in the preceding section.

\[ \Delta \omega_{m_m} = \frac{mQ_e}{m+1} \frac{I_m}{3 B_{m_m} \sqrt{h \cos \beta}} \sum_{p=r}^{+\infty} Z_L((p+Q_e)u_{l_1}) A_m((p+Q_e)u_{l_1}) \]

$B_{m_m}$ is the bunching factor for one bunch.

The approximation $\sum_{p=r}^{+\infty} |p| A_m((p+Q_e)u_{l_1}) = (m+1) \frac{I_m}{3 B_{m_m} \sqrt{h \cos \beta}} \sum_{p=r}^{+\infty} A_m((p+Q_e)u_{l_1})$, valid for diagonal modes only, has been used in (33).

The $h_{m_m}$ functions depend on the interaction $Z_\phi(p)$. If the modes were known exactly, (30) and (31) would give the exact eigenvalues. On the other hand, for a non exact but realistic set of modes, equation (33) can be used to find the approximate eigenvalues for any $Z_\phi(p)$. Exact solutions exhibited in figure 2, fulfill these requirements. They are fitted fairly well by sinusoidal modes which are easier to work with. Therefore, in (33), $h_{m_m}(p)$ can be expressed in the following form

\[ h_{m_m}(u) = (m+1)^2 \frac{4 \xi q - (m+1)^2 \cos (m+1) \xi q}{(\frac{m+1}{m})^2 - (m+1)^2 \xi q} \]

4.2 Transverse

The same $h_{m_m}(u)$ can be used to approximate transverse modes for the bunch with $\xi_q = \frac{2 \xi q}{m}$. The corresponding formula for frequency shifts is then

\[ \Delta \omega_{m_m} = \frac{\beta_1}{4 \pi \beta m_m Q_e u_{l_1}} \sum_{p=r}^{+\infty} Z_L((p+Q_e)u_{l_1}-m Q_e) A_m((p+Q_e)u_{l_1}) \]

When comparing results from (33) and (35) to exact solutions, maximum deviation of 25% have been obtained in a few extreme cases.

5. EQUATIONS FOR COUPLED-BUNCH COHERENT MOTION

In section 1, particles have been arranged within one bunch to produce a coherent signal. If $M$ equally spaced bunches are present in the ring, there are $M$ different possibilities to couple adjacent bunches for all the particles in the ring to contribute coherently to the signal.

With the index $n$ running from $0$ to $M-1$, coupled-bunch mode $n$ will correspond to a phase shift of $2n\pi/M$ between the distributions of two adjacent bunches.

The spectral amplitude is $M$ times larger but only every $M$th line occurs. Exact solutions are derived from

\[ \Delta \omega_{m_m}^n \xi_m,(d) = \frac{m Q_e}{m+1} \frac{I_m}{3 B_{m_m} \sqrt{h \cos \beta}} \sum_{p=r}^{+\infty} Z_L((p+Q_e)u_{l_1}-m Q_e) A_m((p+Q_e)u_{l_1}) \]

with

\[ A_{m_m}^n = \frac{m Q_e}{m+1} \frac{I_m}{3 B_{m_m} \sqrt{h \cos \beta}} \sum_{p=r}^{+\infty} Z_L((p+Q_e)u_{l_1}-m Q_e) A_m((p+Q_e)u_{l_1}) \]

\[ \int_{-\infty}^{\infty} \frac{2 \xi q}{d} \frac{J_m(n p M) u_{l_1}}{(n p M) u_{l_1}} \frac{J_n(m + p M) u_{l_1}}{(m + p M) u_{l_1}} d\xi q \]
As far as the approximate formulae are concerned, they remain the same except the summations are over the coupled-bunch mode spectra

\[ \Delta \omega_{mn}^{ll} = \frac{j}{m+1} \frac{I_o}{\omega - 5 B_e \gamma \cos \beta} \sum_{n=-\infty}^{\infty} \frac{Z_f((n+p)\omega_1 \pm m\omega_2) \delta_{mm}(n+p)m \omega_2)}{\delta_{mm}(n+p)m \omega_2} \]

\[ \Delta \omega_{mm}^{ln} = \frac{j}{m+1} \frac{I_o}{4 \pi R B_{m} \gamma Q_{u_2}} \sum_{n=-\infty}^{\infty} \frac{Z_f((n+p)\omega_1 \pm m\omega_2) \delta_{mm}(n+p)m \omega_2)}{\delta_{mm}(n+p)m \omega_2} \]

Equations (36), (37), (38) and (39) are the basic equations for dealing with low intensity bunched beam instability. Given any coupling impedance, its effect on the beam can be computed. What is usually missing are measurements or calculations of \( Z(\omega) \).

6. COUPLING IMPEDANCE OF A RING

Numerous attempts to measure accelerator impedance have been made in the past five years. The overall results confirm the presence of the following three major components of the impedance (Fig. 3).

i) A broad band component which can be simulated by a low \( Q \) \((Q_{\text{res}} \approx 1)\) resonator centered around the lowest TM cutoff frequency \((\sim \text{a few GHz})\) with a peak value of \( Z_f/p \) situated between 5 and 30 ohms. Below cutoff, relation (18) between longitudinal and transverse broad band components is assumed.

ii) A resistive wall component which dominates at low frequencies

iii) Parasitic resonators corresponding to higher order transverse or longitudinal modes in cavities for instance.

7. USUAL TYPES OF INTERACTION

In this section, different types of coupling impedances are reviewed and their effect on the beam are discussed. (25) or (29) will define the bunch depending on whether longitudinal or transverse instabilities are being considered.
# Longitudinal

## 7.1 Inductive walls and space charge

For $Z_f(p)/p$ constant over the bunch spectrum, the summations drop out of (38) and the result is the same for a single bunch or several bunches in the ring. With this type of impedance, there is no wake at all, it is a local or instantaneous interaction

$$\Delta \omega_{\text{int}} = \frac{m_e q_s}{m_s} \frac{I_s}{3 eta_i^2 V \cos \phi_s} j \frac{Z_f(p)}{p}$$

Space charge and inductive walls correspond to this type of interaction with

$$j \frac{Z_f(p)}{p} = \frac{Z_s \gamma}{2 \beta_i^4} - L \omega_s$$

$Z_s = 377$ ohms

$s \ell$ total inductance per turn in henrys.

Note that frequency shifts are real and depend strongly on the bunching factor.

## 7.1.2 Resonator

Resistance may cause instability since it contributes an imaginary frequency shift.

Above the transition, positive frequency lines have a destabilizing effect, negative frequency lines have a stabilizing effect, with the opposite below the transition.

For a single bunch or two bunches, the resulting effect is weak unless the resistance varies considerably in a $2m \omega_s$ interval, which happens with narrow band impedances such as RF cavities. In this case a suitable detuning is required for stability (Robinson criterion).

With a large number of bunches, stabilizing and destabilizing frequency lines, corresponding to a given coupled mode, are associated with very different values of the resistance. The resonator will drive a given coupled mode $n$ and damp the complementary mode $M-n$ (Fig. 4).

The overall result is given by (33) which can be written in the equivalent form

![Figure 4](image_url)
\[
\Delta \omega_{mn} = \frac{m \omega_n}{m + 1} \frac{M I_s}{3 B \sqrt{\pi} \cos \psi_s} \frac{R_s \omega_n}{\omega_{res}} F_m(\omega, \nu) D
\]  
(42)

\[
F_m = \frac{1}{MB \sum F_m(n)}
\]  
is a form factor that specifies the efficiency with which the resonator drives a given mode (Fig. 5).  \( R_s \) is defined by (53).

\[
D = \alpha \left( \frac{1}{\lambda - e^{i \pi}} - \frac{1}{\lambda - e^{i \pi}} \right) \text{ with } \alpha = 2 \frac{M}{Q_i} \left( \frac{\omega_n}{\omega_{res}} \right) \quad \text{and} \quad \alpha = \frac{\pi}{M \omega_{res}} \omega_{res}
\]  
(44)

7.1.3 Resistive walls

Resistive wall impedance is large at low frequencies where the spectral amplitude of coherent modes tends to zero, that is why this contribution to frequency shifts is disregarded as far as longitudinal plane is concerned.

7.2 Transverse

7.2.1 Resistive part of the impedance

The resistive part of \( Z_l(\omega) \) causes instability for negative resistance and damping for positive resistance. To prevent the mode spectra from overlapping negative resistance, a positive value of \( \omega_l \) is required (positive chromaticity above transition or negative chromaticity below transition).

Usually, damping does not require a large value of \( \omega_l \). This is illustrated in Fig. 6 for mode \( m = 2 \). The broad positive part of the spectrum contributes more than the narrow region of negative frequencies near the origin. The additional damping provided by the broad band impedance helps in stabilizing high order modes and is very efficient with short bunches.
In Fig. 7 mode $m=0$ is considered with a small $\omega_\perp$ value and $Q_\perp$ just below an integer. Here a single line contributes to the long range part of the wake, namely the line $\omega = (n+pM_\perp Q_\perp)\omega_\perp$ nearest the origin in the narrow band region of $Z_\perp$. From (39) one can write

$$\Delta \omega_{\perp \text{mm}} = \frac{1}{m+1} \frac{\epsilon \beta M_\perp}{4 \pi R m_\gamma Q_\perp \omega_\perp} \int Z_\perp(p) F_m\left((\omega - (n+pM_\perp Q_\perp)\omega_\perp)\tau_\perp\right) dp$$

where $F_m$ is given by (43).

7.2.2 Smooth impedance

When $Z_\perp(p)$ is sufficiently smooth over the bunch spectrum of mode $m$, it can be removed from the summation in (39) and one finds

$$\Delta \omega_{\perp \text{mm}} = \frac{1}{m+1} \frac{\epsilon \beta I_\perp}{4 \pi R B_\perp m_\gamma Q_\perp \omega_\perp} \int Z_\perp(\omega_\perp) dp$$

For $m=0$, this is just the coasting beam result.

7.2.3 Resonator

In the transverse case, it is difficult to derive a general formula equivalent to equation (42). When a single line contributes (45) applies, otherwise equation (39) leads to the approximate value of the frequency shift.

8. STABILITY CRITERION

One can take into account the within-bunch frequency spread to derive a dispersion...
relation. The resulting stability criteria are briefly listed here under.

8.1 Longitudinal

Define $\delta S_y$ as the full spread in $\omega_y$ between center and edge of the bunch due to non-linear synchrotron forces. Then stability requires

$$\delta S_y > \frac{4}{\sqrt{m}} |\Delta \omega_{ym}|$$

(47)

8.2 Transverse

Define $\delta S_\perp$ as the full spread at half height of within-bunch betatron angular frequencies $Q_y \omega_y$. Then stability requires

$$\delta S_\perp > |\Delta \omega_{\perp mm}|$$

(48)

8.3 Decoupling criterion

A sufficient spread in the frequencies of the individual bunches prevents coupled-bunch motion. To decouple bunches, the r.m.s. spread in individual bunch coherent frequencies should exceed the growth rate. This may arise naturally from a difference $\Delta N_b$ in bunch population for instance, in which case

$$\frac{\Delta N_b}{N_b \text{ r.m.s.}} |\text{Re}(\Delta \omega_{mm})| > |\text{Im}(\Delta \omega_{mm})| \cdot |\varepsilon|$$

(49)

9. HIGH INTENSITY COHERENT OSCILLATIONS, MICROWAVE INSTABILITY AND BUNCH LENGTHENING

For low intensity, the coherent shifts are small when compared to the synchrotron frequency, and modes with different $m$ oscillate independently at different frequencies, $\omega_{mm} + m \omega_y + \Delta \omega_{\perp mm}$. Now, to introduce coupling between modes, let us analyse the situation represented in Fig. 8.

Figure 8

Above the transition energy, two adjacent modes $m$ and $m+4$ overlap the impedance of a resonator. The real frequency shift for mode $m$ $\Delta \omega_{y mm}$ is positive due to the mostly inductive contribution of the impedance. On the contrary, for mode $m+4$ $\Delta \omega_{y m+4 m}$
negative. Therefore the two coherent frequencies of these modes tend to cross when increasing current. Modes \( m \) and \( m' \) oscillate at the same frequency, therefore they can exchange energy.

Coherent frequencies are solutions of the \( 2 \times 2 \) determinant

\[
0 = \begin{vmatrix}
\omega_m - \omega_{m'm'} & \Delta \omega_{m'm'} \\
\Delta \omega_{m'm'} & \omega_m - (m' + 1) \omega_{m'} - \Delta \omega_{m'm'}
\end{vmatrix}
\]

(50)

where \( \Delta \omega_{m'm'} \) gives the effect of mode \( m' \) on mode \( m \). For sinusoidal modes,

\[
\Delta \omega_{m'm'} = \frac{\omega_m}{J \beta^m_m + \omega_m} \left( \sum_{p=1}^{\infty} \frac{2}{p} \frac{Z_{m'}(p)}{Z_{m}(p)} \right)
\]

(51)

with

\[
\omega_m = J \beta^m_m \left( \frac{\omega}{2} - \frac{(m+1) \pi}{2} \right) + (1)^m \left( \frac{\omega}{2} + \frac{(m+1) \pi}{2} \right)
\]

(52)

\[ R_s \] shunt impedance of the resonator
\[ S_{\omega, \omega_{res}} \] is the resonator bandwidth
\[ Q_{res} \] quality factor

The threshold current for instability can be computed for different resonator bandwidths and results are summarized in Fig. 9. The main thing to note is that threshold is near the coasting beam value obtained when using local current and momentum spread in the Keil-Schnell formula.  

\[
E = \frac{Z_0}{2 B^2 \omega_{res}^3 \cos \phi} \frac{R_s}{\omega_{res}}
\]

(53)

Figure 9

Once the threshold current is reached, the bunch lengthens to remain safely under the threshold. Note that this instability does not occur below transition energy.

Obviously the same procedure applies for transverse motion and has been recently extended in order to explain "PETRA instability". Just replace longitudinal by transverse and (51) by

\[
\Delta \omega_{m'm'} = \frac{4}{\pi} \beta L_e \left( \sum_{p=1}^{\infty} \frac{2}{p} \frac{Z_{m'}(p)}{Z_{m}(p)} \right)
\]

(54)
CONCLUSION

I shall close this talk here. My goal would be fully achieved if I had contributed to a better understanding of F.J. Sacherer's theory.

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