OCTOBER 1980 PPPL-1705

UC-20g

MHD DESCRIPTION OF PLASMA

BY

R. M. KULSRUD

MASTER

PLASMA PHYSICS LABORATORY

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-76-CHO 3073.

MHD Description of Plasma

"Handbook of Plasma Physics"

Editors R. N. Sagdeev and M. N. Rosenbluth

Russell M. Kulsrud

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544

ABSTRACT

The basic sets of MHD equations for the description of a plasma in various limits are derived and their usefulness and limits of validity are discussed. These limits are: the one fluid collisional plasma, the two fluid collisional plasma, the Chew-Goldberger Low formulation of the guiding center limit of a collisionless plasma and the double-adiabatic limit. Conservation relations are derived from these sets and the mathematics of the concept of flux freezing is given. An example is given illustrating the differences between guiding center theory and double adiabatic theory.

1. MODES OF DESCRIPTION OF A PLASMA

A plasma is a collection of charged particles. These charged particles generate electromagnetic fields through their elementary charges and currents. In order to evaluate these fields it would be necessary to know the position and velocity of every particle at all times. The motions of tne charges themselves must be followed in the fields they generate and those externally imposed. This program is beyond what is possible except in the simplest possible situations.

Fortunately there is a cruder description of the plasma that is often sufficiently accurate to give gross behavior to the extent desired.

Instead of specifying the plasma in terms of each of its particles one can pursue a more macroscopic description of the plasma in which the emphasis is on its fluid nature. Depending on circumstances that we discuss below this fluid description may be a one fluid, a two fluid, or a many fluid approach.

Let us first consider the one fluid approach. We know that every cubic centimeter of plasma must contain a definite number ρ grams of plasma. The rate of change of this density is controlled by mass flow $\vec{U}$ out of the walls of this cubic centimeter. The momentum $\rho\vec{U}$ in any cubic centimeter is itself controlled by the forces acting on it. These are normally electrical, magnetic, and gravitational forces acting on its volume, and pressure forces acting on its walls. Because the plasma is a conducting fluid its current can be found from Ohm's law in some form, while the direct electrical forces are usually small. The current can be used to find the magnetic field by the Biot-Savert law and the changing magnetic field gives the induced part of the electric field, while the remainder, the electrostatic part, follows from the condition that the current driven by the electric field be divergence free. The determination of the pressure forces is often the weakest part of this one fluid description since the pressure is not usually a scalar, particularly if the plasma is collisionless. In addition the heat flow is often quite large. (Microscopically, particles together in a small cube remain together for only a short time.) However, many plasma phenomena of interest do not depend on the pressure in any essential way so that even an inappropriate treatment by an assumed equation of state for a scalar pressure can give a reasonable description of the phenomena in its grosser aspects. (The more basic properties of the plasma are governed by its electrical nature.)

For a more detailed description of plasmas in which interest is centered on plasma temperatures and energy densities, the two-fluid description is more appropriate. In this description the electron and ion fluids are treated separately. Although the mean velocities are nearly equal the electron and ion temperatures are often quite different due to the weak energy exchange rates between ions and electrons. The two fluid approach is also appropriate for a weakly ionized plasma. Here the ion cyclotron frequency may be less than the ion neutral frequency, while the electron cyclotron frequency is greater than the electron neutral collision frequency. The resulting electron and ion flows can be quite different under these circumstances.

Finally, when the plasma is nearly collisionless but the pressure terms play a central role, an even more detailed, but still approximate, description becomes appropriate, the guiding center description. In this description the magnetic field is strong enough that the plasma is still hydromagnetic in a direction perpendicular to the magnetic field, since the gyration frequency is large for both species. However, the particle flows along the lines need not be fluid-like, so it is necessary to keep track of the distribution of velocities parallel to the line by a one-dimensional kinetic equation. Even in this case the description may be simplified to a fluid description that preserves the independent plasma behavior along and across the lines. Two equations of state for the two independent components of the pressure tensor are needed, and this is supplied by the Chew-Goldberger-Low or double adiabatic equations.

In summary, although any real plasma is extremely complicated, some of its main properties may often be captured by simple macroscopic sets of equations. These can only describe the slower more macroscopic properties of a plasma that occur on long enough time and space scales that microscopic processes such as collision and gyrations can establish sufficient consistency in the plasma to enable it to be considered as a coherent fluid.

2. COLLISIONAL PLASMA

As described in the introduction, the fluid picture of a plasma is most appropriate when the plasma is at least somewhat collisional. Then the electrons and ions separately relax to a local thermodynamic equilibria on a time short compared to that on which substantial changes in plasma condition occur, and in regions sma.l compared to the size of the plasma. Thus, we may assign a density ρ, mean velocity $\vec{U}$, and scalar pressure p to each of the plasma components.

In the simplest description of the one fluid plasma we may ignore the differences in the electron and ion properties and simply lump them together. We consider this description first.

2.1 One fluid description

On this level the plasma is in many ways like a highly conducting molten metal. The fluid equations describing its density, velocity and pressure are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \quad , \tag{1}$$

$$\rho \frac{\partial \vec{U}}{\partial t} + \rho \vec{U} \cdot \nabla \vec{U} = \vec{j} \times \vec{B} - \nabla p + \rho \vec{g} \quad , \tag{2}$$

$$\frac{d}{dt} \left(\frac{p}{\rho^{\gamma}} \right) = 0 \quad . \tag{3}$$

Equation (1) is the equation of continuity. Equation (2) is Euler's equation for fluid motion. The left hand side represents the mass of a cubic centimeter of material times its acceleration at any instant. The acceleration is produced by the magnetic and gravitational forces acting on the same cubic centimeter and the surface force term represented by the pressure gradients. $\vec{B}$ is the magnetic field, $\vec{j}$ the plasma current, and $\vec{g}$ a fixed gravitational field. The pressure is the sum of the separate partial pressures of the ions and electrons whose gradients are assumed to act together on the plasma rather than on each species separately.

In the third equation $d/dt \equiv (\partial/\partial t) + \vec{U}\cdot\nabla$ is the convective derivation and γ is the ratio of specific heats of the plasma. This last equation is the equation of state for each separate fluid element following the motion. It is only valid under conditions where the heat flow is small. Note p/ρ^{γ} is related to the entropy per unit mass of a fluid element. If more general conditions prevail, e.g., ionization, radiation pressure, etc., are important, then Eq. (3) should be replaced by the condition of constant entropy following each fluid element. However, in most cases where the one fluid theory is employed the simple power law assumption is generally adequate. Note further that various limiting cases arise by taking $\gamma = 1$, isothermal, or $\gamma = \infty$ incompressible. It can be easily worded as "p/ρ^{γ} is a constant following the motion, but in general is different for different fluid elements."

It should be noted that we have dropped the electrical force $\rho_E\vec{E}$, where ρ_E is the electrical charge and $\vec{E}$ the electric field, in Eq. (2). This is because, as will soon appear, these forces are relativistically small compared to magnetic forces and must be neglected for consistency, since our theory is nonrelativistic.

We see that knowing $\vec{B}$ and $\vec{g}$, Eqs. (1) - (3) form a complete set giving the forward time evolution of the fluid quantities ρ, $\vec{U}$ and p. The velocity $\vec{U}$ needed in Eq. (1) to advance ρ in time is determined by Eq. (2). The pressure needed in Eq. (2), to advance $\vec{U}$, is given by Eq. (3), etc.

The electromagnetic fields are controlled by Maxwell's equations

$$\nabla \times \vec{B} = 4\pi\vec{j} \quad , \tag{4}$$

$$\frac{\partial\vec{B}}{\partial t} = -c\nabla \times \vec{E} \quad , \tag{5}$$

$$\nabla\cdot\vec{B} = 0 \quad , \tag{6}$$

$$\nabla\cdot\vec{E} = 4\pi\rho_E \quad , \tag{7}$$

where c is the speed of light. We have dropped the displacement current in Eq. (4) since, as will appear, its effects are also relativistically small. Further, we have no need for Eq. (7) since the charge density ρ_E appears nowhere else in the equations.

The electromagnetic and fluid equations are coupled by Ohm's law, which in its simplest form can be written (Spitzer 1962)

$$\vec{E} + \frac{\vec{U} \times \vec{B}}{c} = \eta\vec{j} \quad , \qquad (8)$$

where η is the plasma resistivity. The combination $\vec{E}' = \vec{E} + \vec{U} \times \vec{B}/c$ is the electric field seen by the plasma in its moving frame $\vec{U}$, and Eq. (8) states that in this frame $\vec{j}$ is parallel to and proportional to $\vec{E}'$

Equation (8) is not strictly accurate for a plasma. Because of the anisotropy of the field there will be Hall currents flowing perpendicular to $\vec{E}$ and $\vec{B}$ that may actually be larger than that predicted by Eq. (8). However, the current in Eq. (8) is parallel to $\vec{E}'$ and represents dissipation of energy whereas the Hall currents do not. Thus the secular effects produced by this term are generally more significant than those due to the Hall terms. It is customary in the simplest form of the one fluid MHD equations to employ Ohm's law in the form Eq. (8).

Equations (4), (5), and (8) represent three vector equations for the three vectors $\vec{E}$, $\vec{B}$, and $\vec{j}$. They may be combined into two equations by solving Eq. (8) for $\vec{E}$ and substituting from (4) to eliminate $\vec{j}$. We get

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) - \frac{c}{4\pi} \nabla \times (\eta \nabla \times \vec{B}) \quad . \tag{9}$$

If η is a constant, the last term becomes simply $(\eta c/4\pi)\nabla^2 B$ so

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) + \frac{\eta c}{4\pi} \nabla^2 \vec{B} \quad . \tag{9a}$$

The first term on the right gives the change in magnetic field produced by convection of lines of force by the plasma. The second term gives the magnetic diffusion term, which tends to smooth out irregularities in the plasma perhaps induced by the first term. If there were no plasma motions, the diffuse term would smooth out any irregularities, in a characteristic time of order $4\pi L^2/\eta c$ where L is the irregularity size. (This is essentially the "L/R time" for a plasma considered as a lumped circuit.) This decay time is of order $10^{-7}T^{3/2}L^2$ sec where T is the temperature of the plasma in electron volts. For high temperatures or large plasmas this time may be very long. The changes in $\vec{B}$ produced by the convective term often occur on a time so short compared to this diffusive term that the magnetic diffusion can be ignored altogether. That is we may replace Eq. (9a) by the "infinite conductivity" equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) \quad . \tag{10}$$

The subset of the above Eqs. (1), (2), (3), (4), and (10) constitute the so-called ideal MHD equations. They are clearly an approximation to the true plasma equations, but they have so many nice properties that they are the preferred set for describing macroscopic plasma phenomena. Eq. (10) gives the evolution of $\vec{B}$ as a result of plasma motions. Then making use of Eq. (4) we can determine $\vec{j}$, and thus $\vec{j} \times \vec{B}$ to determine the evolution of the fluid quantities under the action of the electromagnetic forces.

We no longer need the electric field $\vec{E}$ in this description but it may be obtained from the infinite conductivity limit of Ohm's law

$$\vec{E} + \frac{\vec{U} \times \vec{B}}{c} = 0 \quad . \tag{11}$$

Then the electric force on the plasma $\rho_E \vec{E}$ can be estimated from Eq. (7) to be

$$\rho_E \vec{E} = \frac{\vec{E}\nabla \cdot \vec{E}}{4\pi} \approx \frac{U^2}{4\pi L c^2} B^2 \quad ,$$

and it is seen, as mentioned earlier, that it is relativistically small compared to the magnetic force $\vec{j} \times \vec{B} \approx B^2/4\pi L$. In the same way we may show that inclusion of the displacement current $(1/c)(\partial\vec{E}/\partial t)$ has a relativistically small effect on the equations. Adding it to Eq. (4) will alter $\vec{j}$ by the small amount $\delta\vec{j}$ and this will produce an additional contribution to the electromagnetic force term in Eq. (2)

$$\delta\vec{j} \times \vec{B} = \frac{1}{4\pi c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} = - \frac{\partial}{\partial t} \left[\frac{\vec{U} \times \vec{B}}{4\pi c^2}\right] \times \vec{B} \approx \frac{UB^2}{4\pi t c^2} \quad ,$$

where t is a macroscopic time. Comparing this with the inertia term on the left we see that it is smaller by $B^2/4\pi\rho\, c^2$. In fact,the addition of this term can be thought of as adding the "mass" of the magnetic field to the mass of the plasma.

The ideal equations of MHD are best thought of as exactly describing an ideal infinitely conducting fluid with an adiabatic equation of state whose properties are sufficiently close to a plasma to be of interest, rather than an appropriate system of equations for a real plasma. For he moment imagine that we have such an ideal infinitely conducting fluid to study. It is immersed in some magnetic field. Then by the condition of flux freezing the evolution of the field may be expressed in terms of the distribution of magnetic lines of force bodily transmitted by the velocity $\vec{U}$. This means the field only depends on the net displacement of each element of the fluid and not on the history of the fluid displacements. The $\vec{j} \times \vec{B}$ force can readily be thought of as the magnetic tension and pressure contained in these lines of force. Similarly, ρ is given purely by the displacement of the fluid elements and further the pressure is also thus determined. This means that at least in principle the force on a fluid element is determined holonomically by its displacement and the displacement of its neighbors. It is this fact, plus the fact that the system is dynamical (given by a Lagrangian) that leads to the many very satisfying properties of this ideal system. In fact a considerable amount of macroscopic plasma physics is devoted to determining to what extent a real plasma can differ from its ideal counterpart. Some of these questions, magnetic reconnection for example, are among the most important of modern day research problems (Petschek 1964).

2.2 The two fluid description

An alternative and more precise treatment of a fully ionized plasma is contained in the two-fluid description. The two fluids are the electrons and ions. If there is a single species of ions, we can assign a density, velocity and pressure to the electrons and to the ions. Then the three equations for a single fluid, Eqs. (1) - (3), must be replaced by six equations, three for each fluid, describing the six independent quantities $\rho_i, \rho_e, \vec{U}_i, \vec{U}_e, p_i, p_e$. Now the one fluid equations were written down on phenomonological grounds and were not extremely accurate except in the limit $\omega_{ce}\,\tau_e$ very small where ω_{ce} is the electron cyclotron frequency and τ_e the electron collision frequency. On the other hand considerable work has been devoted to deriving a set of equations accurate for any collision rate faster than the dynamic rates of change of ρ_i, ρ_e etc. The generally accepted set of equations are those of Braginski (1965), that are now taken as standard. We give them here for reference.

The two continuity equations are

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{U}_i) = 0 \quad , \tag{12}$$

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\vec{U}_e) = 0 \quad , \tag{13}$$

where n_i and n_e are the electron and ion particle densities. These equations are linked by the charge neutrality condition, $Zn_i = n_e$, where Z is the ion charge number.

The two vector equations of motion are

$$\rho_i\left(\frac{\partial \vec{U}_i}{\partial t} + \vec{U}_i\cdot\nabla\vec{U}_i\right) = -\nabla p_i - \nabla\cdot\underline{\underline{\pi}}_i + Zen_i\left(\vec{E} + \frac{\vec{U}_i \times \vec{B}}{c}\right) , \qquad (14)$$

$$- \vec{R}_{ei} + \rho_i\vec{g}$$

$$\rho_e\left(\frac{\partial \vec{U}_e}{\partial t} + \vec{U}_e\cdot\nabla\vec{U}_e\right) = -\nabla p_e - \nabla\cdot\underline{\underline{\pi}}_e - n_e e\left(\vec{E} + \frac{\vec{U}_e \times \vec{B}}{c}\right) , \qquad (15)$$

$$+ \vec{R}_{ei} + \rho_e\vec{g} \quad .$$

In these equations p_i and p_e are the ion and electron scalar pressures, $\underline{\underline{\pi}}_i$ and $\underline{\underline{\pi}}_e$ are the nonscalar parts of the stress tensors, $\vec{R}_{ei}$ is the rate of transfer of momentum from ions to electrons by collisions. They in turn are linked by the equation defining the current $\vec{j} = (Zn_i e/c)(\vec{U}_i - \vec{U}_e)$, where e is the electronic charge. We assume that Zn_i is much closer to n_e than $\vec{U}_i$ is to $\vec{U}_e$. Because, $\vec{j}$ cannot be too large without producing electromagnetic effects we can say that $\vec{U}_i$ and $\vec{U}_e$ are also close together.

The two energy equations are:

$$\frac{3}{2}\, n_i\left(\frac{\partial T_i}{\partial t} + \vec{U}_i\cdot\nabla T_i\right) + p_i\nabla\cdot\vec{U}_i = -\nabla\cdot\vec{q}_i - \underline{\underline{\pi}}_i:\nabla\vec{U}_i + Q_i , \qquad (16)$$

$$\frac{3}{2}\, n_e\left(\frac{\partial T_e}{\partial t} + \vec{U}_e\cdot\nabla T_e\right) + p_e\nabla\cdot\vec{U}_e = -\nabla\cdot\vec{q}_e - \underline{\underline{\pi}}_e:\nabla\vec{U}_e + Q_e , \qquad (17)$$

where the temperatures are defined by $p_i = n_i T_i$, $p_e = n_e T_e$ and the units of T are chosen to make Boltzman's constant unity. The second term on the left of each equation is the p dV work done by compression. $\vec{q}_i$ and $\vec{q}_e$ are the heat flows, $\underline{\underline{\pi}}_i:\nabla\vec{U}_i$ and $\underline{\underline{\pi}}_e:\nabla\vec{U}_e$ are the frictional heating terms due to nonuniform velocities while Q_i and Q_e represent energy exchange between the species and joule heating.

Equations (14) - (17) become more accurate as the collision time τ goes to zero. They consist of "fluid" terms and dissipative terms and the latter are smaller than the former roughly by τ/t. Thus, if τ were zero, collisions would be sufficient to maintain an isotropic velocity distribution in the frame moving with the fluid and the π terms would be small. However, because $\vec{U}$ is inhomogeneous, an isotropic distribution at one point does not match the isotropic distribution a mean free path away, and a certain mixing of these distributions leads to anisotropy of the distribution and to off diagonal terms in the stress tensor. The other dissipative term $\vec{R}_{ei}$ is produced by unlike particle collisions and is the friction force between electrons and ions. Since the difference between the electron and ion velocities is the current, this friction includes the resistivity as well as thermoelectric effects. In most cases in practice $\vec{U}_i$ is close to $\vec{U}_e$ and can be identified with the mass flow of the plasma. If Eq. (14) is added to Eq. (15), the electron ion friction force cancels out and the electron inertial term and gravitational terms are negligible. Thus, except for the viscosity terms $\underline{\underline{\pi}}_i$ and $\underline{\underline{\pi}}_e$, we recover the one fluid equation of motion, Eq. (2). On the other hand, if we express $\vec{U}_e$ in terms of $\vec{U}_i$ and $\vec{j}$ by solving

$$\vec{j} = n_i Z_e(\vec{U}_i - \vec{U}_e) \quad , \tag{18}$$

we obtain a form of Ohm's law usually denoted as the generalized Ohm's law (Spitzer 1962)

$$\vec{E} + \frac{\vec{U} \times \vec{B}}{c} = \frac{c}{n_e e}\vec{j} \times \vec{B} - \frac{\nabla p_e}{n_e e} - \frac{\nabla \cdot \underline{\underline{\pi}}_e}{n_e e} + \frac{\vec{R}_{ei}}{n_e e} \quad . \tag{19}$$

Equations (12) - (17) are the equations describing the electron and ion fluids separately. To complete them, we must add Maxwell's equations from Section 2.1, Eqs. (4) - (6), where $\vec{j}$ is defined by Eq. (18). Again, we may consistently neglect the displacement current term in Eq. (4) and take $Zn_i = n_e$ so Eq. (13) is not needed. (This is the case for low frequency phenomenon. Although it is the case that the two fluid equations may be used to derive some high frequency wave phenomena provided thermal effects are small, these derivations are not really sound.) We also need the expressions for the various dissipation terms. These are given in Braginski's article (1965). Let the ion and electron collision times be defined as

$$\tau_e = \frac{3\sqrt{m_i}\, T_i^{3/2}}{4\sqrt{\pi}\, \log \Lambda\, e^4 Z^2 n_i} \quad , \tag{20a}$$

$$\tau_e = \frac{3\sqrt{m_e}\; T_e^{3/2}}{4\sqrt{2\pi}\, \log\Lambda\; e^4 Z n_e} \quad , \tag{20b}$$

where $\log\Lambda$ is the Coulomb logarithm and $m_{i,e}$ the particle masses. Let us further limit ourselves to the case $Z = 1$ and to the limit $\omega_{cs}\tau_s >> 1$, where s indicates the particle species, i or e. Then from Braginski's article we have

$$\begin{aligned} \underline{\underline{\pi}}_s = {} & \eta_s^0 (\vec{b}\cdot\nabla\vec{U}_s\cdot\vec{b} - \tfrac{1}{3}\nabla\cdot\vec{U}_s)(\underline{\underline{I}}_\perp - 2\underline{\underline{bb}}) \\ & + \eta_s^1 (\vec{b}\cdot\nabla\vec{U}_s\cdot\vec{b} + \tfrac{1}{3}\nabla\cdot\vec{U}_s)\, \underline{\underline{I}}_\perp \\ & + (\eta_s^1 \underline{\underline{I}}_\perp - \eta_s^2 \vec{b}\vec{b})\cdot\nabla\vec{U}_s\cdot\underline{\underline{I}}_\perp + \underline{\underline{I}}_\perp\cdot\nabla\vec{U}_s\cdot(\eta_s^1\underline{\underline{I}}_\perp - \eta_2^s \vec{b}\vec{b}) \\ & + (\vec{b}\times\nabla\vec{U}_s)\cdot(\eta_s^3\underline{\underline{I}}_\perp - \eta_s^4\vec{b}\vec{b}) - (\eta_s\underline{\underline{I}}_\perp - \eta_s^4\vec{b}\vec{b})\cdot(\vec{b}\times\nabla\vec{U}_s) \\ & + \frac{\eta_s^3}{3}\nabla\cdot\vec{U}_s(\vec{b}\times\underline{\underline{I}}_\perp - \underline{\underline{I}}_\perp\times\vec{b}) \quad , \end{aligned} \tag{21}$$

where

$$I_\perp \equiv \underline{\underline{I}} - \underline{\underline{bb}} \; , \qquad \eta_i^0 = 0.96\, n_i T_i \tau_i \; , \qquad \eta_e^0 = 0.73\, n_e T_e \tau_e \; ,$$

$$\eta_i^1 = 0.3\, n_i T_i/\omega_{ci}^2\tau_i \; , \; \eta_i^e = 0.51\, n_e T_e/\omega_{ce}^2\tau_e \; , \; \eta_s^2 = 4\eta_s^1 \quad , \tag{22}$$

$$\eta_i^3 = 0.5\, n_i T_i/\omega_{ci}\tau_i \; , \; \eta_e^3 = -0.5\, n_e T_e/\omega_{ce} \; , \quad \eta_s^4 = 2\eta_s^3 \; .$$

For $\vec{R}_{ie}$ we have

$$\vec{R}_{ie} = en_e \left(\frac{\vec{j}\cdot\vec{b}}{\sigma_{\parallel}} + \frac{\vec{j}_{\perp}}{\sigma_{\perp}} \right) - 0.71\ n_e \vec{b}\cdot\nabla T_e - \frac{3}{2} \frac{n_e}{\omega_{ce}\tau_e} (\vec{b} \times \nabla T_e), \quad (23)$$

where $\sigma_{\perp} = e^2 n_e \tau_e / m_e$, $\sigma_{\parallel} = 1.96\ \sigma_{\perp}$, and the last two terms of Eq. (23) represent thermal forces.

The heat flow terms $\vec{q}_s$ are given by

$$\vec{q}_s = -K_{s\parallel} \vec{b}\cdot\nabla T_s - K_{s\perp} \underline{\underline{I}}_{\perp}\cdot\nabla T_s + \frac{5}{2} \frac{n_s T_s}{\omega_{cs} m_s} \vec{b} \times \nabla T$$

$$+ \left(0.71\ n_e T_e (\vec{U}_i - \vec{U}_e) + \frac{3}{2} \frac{n_e T}{\omega_{ce}\tau_e} \vec{b} \times (\vec{U}_i - \vec{U}_e) \right) \delta_{es} \quad , \quad (24)$$

where

$$K_{e\parallel} = 3.16\ n_e T_e\ {}_e/m_e \ , \quad K_{i\parallel} = 3.9\ n_i T_i \tau_i / m_i \ ,$$

$$K_{e\perp} = 4.66\ n_e T_e / m_e\ \omega_{ce}^2\ \tau_e \ , \quad K_{i\perp} = 2 n_i T_i / m_i \omega_{ei}^2 \tau_i \ , \quad (25)$$

and the factor multiplying the bracket indicates that this term (the thermoelectric term) is present only for $\vec{q}_e$.

The internal heating terms Q are given by

$$Q_e = -\vec{R}_{ei}\cdot(\vec{U}_i - \vec{U}_e) - Q_{\Delta} \ , \quad (26)$$

where the first term is the joule heating term and the second

$$Q_i = Q_\Delta = 3 \frac{m_e}{m_i} \frac{n_e}{\tau_e} (T_e - T_i) \quad , \tag{27}$$

the energy exchange term.

Equations (12) - (17) are a complete set of equations for the plasma quantities $n_i = n_e$, $\vec{U}_i$, $\vec{U}_e$, p_i, and p_e, all the quantities on the right being defined in terms of them. They allow a much richer set of plasma phenomena to be described than the one fluid equations, particularly in the allowance for different electron and temperatures and the inclusion of non ideal effects such as thermal conductivity, viscosity and resistivity and thermoelectric effects. Thus, they are more useful for describing long term phenomena in which nonideal effects play a significant role. It is possible to include such nonideal terms in the one fluid equation. However, because ions and electrons transport play different roles and because the temperature sensitivity of these is important, the modified one fluid approach is usually highly inaccurate and misleading. Thus, one could possibly distinguish between the usefulness of the one fluid and two fluid approaches as follows. The one fluid approach is preferable for short time hydrodynamic effects in which nonideal effects play a minor role. Its great advantage is that its equations are considerably simpler to handle than the two fluid approach. Finally, it can be used in longer time problems to get an idea of at least some of the plasma behavior.

The two fluid equations are more accurate and necessary for any precision in the discussion of phenomena where plasma transport or dissipation is involved. They are too complex to solve, however, for any problems except those with simple geometries. They, of course, can be used to form a good idea as to the accuracy of calculations based on the one fluid approach.

3. COLLISIONLESS PLASMA

In Section 2 we discussed plasmas in which the collision time was the shortest time in the problem with the possible exception of the gyration period. Thus, a small element of mass of a plasma will relax quickly to a Maxwellian before it can change its oroperties, and a local description in terms of the parameters characterizing this Maxwellian is appropriate. This consistency justifies a fluid description. But in many important plasmas the collision time is so long that one should ignore collisions. It would appear that for such "collisionless" plasmas a fluid theory is not appropriate. However, even for weak magnetic fields, the cyclotron period is still shorter than any macroscopic period, and the plasma does have a two-dimensional consistency perpendicular to the magnetic field. This restores the possibility of a fluid theory to a limited extent and is the basis for the guiding center description of a plasma.

3.1 The guiding center limit of the Vlasov equation

A collisionless plasma is completely described by giving its velocity distribution functions f_s [$f_s(t, \vec{r}, \vec{v}) d^3r d^3x$ is the number of particles in an element $d^3r d^3v$ at position $\vec{r}$ and velocity $\vec{v}$ at time t]. Its time behavior is governed by the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f + \frac{e_s}{m_s}\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right)\cdot\nabla_{\vec{v}} f_s = 0. \quad , \tag{28}$$

where $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are the mean electric and magnetic fields produced by the smoothed out plasma distributions f_s

$$\nabla \times \vec{B} = 4\pi \sum_s \frac{e_s}{c} \int f_s \vec{v}\, d^3v + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad , \tag{29a}$$

$$\nabla\cdot\vec{E} = 4\pi \sum e_s \int f_s\, d^3v \quad , \tag{29b}$$

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E} \quad , \tag{29c}$$

$$\nabla\cdot\vec{B} = 0 \quad . \tag{29d}$$

These equations are more complicated than the fluid equations because they involve seven independent variables t, $\vec{r}$, $\vec{v}$ rather than four, t, $\vec{r}$. However, by an asymptotic expansion in the smallness of the gyration radiation $\rho = mcv/eB$ compared to the scale size of the plasma the effective number of variables in the kinetic equation can be reduced by two, because the gyration phase variable is irrelevant and the scalar perpendicular velocity is controlled by a constant of the motion, the adiabatic invariant (Chew, Goldberger, and Low 1955; Kulsrud 1962).

Further, we know that to lowest order, the motion of the particles consists of an $\vec{E} \times \vec{B}$ velocity perpendicular to the magnetic field common to all particles, regardless of their peculiar velocities or species, and a parallel motion along the field. If the parallel electric field $E_{\parallel} = \vec{b}\cdot\vec{E}$, where $\vec{b} \equiv \vec{B}/B$, is small [cf., the discussion after Eq. (34)], it is well known that the magnetic lines of force can be assigned the same $\vec{E} \times \vec{B}$ velocity perpendicular to themselves (Newcomb 1958). Thus, all particles will stay on the same line and it should be possible to concentrate our attention on a single line and derive a kinetic equation involving only two particle variables, position along the line and parallel velocity.

To derive the equations for this reduced system we may carry out a formal expansion in the quantity m/e (Kruskal 1960). (If we regard macroscopic lengths and times to be fixed, then the small gyration radius limit is reached by taking a sequence of fictitious charged particles with different atomic properties m/e approaching zero. In this imagined series of experiments one expects results to be near their asymptotic value when the true values of m/e are reached, if the ratio of gyration radius to scale size is sufficiently small.) In point of fact, it turns out to be slightly more convenient to expand all quantities $\vec{E}$, $\vec{B}$, f in just the reciprocal charge, the quantity 1/e (Rosenbluth and Rostoker 1958).

Consider first the Vlasov Eq. (28) and set $f = f_c + f_1$ where $f_1 = O(1/e)$ etc. From this point on we drop the subscript s when no confusion results. Then to lowest order

$$\left(\vec{E} + \frac{\vec{v} \times B}{c}\right) \cdot \nabla_{\vec{v}} f_o = 0 \quad . \tag{30}$$

Introduce the $\vec{E} \times \vec{B}$ velocity

$$\vec{U}_E = c \, \frac{\vec{E} \times \vec{B}}{B^2} \quad , \tag{31}$$

and set $\vec{v} = \vec{v}' + \vec{U}_E$. Equation (28) then becomes

$$\frac{\vec{v}' \times \vec{B}}{c} \cdot \nabla_{v'} f_o + E_{||} \vec{b} \cdot \nabla f_o = 0 \quad . \tag{32}$$

Next introduce cylindrical coordinates $v_\perp$, ϕ and $v_{||}$ in $\vec{v}'$ space, by

$$\vec{v}' = \vec{\hat{x}} v_\perp \cos\phi + \vec{\hat{y}} v_\perp \sin\phi + \vec{\hat{z}} v_{||} \quad . \tag{33}$$

Then Eq. (32) becomes

$$- \frac{B}{c} \frac{\partial f_o}{\partial \phi} + E_{||} \frac{\partial f_o}{\partial v_{||}} = 0 \quad . \tag{34}$$

If $E_{||} \neq 0$, then Eq. (34) implies f_o is constant along a helix in velocity space extending to infinite velocities, which is unphysical. Therefore, Eq. (30) has reasonable solutions only if $E_{||}$ is expanded in $1/e$ also. That is $E_{||} = O(1/e)E$. (If this were not the case, the greatly more effective $E_{\|}$ would accelerate particles on a cyclotron period time scale until $E_{\|}$ is shorted out to the lowest order.) The resulting greatly reduced $E_{\|}$ can then produce a force comparable with the other forces. [See Eq. (19)]. It is simpler not to expand $\vec{E}$ and $\vec{B}$ further, but simply to regard $E_{\|}$ as smaller by one power of e.

Dropping the $E_{\|}$ term in Eq. (24), we see that the lowest order the Vlasov equation says that f_o is independent of ϕ , but gives no further information on its dependence on t, $\vec{r}$ $v_\perp$ and $v_{\|}$. Proceeding to first order we have

$$\frac{\partial f_o}{\partial t} + \vec{v}\cdot\nabla f_o + \frac{e}{m} (\vec{E}_o + \vec{v} \times \vec{B})\cdot\nabla_{\vec{v}} f_1 + \frac{e}{m} E_{\|} \frac{\partial f_o}{\partial v_{\|}} = 0 \quad . \tag{35}$$

Transforming to the cylindrical variables $v_\perp$,$v_{\|}$, yields

$$\frac{eB}{mc} \frac{\partial f_1}{\partial \phi} = \left(\frac{\partial f_o}{\partial t} + \vec{v}\cdot\nabla f_o\right) + \frac{e}{m} E_{\|} \frac{\partial f_o}{\partial v_{\|}} \quad . \tag{36}$$

(The terms in parenthesis are not yet so transformed but they must be.) This transformation is somewhat complex since at fixed $\vec{v}$, $v_\perp$, and $v_{\|}$ are dependent on $\vec{r}$ and t, because $\vec{B}$ and $\vec{U}_E$ are through Eq. (31). It is easy to see that actually the transformation of the quantities in parenthesis leads to a series of terms that are sines and cosines in ϕ. Once this transformation is accomplished it is easy to solve Eq. (36) for f_1. However, any constant term leads to an f_1 linear in ϕ and therefore not periodic with period 2π. Thus, in order to have a proper solution for f_1 a necessary and sufficient condition is that the average of the right-hand side of Eq. (36) vanish. Imagine the right hand side transformed to $v_\perp$, $v_{||}$ variables and averaged over ϕ. The details of this calculation are straightforward and the result is that Eq. (36) can be solved for f_1, if and only if

$$\frac{\partial f_o}{\partial t} + (\vec{U}_E + v_{||}\vec{b})\cdot\nabla f_o - \frac{v_\perp}{2}\left(\nabla\cdot\vec{U}_E - \vec{b}\cdot\nabla\vec{U}_E\cdot\vec{b} + v_{||}\nabla\cdot\vec{b}\right)\frac{\partial f_o}{\partial v_\perp} ,$$

$$+ \left(-\vec{b}\cdot\frac{D\vec{U}_E}{Dt} + \frac{v_\perp^2}{2}(\nabla\cdot\vec{b}) + \frac{e}{m}E_{||}\right)\frac{\partial f_o}{\partial v_{||}} = 0 , \tag{37}$$

where $D\vec{U}_E/Dt \equiv \partial\vec{U}_E/\partial t + (\vec{U}_E + \vec{b}v_{||})\ \nabla\vec{U}_E$. This condition thus gives the time evolution of f_o. Strictly speaking we should go ahead and solve for f_1 once we are assured by Eq. (37) that this can be done. But it will appear shortly that we do not need f_1 for a lowest order description of a guiding center plasma.

To complete the system we must add the equations for $\vec{E}$ and $\vec{B}$, Maxwell's Eqs. (29a) - (29d). They involve f so that they also must be expanded in our small "parameter" 1/e. To lowest order we have

$$0 = 4\pi \sum_s \frac{e_s}{c} \int f_{so}\vec{v}\, d^3v , \tag{38a}$$

$$0 = 4\pi \sum e_s \int f_{so}\, d^3v . \tag{38b}$$

Equation (38b) is the charge neutrality condition which states that to lowest order in 1/e the total charges of each species must be equal. For a Z = 1 ion species this reduces to equality of the species densities. (Any finite charge density is produced by first order differences in charge density because of the factor 1/e.) Similarly Eq. (38a) is the current neutrality condition. If we transform the velocity integration to cylindrical coordinates, we get for Eq. (38a)

$$0 = 4\pi \sum \frac{e_s n_{so}}{c} \vec{U}_E + 4\pi \sum \frac{e_s}{c} \int f_o v_{\parallel} 2\pi v_{\perp} dv_{\perp} dv_{\parallel} ,$$

and the first term vanishes by virtue of Eq. (38b) so we have

$$0 = \sum \vec{j}_{s-1} \cdot \vec{b} = \sum \frac{e_s}{c} \int f_o v_{\parallel} d^3v . \tag{39}$$

[Equations (38b) and (39) are related by the continuity equation derivable from Eq. (37) or even from Eq. (28)],

$$\sum e_s \left(\frac{\partial n_{os}}{\partial t} + \vec{B} \cdot \nabla \frac{n_{os} (\vec{U}_s \cdot \vec{b})}{B} \right) \tag{40}$$

so that if Eq. (39) is satisfied at some initial time t, and Eq. (38b) is satisfied (and the other guiding center equations are satisfied), then Eq. (39) will be satisfied for all t. Alternatively, if the charge neutrality condition is satisfied and Eq. (39) is satisfied at one point on each line at every time it will be satisfied everywhere.

Equations (38b) and (39) are extra conditions imposed on f_o and do not serve to advance $\vec{E}$ and $\vec{B}$ in time. These conditions are essentially thought to be a control on the magnitude of $E_{\parallel}$, which is usually chosen to ensure that they are satisfied. To complete our equations we must include Eqs. (29c) and (29d) and proceed to one higher order in the expansion of Eqs. (29a) and (29b). Thus, Eqs. (29a) and (29b) become

$$\nabla \times \vec{B} = 4\pi \sum \frac{e_s}{c} \int \vec{v} f_{1s}\, d^3v + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad , \tag{41a}$$

$$\nabla \cdot \vec{E} = 4\pi \sum e_s \int f_{1s}\, d^3v \quad . \tag{41b}$$

It would appear that it is necessary to evaluate f_1 from Eq. (36) after all. However, full information on the dependence of f_1 is not needed. Transformation of Eq. (41a) to cylindrical coordinates shows we only need $\int f_1\, d\phi$, $\int f_1 \sin\phi\, d\phi$, and $\int f_1 \cos\phi\, d\phi$. These may be obtained by multiplying Eq. (36) by 1, $\sin\phi$ and $\cos\phi$ and integrating over ϕ. An equivalent set of moments can be carried out on the exact Vlasov Eq. (28) and passing to the zeroth order limit. But these are simply the MHD equations of sections (1) and (2). Thus, $\vec{j}$ to zeroth order is determined by

$$\sum n_s m_s \left(\frac{\partial \vec{U}_s}{\partial t} + \vec{U}_s \cdot \nabla \vec{U}_s \right) = -\nabla \cdot \underline{\underline{P}} + \vec{j} \times \vec{B} \quad , \tag{42}$$

where the mass velocity $\vec{U}_s$ and the stress tensor P are defined by

$$n_s\vec{U}_s = \int f_s\vec{v}\, d^3v \quad , \tag{43}$$

$$\underline{\underline{P}} = \sum_s m_s \int f_s(\vec{v} - \vec{U}_s)(\vec{v} - \vec{U}_s) \quad .$$

Note that the component of $\vec{U}_S$ perpendicular to $\vec{b}$ is $\vec{U}_E$, while by Eq. (39) the parallel mass velocities are the same for both species. Thus $\vec{U} = \vec{U}_s$. On transforming to cylindrical coordinates the stress tensor may be written

$$\underline{\underline{P}} = p_\perp(\underline{\underline{I}} - \vec{b}\vec{b}) + p_{\parallel}\,\vec{b}\vec{b} \quad , \tag{44a}$$

where $\underline{\underline{I}}$ is the unit dyadic and

$$p_\perp = \sum m_s \int f_s \frac{v_\perp^2}{2}\, d^3v \quad , \tag{44b}$$

$$p_{\parallel} = \sum m_s \int f_s(v_{\parallel} - \vec{U}\cdot\vec{b})^2\, d^3v \quad . \tag{44c}$$

As advertised, Eq. (42) determines the part of $\vec{j}$ perpendicular to b. The parallel part of $\vec{j}$ is a different moment of f_1 but can also be found from Maxwell's equations. We may continue this scheme but it is more efficacious at this point to change the emphasis from $\vec{E}$ to $\vec{U}$, regarding $\vec{U}$ as the primary variable and $\vec{E}$ as a secondary variable;

$$\vec{E} = -\frac{\vec{U} \times \vec{B}}{c} \quad , \tag{45}$$

from Eq. (31). This is particularly true since $\vec{E}$ is restricted to be perpendicular to $\vec{b}$, while $\vec{U}$ is not and determines $\vec{E}$ automatically to satisfy this condition.

Solving Eq. (29a) for $\vec{j}_o$, substituting into Eq. (42) and making use of Eq. (45) we have

$$\rho\left(\frac{\partial\vec{U}}{\partial t} + \vec{U}\cdot\nabla\vec{U}\right) = -\nabla\cdot\underline{\underline{P}} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi} + \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{U} \times \vec{B}) \times \vec{B} + \frac{(\vec{U} \times \vec{B})\nabla\cdot(\vec{U} \times \vec{B})}{c^2} , \tag{46}$$

where $\rho = \sum n_s m_s$. Then substituting Eq. (45) into Eq. (29c) we have

$$\frac{\partial\vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) . \tag{47}$$

Equations (45) and (47) are nearly self-contained except we need f_{os} to compute and $\underline{\underline{P}}$. ρ is given by the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{U}) = 0 , \tag{48}$$

but we cannot obtain $\underline{\underline{P}}$ in any other way than from f_o. Thus, the equation determining f_o and thus $\underline{\underline{P}}$, Eq. (37), may be considered to determine the "equation of state" of the plasma. Finally, inspection of Eq. (37), shows it brings in $E_{\parallel}$, that must be determined by the charge neutrality condition Eq. (38a) or alternatively the parallel current condition of Eq. (39). It is possible by combining the separate moment equations to show that

$$E_{\parallel} = \frac{\sum (e_s/m_s)\vec{b}\cdot\nabla\cdot\underline{\underline{P}}_s}{\sum (n_s e_s^2/m_s)} \quad . \tag{49}$$

However, this is a little misleading since Eq. (49) arises from the second term derivative of the charge neutrality condition Eq. (38) and in fact if one seeks equilibria, $E_{\parallel}$ actually drops out of Eq. (49).

Our complete system of guiding center equations are Eqs. (45) - (48) with $\underline{\underline{P}}$ defined by Eqs. (44a) - (44c) and f_o and $E_{\parallel}$ determined by Eqs. (37) and (38a). Again as in the one fluid theory we see that the last two terms of Eq. (45) may be dropped as relativistically small. The system then reduces to that of a one fluid description with the main complication occurring through the equation of state. This complication can only be removed by solving an apparently five-dimensional equation for f_o. However, these five dimensions t, $\vec{r}$, $v_\perp$, $v_{\parallel}$ can be reduced to four by replacing $v_\perp$ by the new variable

$$\mu \equiv v_\perp^2/2B \quad , \tag{50}$$

equal to the magnetic moment of the particle. Equation (37) then reduces to

$$\frac{\partial f_o}{\partial t} + (\vec{U}_E + v_{\parallel}\vec{b})\cdot\nabla f_o + \left(-\vec{b}\cdot\frac{D\vec{U}_E}{Dt} + \mu B\nabla\cdot b + \frac{e}{m} E_{\parallel}\right)\frac{\partial f_o}{\partial v_{\parallel}} = 0 \tag{51}$$

and μ does not enter into any derivative. It occurs merely as a parameter in Eq. (52) and $v_{||}$ is the only real variable in addition to $\vec{r}$ and t. Note

$$\vec{U}_E = \vec{U}_\perp \equiv \vec{U} - \vec{b}\vec{b}\cdot\vec{U} \quad . \tag{52}$$

The guiding center theory demonstrates how in the absence of collisions the magnetic field acts to give the plasma almost enough consistency for a hydrodynamic description. It interferes strongly with motions across itself forcing all particles to move together so that all particles in one tube of force stay in that one tube of force.

Equation (51) may be reduced by two more dimensions in line with our remarks at the beginning of this section. To do this we make use of the Clebsch form for any divergence free field as shown in Section 4.2, for any vector field $\vec{B}$ such that $\nabla\cdot\vec{B} = 0$ one can find two scalars α and β such that

$$\vec{B} = \nabla\alpha \times \nabla\beta \quad , \tag{53}$$

α and β are not uniquely determined, but if they once give $\vec{B}$ at some initial time t_o, they will continue to represent $\vec{B}$ by Eq. (53) for all time, provided they satisfy

$$\frac{\partial\alpha}{\partial t} + \vec{U}\cdot\nabla\alpha = 0 \ ; \quad \frac{\partial\beta}{\partial t} + \vec{U}\cdot\nabla\beta = 0 \quad , \tag{54}$$

or, in other words, provided they are "frozen" in the fluid. Since α and β are flux labels, a line of force is always given by

α = const, β = const. This result is a precise mathematical expression of the fact that lines of force are frozen in a plasma. If we replace the general position variable $\vec{r}$ by new coordinates α, β, and ℓ, a parameter characterizing position along a line of force, then Eq. (52) can be reduced to a "one dimensional" kinetic equation by transforming to these variables α, β, ℓ, μ, $v_{||}$, ϕ. It becomes

$$\frac{\partial f_o}{\partial t} + v_{||}\left(\frac{\partial \ell}{\partial s}\right)\frac{\partial f_o}{\partial \ell} + \left(-\vec{b}\cdot\frac{D\vec{U}_E}{Dt} + \mu B\nabla\cdot\vec{b} + \frac{eE_{||}}{m}\right)\frac{\partial f_o}{\partial v_{||}} = 0 \quad , \tag{55}$$

provided only that ℓ satisfies $(\partial\ell/\partial t + \vec{U}_E\cdot\nabla\ell) = 0$.

For completeness we collect together the full systems of guiding center equations for the fundamental variables ρ, $\vec{U}$, $\vec{B}$, f_o, and $E_{||}$.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{U}) = 0 \quad , \tag{48}$$

$$\rho\left(\frac{\partial U}{\partial t} + \vec{U}\cdot\nabla\vec{U}\right) = \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi} - \nabla\cdot\underline{\underline{P}} \quad , \tag{46}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) \quad , \tag{47}$$

$$\underline{\underline{p}} = p_\perp \underline{\underline{I}} + (p_{\parallel} - p_\perp)\vec{b}\vec{b} \quad , \tag{44a}$$

$$p_\perp = \sum_s m_s \int f_{os} \frac{v_\perp^2}{2} d^3v \; ; \; p_{\parallel} = \sum_s m_s \int f_{os}(v_{\parallel} - \vec{U}\cdot\vec{b})^2 \, d^3v$$

$$\frac{\partial f_{os}}{\partial t} + (\vec{U}_E + v_{\parallel}\vec{b})\cdot\nabla f_{os} - v_\perp(\nabla\cdot\vec{U}_\perp - \vec{b}\cdot\nabla\vec{U}\cdot\vec{b} + v_{\parallel}\nabla\cdot b)\frac{\partial f_{os}}{\partial v_\perp}$$

$$+ \left(-\vec{b}\cdot\frac{D\vec{U}_E}{Dt} + \frac{v_\perp^2}{2}\nabla\cdot\vec{b} + \frac{e}{m}E_{\parallel}\right)\frac{\partial f_{os}}{\partial v_{\parallel}} = 0 \tag{37}$$

$$\sum e_s \int f_{os} \, d^3v = 0 \; . \tag{39a}$$

3.3 The double adiabatic theory

As remarked in Section 3.2, a collisionless plasma is subject to description by fluid equations with the single difficulty involving the determination of the evolution of the two pressure components $p_\perp$ and $p_{\parallel}$. Chew, Goldberger, and Low (1956) showed that these quantities themselves can be expressed in terms of two equations of state

$$\frac{d}{dt}\left(\frac{p_\perp}{\rho B}\right) = 0 \quad , \tag{56a}$$

$$\frac{d}{dt}\left(\frac{p_{\parallel}B^2}{\rho 3}\right) = 0 \quad , \tag{56b}$$

which apply under the same restrictions as the adiabatic theory of Section 3.2 but with an important additional restriction. The system must vary sufficiently slowly along the lines of force that little communication of particles from points of different behavior along the lines occurs. More explicitly, see Fig. 1, let points P_1 and P_2 be two points on a line of force at which the plasma properties, ρ, T, $\vec{B}$, etc., are significantly different. Then in a time $t \approx \ell/v$, particles from 1 and 2 will mix together and they can no longer be considered separate units. However, if significant changes occur at P_1 in a time short compared to t, the behavior at P_2 can exert no appreciable affect on P_1. Particles at P_1 can be considered to remain intact and the two particle adiabatic invariants may be employed to determine the behavior at P_1. $p_\perp$ is proportional to $v_\perp^2$ averaged over all the particles and to the density ρ, while $\langle v_\perp^2 \rangle$, by the invariance of μ, is proportional to B, so we have

$$p_\perp \propto \langle v_\perp^2 \rangle \rho \propto \rho B \quad .$$

This, of course, is true following the motion since it is the particles and not their location that is of importance.

The second invariant is not so familiar. It is $v_\parallel \ell$ where ℓ is the "extension" of a fluid element along the line. The quantity ℓ has an amount of uncertainty in its definition since the particles are dispersing at a considerable rate. However, it is known that even in free expansion of a one dimensional gas the mean square dispersion of velocities decreases as the density does and moreover goes as one over the length of the element of gas squared. (This can be seen for a gas initially of finite length, the particles of slowest velocity staying near the initial position.) For our case the length ℓ is proportional to B/ρ since the volume of a tube of force, is inversely proportional to ρ, while the cross sectional area is inversely proportional to B. Thus, the parallel pressure goes as

$$p_{\parallel} \propto \rho \langle v_{\parallel}^2 \rangle \propto \rho/\ell^2 \propto \rho^3/B^2 \quad .$$

A more formal derivation is as follows: The condition that points P_1 and P_2 remain intact clearly means that there is no significant heat exchange between points P_1 and P_2. Thus, in the third moment of the Vlasov equation we may neglect $\underline{\underline{Q}}$ the heat flow tensor. Multiply Eq. (28) by $m_s(\vec{v} - \vec{U}_s)(\vec{v} - \vec{U}_s)$, integrate over all velocities at a fixed point $\vec{r}$. By charge and current neutrality $\vec{U}_s$ is the same for ions and electrons if we assume a single ion species. Then we obtain

$$\frac{d}{dt}\underline{\underline{P}}_s + \nabla\cdot\underline{\underline{Q}}_s + P_s\nabla\cdot\vec{U} + \underline{\underline{P}}\cdot\nabla\vec{U} + (\underline{\underline{P}}\cdot\nabla\vec{U})^{tr}$$

$$+ \frac{e_s}{m_s c}(\vec{B}\times\underline{\underline{P}}_s + \underline{\underline{P}}_s\times\vec{B}) = 0 \quad , \tag{57}$$

where the superscript tr indicates transpose of the diadic, P_s is defined as in Eq. (43), and Q_s is the triad.

$$\underline{\underline{Q}}_s \equiv m_s \int (\vec{v} - \vec{U}_s)(\vec{v} - \vec{U}_s)(\vec{v} - \vec{U}_s)\ f\ d^3v \quad . \tag{58}$$

As before, we regard the last two terms as dominant because of the factor e/mc (the small gyration radius expansion). Thus, to lowest significant order, the pressure $\underline{\underline{P}}_{so}$ must satisfy

$$\vec{B} \times \underline{\underline{P}}_{so} = \underline{\underline{P}}_{so} \times \vec{B} \quad . \tag{59}$$

The most general solution of this equation is

$$P_{so} = p_{\perp s}(\underline{\underline{I}} - \vec{b}\vec{b}) + p_{\parallel s}\vec{b}\vec{b} \quad , \tag{60}$$

where the two scalars (so far) are arbitrary functions of time and space.

Let us denote the left hand side of Eq. (57) by $L\,\underline{\underline{P}}^{o}$. Then to next significant order in our expansion, Eq. (57) reads

$$L\underline{\underline{P}}_{os} = \frac{e_s}{m_s c}\,(\underline{\underline{P}}_{s1} \times \vec{B} - \vec{B} \times \underline{\underline{P}}_{s1}) \quad , \tag{61}$$

where $\underline{\underline{P}}_{s1}$ is the first order pressure. The necessary sufficient conditions that we can solve for $\underline{\underline{P}}_{s1}$ is that the trace of this equation vanish and also that it vanish when dotted with $\vec{b}$ on the right and left sides. Performing these operations, dropping $\underline{\underline{Q}}$ and summing over s, we obtain

$$\frac{d}{dt}(2p_\perp + p_{||}) + (2p_\perp + p_{||})\nabla\cdot\vec{U} + 2p_\perp(\nabla\cdot\vec{U} - \vec{b}\cdot\nabla\vec{U}\cdot\vec{b}) + 2p_{||}\vec{b}\cdot\nabla\vec{U}\cdot\vec{b} = 0 \quad , \tag{62a}$$

$$\frac{d}{dt}p_{||} + p_{||}\nabla\cdot\vec{U} + 2p_{||}\vec{b}\cdot\nabla\vec{U}\cdot\vec{b} = 0 \quad . \tag{62b}$$

We relate $\vec{U}$ to the rate of change of ρ and B by Eqs. (48) and (47).

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{U} \quad , \tag{63}$$

and

$$\frac{dB}{dt} = \vec{b}\cdot\frac{d\vec{B}}{dt} = \vec{b}\cdot[\nabla \times (\vec{U} \times \vec{B}) + \vec{U}\cdot\nabla\vec{B}] = B(\vec{b}\cdot\nabla\vec{U}\cdot\vec{b} - \nabla\cdot\vec{U}) \quad , \tag{64}$$

so that Eq. (62) becomes

$$\frac{dp_{||}}{dt} = -\frac{3p_{||}}{\rho}\frac{d\rho}{dt} + \frac{2p_{||}}{B}\frac{dB}{dt} \quad . \tag{65}$$

This reduces immediately to Eq. (56b). Subtracting Eq. (62b) from Eq. (62a) and using Eqs. (63) and (64) again yields

$$\frac{2dp_\perp}{dt} - \frac{2p_\perp}{\rho}\frac{d\rho}{dt} - \frac{2p_\perp}{B}\frac{dB}{dt} = 0 \quad ,$$

which reduces to Eq. (55a).

Thus, the double adiabatic equations of state result from the guiding center equations and the dropping of the heat flow. We can reduce the expression for $\underline{\underline{Q}}$ by making use of the special form of f_o, derived in the previous section from Eq. (34), that is its independence of gyration phase ϕ. $\underline{\underline{Q}}$ can be written

$$Q = 2q_\perp'(\underline{\underline{I}}\vec{b} + \vec{b}\underline{\underline{I}} + tr) + 2q_{||}'\vec{b}\vec{b}\vec{b} \quad , \tag{66}$$

where

$$q_\perp' = \sum m_s \int \frac{v_\perp^2}{2}(v_{||} - \vec{U}\cdot\vec{b})\ f\ d^3v \quad , \tag{66a}$$

$$q_{||}' = \sum m_s \int (v_{||} - \vec{U}\cdot\vec{b})^3\ f\ d^3v \quad , \tag{66b}$$

and the symbol tr denotes the third possible transposition of the triad $\underline{\underline{I}}\vec{b}$. $q_\perp'$ is the parallel heat flow of perpendicular energy while $q_{||}'$ is the parallel flow of parallel energy. They are only small if f is nearly symmetric, the situation arising when macroscopic plasma parameters vary slowly along $\vec{B}$. Also

$$\text{trace}\ \nabla\cdot\underline{\underline{Q}} = \vec{b}\cdot\nabla(10q_\perp' + 2q_{||}') - (10q_\perp' + 2q_{||}')\frac{\vec{b}\cdot\nabla B}{B} \quad , \tag{67a}$$

and

$$\vec{b}\cdot(\nabla\cdot\underline{\underline{Q}})\vec{b} = \vec{b}\cdot\nabla(6q_\perp' + 2q_{||}') - 2(q_\perp' + q_{||}')\frac{\vec{b}\cdot\nabla B}{B} \quad , \tag{67b}$$

so the derivative reduces the heat flow term by an additional factor proportional to the slowness of variation along $\vec{B}$.

To summarize the double adiabatic formalism, it is identical with the single fluid theory, Eqs. (1)-(4) and Eq. (10), with the single change that p is replaced by the divergence of the tensor pressure $\underline{\underline{P}}$, with the two scalars $p_{\perp}$, $p_{\parallel}$ determined by the double equations of state, Eqs. (56a) and (56b). Again it can be seen that the double adiabatic formalism is holonomic, all quantities can be expressed in terms of the displacement vector and can be reduced to a Lagrangian formalism.

These nice properties plus the apparent generalization allowed by a nonscalar pressure have made the double adiabatic theory quite popular. Unfortunately, the stringent conditions of very slow variation along magnetic lines of force imposed by the neglect of $\underline{\underline{Q}}$, greatly limit its applicability, at least when accurate results are desired. On the other hand the equations can be applied to solve problems beyond their limits of applicability, and the answers obtained are grossly inaccurate. This will be illustrated by an example in Section 5; namely, the computation of the criteria for stability against the mirror instability when a homogeneous magnetized plasma has unequal perpendicular and parallel pressures. This easy applicability of the formalism beyond the range of its validity makes it somewhat dangerous.

4. CONSEQUENCES OF THE MHD DESCRIPTION

The ideal MHD equations and, to a lesser extent, the double adiabatic equations and the guiding center equations possess some nice properties that often may be employed to draw some intuitive conclusions concerning plasma behavior without solving the equations in detail. They consist of some general global relations, conservation equations, and virial theorems, and also of the flux and line conservation equations which may be thought of as detailed conservation equations.

4.1 Conservation relations

The three quantities conserved by a plasma are linear momentum, energy, and angular momentum. To write them down for the ideal one fluid system let us first rewrite the force equation:

$$\rho \frac{\partial \vec{U}}{\partial t} = -\rho\vec{U}\cdot\nabla\vec{U} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi} - \nabla p - \rho\nabla\phi \quad , \tag{68a}$$

where we have made use of Eq. (4) to eliminate $\vec{j}$ and introduced the gravitational potential ϕ with $\vec{g} = -\nabla\phi$. Multiplying the continuity equation by $\vec{U}$ and adding we get

$$\frac{\partial}{\partial t}(\rho\vec{U}) = -\nabla\cdot\vec{T} - \rho\nabla\phi \quad , \tag{68b}$$

where

$$\underline{\underline{T}} = -\rho\vec{U}\vec{U} + \left(\frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{\underline{BB}}}{4\pi}\right) - p\underline{\underline{I}} \quad . \tag{69}$$

$\underline{\underline{T}}$ represents stresses exerted on any surface, the first terms are Raleigh stresses, the second, magnetic stresses, magnetic pressure and tension, while the third term is the pressure stress. Integrating Eq. (69) over a fixed volume V, and employing Gauss's theorem we have

$$\frac{\partial}{\partial t}\int_V \rho\vec{U}\, d\tau = -\int_S \underline{\underline{T}}\cdot ds + \int_V \rho\vec{g}\, d\tau \quad . \tag{70}$$

The term on the left is the rate of change of the plasma momentum in the volume, the first term on the right represents changes in this momentum due to forces exerted on the surfaces, and the last, changes in this momentum due to gravitational forces. If the system were isolated and $\vec{g}$ zero, then the total linear momentum would be conserved. [This is actually impossible (see the virial theorem below) but if the gravitational force is self consistent, produced by the plasma, the gravitational force can be written as a divergence and the linear momentum is actually conserved, as for example in an isolated star.] In any event the linear momentum density of a plasma is simply $\rho\vec{U}$ and includes no magnetic field contribution. Its change may be estimated by the forces on the surface. The electromagnetic contribution is relativistically small and not included in our equation.

A more significant conservation relation is that of energy. It is obtained by first multiplying Eq. (68a) by $\vec{U}$ and making use of the continuity equation to obtain

$$\frac{\partial}{\partial t}\left(\rho\,\frac{U^2}{2}\right) = -\frac{\vec{U}\cdot\nabla\times\vec{B}\times\vec{B}}{4\pi} - \vec{U}\cdot\nabla p - \rho\vec{U}\cdot\nabla\phi - \nabla\cdot\left(\rho\,\frac{U^2}{2}\,\vec{U}\right) \quad . \tag{71}$$

The left hand side represents the rate of change of kinetic energy per unit volume. The kinetic energy is changed as a result of corresponding changes of the magnetic energy (the first term on the right), pressure energy (the second term) and gravitational energy (the third term). In fact, multiplying Eq. (10) by $\vec{U}$ we have

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = B\cdot\frac{\nabla\times(\vec{U}\times\vec{B})}{4\pi} \quad . \tag{72}$$

From Eqs. (3) and (1) we have

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) = -\frac{U\cdot\nabla p}{\gamma-1} - \frac{\gamma p}{\gamma-1}\,\nabla\cdot U \quad . \tag{73}$$

From Eq. (1) we have

$$\frac{\partial}{\partial t}\,(\rho\phi) = -\nabla\cdot(\rho\vec{U})\phi \quad , \tag{74}$$

(ϕ is assumed independent of time). The quantities on the left of Eqs. (70) - (74) are the rates of change of the magnetic, pressure and gravitational energy densities respectively. Each of these expressions is equal to a term that corresponds to one of the terms on the right hand side of Eq. (71). In other words, any change in these energies can produce changes in the kinetic energy density.

Adding Eqs. (71) through (74), integrating over a fixed volume V, and making use of Gauss's theorem yields

$$\frac{d\mathcal{E}_V}{dt} = \frac{d}{dt} \int \left\{ \frac{\rho v^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma-1} + \rho\phi \right\} d\tau$$

$$= - \int d\vec{s} \cdot \left\{ \frac{\rho U^2}{2} U + \frac{\vec{B} \times (\vec{U} \times \vec{B})}{4\pi} + \frac{\gamma}{\gamma-1} p\vec{U} + \rho\vec{U}\phi \right\} . \qquad (75)$$

Thus, we may safely identify the left hand side with the time rate of change of $\mathcal{E}_V$ the total energy inside the volume V and the integral on the right hand side with the loss of energy through the surface S. The energy consists of four types: kinetic energy, magnetic energy, pressure energy, and gravitational energy. Almost any macroscopic plasma process consists of exchange of various forms of energy together with loss of energy through the surface. From Eq. (75) this loss can be seen to consist of direct loss of kinetic energy (first term), Poynting flux (second term since $\vec{U} \times \vec{B} = -c\vec{E}$), thermal energy and p dV work [since $p\vec{U}/(\gamma - 1) = p\vec{U}/(\gamma - 1) + p\vec{U}$], and finally of gravitational work represented by fluid entering at one potential and leaving at another. (The Poynting flux can also be thought of as loss of magnetic energy plus a magnetic P dV work.)

If the system is effectively isolated, say by rigid infinitely conducting walls at which $\vec{B}\cdot\vec{n} = 0$ at some time, then $\vec{B}\cdot\vec{n}$ will continue to be zero at all times and $\vec{U}\cdot\vec{n} = 0$ so the right hand side of Eq. (75) will vanish and the energy inside the volume will be conserved.

Finally, a conservation relation can be derived for angular momentum in complete analogy to Eq. (70). Take any point O as the origin and let $\vec{r}$ be the radius vector from this point. Then

$$\frac{d}{dt}\int_V \vec{r}\times\rho\vec{U}\,d\tau = \int_S (\vec{r}\times\underline{\underline{T}}) + \int_V \rho\vec{r}\times\vec{g}\,d\tau \;. \tag{76}$$

The angular momentum again resides soley in plasma motions. This relation is of considerable use in discussing outflow of angular momentum from the sun via the solar wind.

Another important integral relation for a plasma is the virial theorem. Define with respect to an origin O the tensor moment of inertia of a plasma inside a fixed volume V

$$\underline{\underline{T}}_V = \int \rho\vec{r}\vec{r}\,d\tau \quad . \tag{77}$$

Differentiate twice with respect to time making use of the ideal MHD equations and neglect surface terms and gravity

$$\frac{dI_V}{dt} = \int_V \frac{\partial\rho}{\partial t}\vec{r}\vec{r}\,d\tau = -\int_V \nabla\cdot(\rho\vec{U})\,\vec{r}\vec{r}\,dT \tag{78}$$

$$= \int \rho(\vec{U}\vec{r} + \vec{r}\vec{U})\,d\tau \quad ,$$

$$\frac{d^2I_V}{dt^2} = -\int [(\nabla\cdot\underline{\underline{T}})\,\vec{r} + \vec{r}\nabla\cdot\underline{\underline{T}}]\,d\tau = 2\int_V T\,d\tau \quad . \tag{79}$$

Then if the plasma remains in a finite region of space over a long period of time, we may time average Eq. (79) and drop the left hand side. There results from Eq. (70)

$$\left\langle \int \left[\rho\vec{U}\vec{U} + \left(\frac{B^2}{8\pi} I - \frac{\vec{B}\vec{B}}{4\pi} \right) + pI \right] d\tau \right\rangle = 0 \ . \tag{80}$$

This is the vector virial theorem. < > denotes a time average. Deviations from this equation can result from surface terms so this equation applies only to an isolated system. Taking the trace of Eq. (80) yields

$$\left\langle \int \left(\rho U^2 + \frac{B^2}{8\pi} + 3p \right) d\tau \right\rangle = 0 \quad . \tag{81}$$

Since the integral is clearly positive this then shows the impossibility of an isolated (without coils) force-free system. On the other hand if a self-consistent gravitational term is included we get

$$\left\langle \int \left(\rho U^2 + \frac{B^2}{8\pi} + 3p + \frac{\rho\phi}{2} \right) d\tau \right\rangle = 0 \ , \tag{82}$$

so gravitational energy which is always negative can balance the other three types of energy. [Note that the first term is twice the kinetic energy, the second term is just the magnetic energy, and the third term is $3(\gamma - 1)$ times the thermal energy, equal to two times for $\gamma = 5/3$, while the last term is the gravitational energy.]

A final important theorem concerning ideal MHD systems is that the system is derivable from a Lagrangian. In order to understand this information most easily it is necessary to regard each plasma fluid element as an entity. Any flow pattern between times t_0 and t_1 should be viewed as a set of time dependent displacements $\vec{\xi}(\vec{r}_o, t)$ of each of the fluid elements from its initial position $\vec{r}_0$ at $t = t_0$ to its final position $\vec{r}_1 = \vec{r}_0 + \vec{\xi}$ at t_1. A possible motion consists of a dependence of the displacement $\vec{\xi}(\vec{r}_0,t)$ on t. Then Hamilton's principle for the ideal MHD equations states that the motion that makes

$$L = \int_{t_o}^{t_1} \mathcal{L}\, dt \quad , \tag{83}$$

stationary, where

$$\mathcal{L} = \int \left(\frac{\rho U^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{8\pi} \right) d\tau \quad , \tag{84}$$

is the true dynamical one that satisfies the ideal MHD equations, and conversely. It is to be understood that for any displacement function $\vec{\xi}(\vec{r}, t)$, dynamical or not, the quantities ρ, p, and $\vec{B}$ are to be determined by solving Eqs. (1), (3), and (10) respectively. We know that these quantities are determined holonomically and do not depend on the detailed time dependence of $\vec{\xi}(\vec{r}_0, t)$.

For the proof of this result let us consider a given motion $\vec{\xi}(\vec{r}_0, t)$ and determine a neighboring motion by specifying the Eulerian function $\delta\vec{\xi}(\vec{r}, t)$ which is defined to be the difference between the position of the fluid element at time t that would have been at $\vec{r}$ under the unperturbed motion, and $\vec{r}$. Then it is easy to see that the perturbations in the quantities ρ, p, $\vec{B}$ at position $\vec{r}$ under the influence of the perturbation of motion are

$$\rho_1 = -\nabla\cdot(\rho\delta\vec{\xi}) \quad , \tag{85a}$$

$$p_1 = -\gamma p\nabla\cdot(\delta\vec{\xi}) - \delta\vec{\xi}\cdot\nabla p \quad , \tag{85b}$$

$$\vec{B}_1 = \nabla \times (\delta\vec{\xi} \times \vec{B}) \quad . \tag{85c}$$

It remains to determine $\vec{U}_1$. The perturbation in the fluid element velocity is

$$\frac{\partial\delta\vec{\xi}}{\partial t} + \vec{U}\cdot\nabla\delta\vec{\xi} \quad ,$$

by definition of $\delta\vec{\xi}$. But this perturbation is at $\vec{r} + \delta\vec{\xi}$ and is therefore also equal to $\vec{U}_1 + \delta\vec{\xi}\cdot\nabla\vec{U}$. Hence

$$\vec{U}_1 = \frac{\partial\delta\vec{\xi}}{\partial t} + \vec{U}\cdot\nabla\delta\vec{\xi} - \delta\vec{\xi}\cdot\nabla\vec{U} \quad . \tag{85d}$$

Substituting these perturbations into the corresponding perturbations of Eqs. (83) and (84) we obtain

$$\delta L = \int \delta \mathcal{L}\, dt = \int dt \int d\tau \left[\vec{\rho} U \cdot \left(\frac{\partial \delta \vec{\xi}}{\partial t} + \vec{U}\cdot\nabla\delta\vec{\xi} - \delta\xi\cdot\nabla\vec{U} \right) \right.$$

$$\left. - \nabla\cdot(\rho\delta\vec{\xi})\ \frac{U^2}{2} + \frac{\gamma p \nabla\cdot\delta\vec{\xi}}{\gamma - 1} + \frac{\delta\vec{\xi}\cdot\nabla p}{\gamma - 1} \right] \quad . \qquad (86)$$

Then integration by parts shows that $\delta L = 0$ for all $\delta\vec{\xi}$ vanishing at t_o, t_1, and spatial boundaries, if and only if Eq. (2) is satisfied.

The existence of this Hamilton's principle for the MHD equations is extremely important. It can be shown to underlie most of the general results on MHD such as self-adjointness, energy principles for stability of static equilibrium, and energy conservation. Further, it has been shown that small scale hydromagnetic waves preserve wave action, that is they can be thought of as quantized, and this also is a direct consequence of this Lagrangian approach (Dewar 1970).

We have so far exclusively discussed the properties of the one fluid ideal MHD equations in this section. All of these properties are also possessed by the double adiabatic formalism if we replace p and γ by the appropriate generation. For example $p/(\gamma - 1)$ should be replaced by $p_\perp + p_\parallel/2$ in Eqs. (75), (80), and (84) while 3p should be replaced by $2p_\perp + p_\parallel$ in Eq. (82). Similar results appear to hold for the guiding center theory, although they have so far only been effectively determined in certain limiting situations. We refer the reader to the literature for details.

4.2 Flux frozen in plasma

Probably the most useful of the intuitive concepts implied by the ideal MHD equations, as well as the guiding center theory and the double adiabatic theory, is that concerning the magnetic flux lines frozen in the plasma. Precisely stated the flux conserving theorem is as follows:

Assume that at some initial time t_o magnetic lines of force are drawn throughout the plasma volume in such a way that their density is proportional to the field strength B, and they are everywhere tangent to $\vec{B}$. (For simplicity we take a finite but very large number of such lines so their density is not precisely determined at each point but can be defined to any desired precision by taking a sufficiently large number of such lines.) Then at time t_o the magnetic field $\vec{B}$ is completely represented by these lines. Let the plasma flow with velocity $\vec{U}$ and let the magnetic field evolve according to Eq. (10). At the same time let the lines of force be bodily transported by this velocity $\vec{U}$ to some new configuration, just as though they were "frozen" in the plasma. Then at any later time t, the configuration of the lines at that time will represent the magnetic field at that time both as to field strength given by line density, and direction given by the tangents to the lines.

This theorem holds true to the extent that Eq. (10) does. That is, if $\vec{B}$ deviates from the field given by Eq. (10) due to finite resistivity, it will deviate from the field given by the line configuration to exactly the same extent. Since the displacement of the lines evolves in a continuous manner, their topology must be preserved. Closed lines remain closed, ergodic lines remain ergodic, magnetic surfaces existing at time t_o continue to exist, etc. This flux freezing concept is often a very critical one and it is important to know under what conditions it can be broken. The plasma can occasionally be kept from reaching a state of much lower magnetic energy by this constraint alone. A change in topology which may be produced by a breakdown in Eq. (10) over a very small region, say near an X point, could conceivably lead to a large conversion of magnetic energy to kinetic energy in a plasma. This possibility is usually termed the reconnection problem and it is a problem of great interest since its resolution could conceivably lead to an explanation for certain observed violent plasma behavior such as disruption in tokomaks, solar flares, etc.

There are two mathematical ways to express the theorem of flux freezing. The first is the Lundqvist identity, while the second makes use of the Clebsch formula (Lundqvist 1951).

The Lundqvist identity expresses the magnetic field at time t and position $\vec{r}$ in terms of its value at time t_o and a different position $\vec{r}_o$

$$\frac{\vec{B}}{\rho}(\vec{r}, t) = \frac{\vec{B}}{\rho}(\vec{r}_o, t_o)\cdot\nabla_o\vec{r}(\vec{r}_o, t) \quad . \tag{87}$$

In this formula $\vec{r}$ is understood to be a function of $\vec{r}_o$, and t which represents the position of the fluid element at time t that occupied the position $\vec{r}_o$ at initial time t_o. The subscript o on ∇_o indicates that derivatives are to be taken with respect to $\vec{r}_o$

at fixed t. Let $\vec{B}_o$ and ρ_o represent $\vec{B}(\vec{r}_o, t_o)$ and $\rho(\vec{r}_o, t_o)$ respectively. To establish the validity of Eq. (87) we first show that it satisfies Eq. (10). Making use of $(\partial\vec{r}/\partial t)_{\vec{r}_o} = \vec{U}$ we have

$$\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = \frac{\vec{B}_o}{\rho_o} \cdot \nabla_o \vec{U} \quad , \tag{88}$$

where $d/dt = \partial/\partial t + \vec{U}\cdot\nabla \equiv (\partial/\partial t)_{\vec{r}_o}$. Also

$$\nabla \times (\vec{U} \times \vec{B}) = \vec{B}\cdot\nabla\vec{U} - \vec{U}\cdot\nabla\vec{B} - \vec{B}\nabla\cdot\vec{U} \quad ,$$

so

$$\frac{\partial\vec{B}}{\partial t} - \nabla \times (\vec{U} \times \vec{B}) = \frac{d\vec{B}}{dt} - \vec{B}\cdot\nabla\vec{U} + \vec{B}\nabla\cdot\vec{U}$$

$$= \frac{\rho}{\rho_o} (\vec{B}_o\cdot\nabla_o\vec{U}) + \frac{1}{\rho}\frac{d\rho}{dt}\vec{B} - \vec{B}\cdot\nabla\vec{U} - \frac{\vec{B}}{\rho}\frac{d\rho}{dt} \tag{89}$$

$$= \frac{\rho}{\rho_o} [\vec{B}_o\cdot\nabla_o\vec{U} - (\vec{B}_o\cdot\nabla_o\vec{r})\cdot\nabla\vec{U}] \quad ,$$

where the first line follows from the definition of d/dt, the second line from Eqs. (88) and (1) and the third from substitution of Eq. (87) for $\vec{B}$. The bracket in the third line of Eq. (89) vanishes because of the chain rule for differentiation. Thus, Eq. (87) satisfies Eq. (10) for the evolution of the magnetic field and is valid initially, so it remains valid for all t. Its relation to flux freezing can be seen geometrically. $(\vec{B}_o\cdot\nabla_o\vec{r})/B_o$ is the shearing of a unit line element along the initial line of force by the flow, so Eq. (87) states that $\vec{B}$ continues to be parallel to the sheared line element. Also the line has been lengthened by the same shear flow, but factor ρ/ρ_o represents the decrease in volume. This combined with the lengthening of the line element gives the shrinking of the cross-sectional area which thus represents the amplification of the density of the lines of force.

The other alternative mathematical method for describing flux conservation involves the Clebsch formula for expressing an arbitrary divergence-free vector field such as $\vec{B}$ in terms of two scalar functions

$$\vec{B} = \nabla\alpha \times \nabla\beta \quad . \tag{90}$$

If such α and β scalars exist, $\vec{B}$ given by Eq. (90) clearly is divergence free. Further, dotting Eq. (90) with $\nabla\alpha$ and with $\nabla\beta$ gives

$$\begin{aligned} \vec{B}\cdot\nabla\alpha &= 0 \quad , \\ B\cdot\nabla\beta &= 0 \quad , \end{aligned} \tag{91}$$

so α and β are constants along lines of force, and indeed a general line of force can be determined by $\alpha = \alpha_o$, $\beta = \beta_o$ where α_o and β_o are constants. Lastly because $J = (\vec{B}\cdot\nabla\alpha \times \nabla\beta/B) = B$ is the Jacobian for a transformation from coordinates $\vec{r}$ to coordinates α, β, ℓ, where ℓ is arc length along the lines, we can see that $d\alpha\ d\beta$ represents the element of flux. That is, if we parameterize a surface S cutting the lines by α and β then $d\alpha\, d\beta$ is the flux through the corresponding element of area (Fig. 2). Thus, if we select the lines of force by a uniform distribution of values of α and β, their density will be proportional to the magnetic field strength B.

The above properties of α and β show how they can actually be found to satisfy Eq. (90). As in Fig. 2, choose α and β' arbitrarily on S and extend them through all space so as to satisfy Eq. (91) and $\vec{B}\cdot\nabla\beta' = 0$; i.e., by keeping them constant on $\vec{B}$ lines. Then

$$\vec{B} \times (\nabla\alpha \times \nabla\beta') = \vec{B}\cdot\nabla\beta'\,\nabla\alpha - \vec{B}\cdot\nabla\alpha\nabla\beta' = 0 \quad ,$$

so

$$\vec{B} = g(\nabla\alpha \times \nabla\beta') \quad ,$$

where g is a scalar. From $\nabla\cdot\vec{B} = 0$ we have

$$(\nabla\alpha \times \nabla\beta')\cdot\nabla g = \frac{\vec{B}\cdot\nabla g}{g} = 0 \quad ,$$

so g is constant along $\vec{B}$ lines, and thus a function of α and β', $g = g(\alpha , \beta')$. Now choose β to satisfy

$$\frac{\partial\beta}{\partial\beta'} = g(\alpha, \beta') \quad . \tag{92}$$

Then for this α and β Eq. (90) is easily verified.

Now α and β are clearly not unique. However, once they are chosen to represent $\vec{B}$ at some initial time t_o, they can be chosen at any later time by demanding they stay constant on any fluid element. That is, let them satisfy

$$\frac{\partial\alpha}{\partial t} + \vec{U}\cdot\nabla\alpha = 0 \quad , \tag{92a}$$

$$\frac{\partial\beta}{\partial t} + \vec{U}\cdot\nabla\beta = 0 \quad . \tag{92b}$$

Then $\vec{B}$ as given by Eq. (90) satifies Eq. (10) and thus continues to give the magnetic field. For

$$\frac{\partial}{\partial t}(\nabla\alpha \times \nabla\beta) - \nabla \times [\vec{U} \times (\nabla\alpha \times \nabla\beta)]$$

$$= \nabla \frac{\partial\alpha}{\partial t} \times \nabla\beta + \nabla\alpha \times \nabla \frac{\partial\beta}{\partial t} - \nabla \times [\vec{U}\cdot\nabla\beta\nabla\alpha - \vec{U}\cdot\nabla\alpha\nabla\beta]$$

$$= -\nabla(\vec{U}\cdot\nabla\alpha) \times \nabla\beta - \nabla\alpha \times \nabla(\vec{U}\cdot\nabla\beta) - \nabla(\vec{U}\cdot\nabla\beta\nabla\alpha) + \nabla(\vec{U}\cdot\nabla\alpha\nabla\beta) = 0 \quad ,$$

where the second line follows from expanding out of the triple vector product in the bracket in the first line, while the third line follows from Eq. (92) and taking the curl of the bracket in the second line.

The properties of α and β clearly correspond to those of magnetic lines in the flux conservation theorem.

5. AN EXAMPLE

We should like to illustrate the guiding cent^r formalism by an example which will also bring out the limitations of the double adiabatic formalism.

Consider a homogeneous, magnetized, ion-electron plasma with unequal perpendicular and parallel temperatures. Take the uniform field $\vec{B}_o$ in the z direction. For simplicity we take the equilibrium distribution to be a bi-Maxwellian with unequal perpendicular and parallel temperatures

$$f_{os} = \frac{n}{(2\pi m_s)^{3/2}\, T_{\perp s} T_{\parallel s}^{1/2}} \exp\left(-\frac{m_s v_\perp^2}{2T_{\perp s}} - \frac{m_s v_\parallel^2}{2T_{\parallel s}}\right) \quad . \tag{93}$$

Consider a sinusoidal perturbation of this plasma proportional to $\exp(-i\ t + ik_x x + ik_z z)$. Under what conditions is this perturbation unstable?

We start with the fluid Eqs. (45) and (47). If we introduce the plasma displacement $\vec{\xi}$, with $\vec{U} = -iw\vec{\xi}$, then these become

$$-\rho\omega^2\vec{\xi} = -\nabla' \underline{\underline{P}}_1 - \frac{1}{4\pi}\nabla(\vec{B}_o\cdot\vec{B}_1) + \frac{\vec{B}_0\cdot\nabla\vec{B}_1}{4\pi} \quad , \tag{94}$$

$$\vec{B}_1 = ik_z\xi_x B_o\hat{\vec{x}} - ik_x\xi_x B_o\hat{\vec{z}} \quad , \tag{95}$$

where the subscript or superscript 1 indicates perturbed quantities. From Eq. (44a) we have for the perturbed pressure

$$\underline{\underline{P}}_1 = p_\perp' \underline{\underline{I}} + (p_\parallel' - p_\perp')\vec{b}\vec{b} + (p_\parallel - p_\perp)(\vec{b}_1\vec{b} + \vec{b}\vec{b}_1) \quad . \tag{96}$$

Now from Eq. (95) we have

$$B_1 = -ik_x \xi_x B_o \quad , \tag{97a}$$

$$\vec{b}_1 = ik_z \xi_x \hat{\vec{x}} \quad , \tag{97b}$$

so we easily find

$$\begin{aligned} \nabla \cdot \underline{\underline{P}}_1 = & \left[ik_x p_\perp' - (p_\parallel - p_\perp)k_z^2 \xi_x\right] \hat{\vec{x}} \\ & + \left[ik_z p_\parallel' - (p_\parallel - p_\perp)k_x k_z \xi_x\right] \hat{\vec{z}} \quad . \end{aligned} \tag{98}$$

Substituting Eq. (98) in the equation of motion, Eq. (94) and taking the x and z components we find two equations

$$-\rho\omega^2 \xi_x = -ik_x p_\perp' + k_z^2 (p_\parallel - p_\perp)\xi_x - (k_x^2 + k_z^2) \frac{B_o^2}{4\pi} \xi_x \quad , \tag{99a}$$

$$-\rho\omega^2 \xi_z = -ik_z p_\parallel' + k_x k_z (p_\parallel - p_\perp) \xi_x \quad , \tag{99b}$$

for ξ_x and ξ_z. In order to complete the system we need equations of state for $p_\perp'$ and $p_\parallel'$.

Up to this point the double adiabatic theory and the guiding center theory coincide. They differ as to the determination of $p_\perp'$ and $p_\parallel'$, however. First let us complete the equations by invoking the two equations of state, Eqs. (56a) and (56b) of the double adiabatic theory, to express $p_\perp'$ and $p_\parallel'$ in terms of ξ_x and ξ_z. Since from the continuity equation, Eq. (48), $\rho_1 = -i(k_x\xi_x + k_z\xi_z)$, we have from this and Eq. (97a)

$$\frac{p_\perp'}{p_\perp} = \frac{\rho_1}{\rho} + \frac{B_1}{B_o} = -2ik_x\xi_x - ik_z\xi_z \quad , \tag{100a}$$

$$\frac{p_\parallel'}{p_\parallel} = \frac{3\rho_1}{\rho} - \frac{2B_1}{B_o} = -ik_x\xi_x - 3ik_z\xi_z \quad . \tag{100b}$$

Substitution of Eqs. (100a) and (100b) in Eqs. (94) and (95) yields two equations for ξ_x and ξ_z alone. Setting the determinant of these equations to zero gives the eigenvalue equation for ω

$$\left[\rho\omega^2 - \left((2k_x^2 + k_z^2)p_\perp + \frac{k^2B_o^2}{4\pi} - k_z^2 p_\parallel\right)\right](\rho\omega^2 - 3k_z^2 p_\parallel) = k_x^2 k_z^2 p_\perp^2 \quad . \tag{101}$$

It is easy to see that the roots of ω^2 are real. We have instability if one of the roots for ω^2 is negative and the condition for this is

$$2k_x^2\left[\frac{B_o^2}{8\pi} + p_\perp\left(1 - \frac{p_\perp}{6p_\parallel}\right)\right] + k_z^2\left(\frac{B_o^2}{4\pi} + p_\perp - p_\parallel\right) < 0 \quad .$$

This is negative if $k_x = 0$ and

$$p_{\parallel} > p_{\perp} + \frac{B_o^2}{4\pi} \quad , \tag{102}$$

the "fire hose instability," or $k_z \to 0$ (it must not vanish) and

$$\frac{p_{\perp}^2}{6p_{\parallel}} > \frac{B^2}{8\pi} + p_{\perp} \quad , \tag{103}$$

the "mirror instability." Equations (102) and (103) are the stability results derived from double adiabatic theory.

Let us now employ guiding center theory to find $p_{\perp}'$ and $p_{\parallel}'$ and to complete Eqs. (99a) and (99b). Actually we can determine $p_{\perp}'$ from ξ_x alone and we need consider only Eq. (99a). $p_{\perp}'$ is found from f' which is given by solving Eq. (52), for example. Let $f = f_o + f_1$. Then, since B is the Jacobian of the transformation to μ, $v_{\parallel}$ variables,

$$p_{\perp} = \sum m_s \int f_s \, \mu B \, (B \, d\mu) \, dv_{\parallel} \, d\phi \quad ,$$

and

$$p_{\perp}' = \sum m_s \int f_s' \, \mu B \, d^3v + \frac{2B_1}{B_o} p_{\perp} \quad . \tag{104}$$

Perturbing Eq. (52) and using Eq. (93) we have

$$f_{1s} = \frac{[-k_x k_z \xi_x \, (v_{\perp}^2/2) + (e_s/m_s) \, E_{\parallel}]}{-i\omega + ik_z v_{\parallel}} \frac{m_s v_{\parallel}}{T_{\parallel s}} f_s \quad . \tag{105}$$

Near the marginal point for stability, we may neglect ω in the denominator [if $\omega << k_z(T\ /m)^{1/2}$] and

$$f_{1s} = ik_x\xi_x \frac{m_s v_\perp^2}{2T_{\|s}} f_s - \frac{ie_s E_\|}{k_z T_{\|s}} f_s \quad . \tag{106}$$

Now from charge neutrality we can determine $E_\|$ to be

$$E_\| = \frac{k_z}{e} (k_x\xi_x) \frac{(T_\perp/T_\|)_i - (T_\perp/T_\|)_e}{(1/T_{\|i}) - (1/T_{\|e})} \quad . \tag{107}$$

For simplicity, we take $(T_\perp/T_\|)_i = (T_\perp/T_\|)_e$ so that $E_\| = 0$. Substituting Eq. (106) (with $E_\| = 0$) into Eq. (104) and making use of Eq. (97a) we find

$$p_\perp' = 2ik_x\xi_x \sum_s \left(\frac{T_\perp^2}{T_\|}\right)_s n - 2ik_x\xi_x p_\perp \quad , \tag{108}$$

and if we further take $T_{\perp i} = T_{\perp e}$

$$p_\perp' = 2ik_x\xi_x \left(\frac{p_\perp^2}{p_\|} - p_\perp\right) \quad . \tag{109}$$

Then for sufficiently small ω (see above) we have from Eq. (99a)

$$\rho\omega^2 \doteq 2k_x^2 \left(p_\perp + \frac{B_o^2}{8\pi} - \frac{p_\perp^2}{p_\|}\right) + k_z^2\left(p_\perp + \frac{B_o^2}{4\pi} - p_\|\right) \quad .$$

Again we have the fire hose instability if $k_x = 0$ and Eq. (102) is satisfied. However, the condition for the mirror instability is changed to $k_z \to 0$ and

$$(p_\perp^2/p_\|) > p_\perp + (B_o^2/8\pi) \quad ,$$

a criteria differing substantially from Eq. (103) (by a factor of 6).

The reason for the different criteria for the guiding center theory of the mirror instability and the double adiabatic theory is that ω must pass through zero so that particle communication sets in over a distance k^{-1} along the lines in a time short compared to ω^{-1}, so the condition necessary for the validity of the latter theory fails.

This example illustrates the dangers inherent in the double adiabatic theory, since the failure of the validity conditions to hold really only becomes evident after the more accurate guiding center theory is carried out. The fire hose instability theory remains valid since, as can be seen from intuitive picture of the instability, parallel heat flow plays no role.

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-76-CHO-3073.

REFERENCES

Braginski, S. I., (1965) Reviews of Plasma Physics, edited by A. Leontovitch (Consultants Bureau, New York 1965), Vol. 1, p. 605.

Chew, G. F., Goldberger, M. L., and Low, F. E., (1955) Los Alamos Lecture Notes on Physics of Ionized Gases, LA-2055.

Chew, G. F., Goldberger, M. L., and Low, F. E., (1956) Proceedings of Royal Society, A 236 112.

Dewar, R. L. (1970) Phys. Fluids 13, 2710.

Kruskal, M. D. (1960) La theorie des Gaz Neutres et Ionises edited by C. Dewitt and J. F. Detoeuf.

Kulsrud, R. M. (1962) Rendiconti della Scuola Internazionale di Fiscia "Enrico Fermi" Course XXV (edited by M. N. Rosenbluth) Academic Press New York, p. 54.

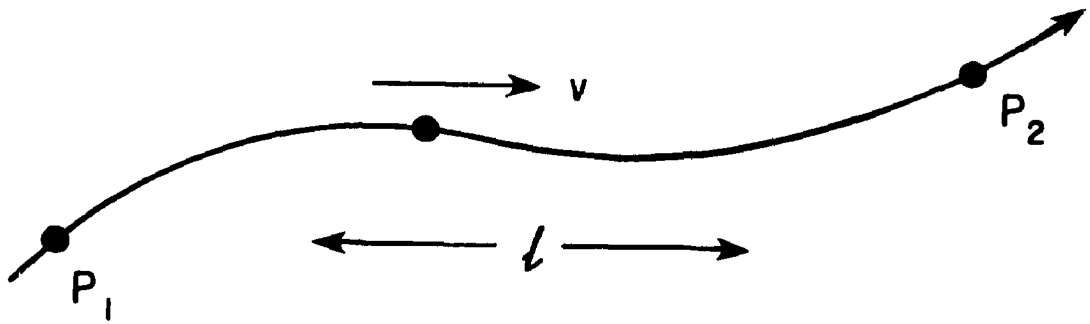
Lundqvist, S. (1951) Phys. Rev. 83 307.

Newcomb, W. A. (1958) Ann. Phys. 3, 347.

Petschek, H. E. (1964) The Physics of Solar Flares
AAS-NASA Symposium (edited by W. N. Hess)
(NASA SP-50, 1964) p. 425.

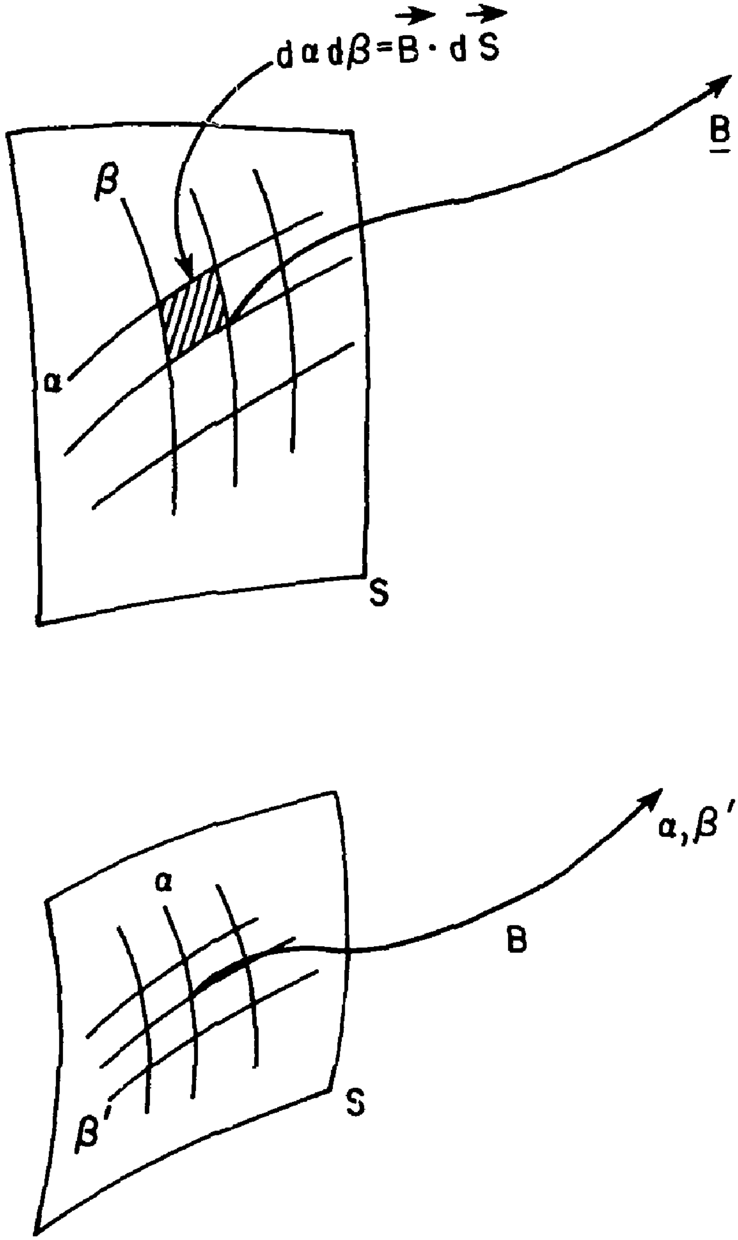
Rosenbluth, M. N. and Rostoker, N. (1958)
Phys. Fluids 2, 23.

Spitzer, L., jr. (1962) Physics of Fully Ionized Gases,
Interscience, New York.



(PPPL-802271)

Fig. 1. A line of force, $\vec{B}$.



(PPPL-802272)

Fig. 2. Clebsch coordinates α and β.