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CENTRAL INSTITUTE OF PHYSICS
INSTITUTE FOR PHYSICS AND NUCLEAR ENGINEERING
*Bucharest, P.O.B.5206, ROMANIA*

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# Four dimensional sigma model coupled to the metric tensor field

*G.Ghika and M.Visinescu*

ABSTRACT: *We discuss the four dimensional nonlinear sigma model with an internal $O(n)$ invariance coupled to the metric tensor field satisfying Einstein equations. We derive a bound on the coupling constant between the sigma field and the metric tensor using the theory of harmonic maps. A special attention is paid to Einstein spaces and some new explicit solutions of the model are constructed.*

## 1. INTRODUCTION

In recent years there has been much interest in the finite action solutions of the Euclidean sigma model in two dimensions [1] . This model bears many similarities to a four dimensional non-Abelian gauge theory like scale invariance, asymptotic freedom, instanton and meron solutions , action bounded by a multiple of a topological charge, etc...

When trying to generalize this model to a four dimensional one there are difficulties. As long as we use a traditional Lagrangian, the sigma field $\Phi$ has non-zero dimension and the constraint $\sum_a (\Phi^a)^2 = \lambda^2$ must introduce a dimensional constant $\lambda$ which destroys the conformal invariance. On the other hand a dimensionless $\Phi$ implies a higher derivative Lagrangian with many troubles at the quantum level.

A very elegant way to avoid these difficulties was found by de Alfaro, Fubini and Furlan [2] constructing a model in which the sigma field is coupled to the metric tensor $g_{\mu\nu}$ in a generally invariant fashion. Their model, having an internal $\mathcal{O}(n)$ invariance, is characterized by the following Lagrangian density

$$(1.1)\quad \mathcal{L} = -\frac{1}{4K}\sqrt{G}\left[R(G) + \frac{3}{2}\Lambda^2\right] - \frac{1}{4}\sqrt{G}\, F\, G^{\mu\nu}\partial_\mu \Phi^a \partial_\nu \Phi^a$$

where $\mu,\nu = 1,2,3,4$ , $a = 1,2,\ldots,n$ is an internal index and the field $\Phi$ is defined on a four dimensional Riemannian manifold $M$ taking values in an Euclidean sphere $S^{n-1}$ :

(1.2) $$\sum_{a=1}^{n} (\Phi^a)^2 = 1$$

In Eq. (1.1) $K$ is the coupling constant of the gravitational field with the usual dimensions of a $(\text{length})^2$, $G_{\mu\nu}$ is metric tensor, $G = \det(G_{\mu\nu})$, $R$ is the scalar curvature of $M$ and $F$ is a coupling constant. For generality a cosmological term $(\frac{3}{2}\Lambda^2)$ was added but we shall deal mainly with the case $\Lambda^2=0$. Introducing as in paper [2] a new set of quantities $g_{\mu\nu}=K^{-1} G_{\mu\nu}$, $\lambda^2=K\Lambda^2$, $f = KF$ we get

(1.3) $$\mathcal{L} = -\frac{1}{4}\sqrt{g}\left[R(g) + \frac{3}{2}\lambda^2 + f g^{\mu\nu}\partial_\mu\Phi^a\partial_\nu\Phi^a\right]$$

In these notations, the metric field has dimensions of a $(\text{length})^{-2}$ $\lambda^2$ and $f$ are pure numbers and $\Phi$ is dimensionless. In the usual sigma model $F$ is a positive constant and interpreting $K$ as the usual gravitational constant it follows that $f$ is nonnegative. But wishing to keep the model as general as possible we shall not impose a restriction on the sign of $f$ from the beginning.

The authors of [2] were able to find explicitely an instanton solution for $f = 3$ and a meron solution for $f = 2$ in the case $\lambda^2=0$. It turns out that for these values of $f$ the order of magnitude of the coupling $F$ is not in the typical range of high energy physics, interpreting $K$ as the usual gravitational constant. But in spite of this problem we consider that the model deserves a closer examination.

Concerning the cosmological constant we mention that in the last time many authors reevaluated its status. One argues [3] that the quantization of gravity with a cosmologi -

cal term is both necessary and feasible and the empirical value of the constant ( very small and possible zero) do not necessarily imply that one should set $\lambda^2 = 0$ in the bare Einstein Lagrangian. On the other hand, in the " spacetime foam" program [4,5] , the spacetime looks nearly flat when viewed on large length scales, but suffers large fluctuations of the metric and topology on scales on the order of the Planck length. The foam-like structure can be described by introducing a cosmological term as a Lagrange multiplier for the 4-volume in the path integral approach[6].

In the present paper we shall use the theory of the harmonic maps [7] as a valuable tool to handle the nonlinear sigma model in four dimensions. The role of the harmonic maps in different fields of particle physics and general relativity was pointed out by Misner [8] . In the last time many papers use the theory of the harmonic maps in order to investigate the nonlinear sigma model [9] , Gribov ambiguity in non-Abelian gauge theory [10] , general relativity [11] , etc.

We observe that the field $\Phi$ is in fact a harmonic map from a Riemannian four dimensional manifold M into a (n-1) dimensional sphere (for an internal $\mathcal{O}$ (n) invariance). On the other hand, from the equations of motion we find that the metric on M is such that the Ricci tensor is related to the "pull-back" of the metric of the final sphere $S^{n-1}$.These observations will permit us to find some constraints on the parameter f if the space M is compact as we shall mainly assume in our paper.

Of course the use of the compact spaces is a technical restriction which allows us to find solutions with finite action. This does not mean necessarily that the actual space-time is compact and one can interpret it as a normalization prescription like introducing a box in ordinary quantum mechanics. On the other hand it is known that starting with a non-compact manifold M it is difficult to obtain finite action solutions. For example Garber et al [9] proved that the d-dimensional $\sigma$ -model ( $\Phi : \mathbb{R}^d \rightarrow S^n$ ) with $d > 2$ has no regular finite solution. We must mention also that starting with a compact manifold it is essential to search solutions satisfying the constraint (1.2). This can be put in connection with the fact that compact, orientable Riemannian manifolds without boundary do not support non trivial harmonic functions [12] Hence this kind of manifolds is not interesting for an Euclidean relativistic theory with energy-momentum tensor constructed from a free scalar field without mass.

In Section 2 we derive the general constraints imposed by the harmonic property of the map $\Phi$ and by the Einstein equations for the metric. We are able to derive a bound on the coupling f which might allow a hope to obtain a solution with acceptable value for F , at least for a vanishing cosmological term. Special attention is paid to Einstein manifolds which form an important class of possible solutions if the metric tensor of M is proportional to the pull-back of the metric of the final sphere. For this class of solutions we express the coupling f in a geometrical way, namely as the ratio of the scalar curvature of M and an eigenvalue of the Laplace - Beltrami operator on this manifold.

In Section 3 we construct several solutions with finite action. They involve for M various topologies like $S^4$, $S^2 \times S^2$, $CP^2$ which appear also in the study of the gravitational instantons . Einstein equations on compact manifold have solutions with multiple topologies which can play an important role in the path integral in the space of metrics [5, 6] . Finally we discuss the meron type solution (infinite action) assuming that the manifold M is conformally flat.

In the last section we make some short comments concerning the model and its possible generalizations. A collection of useful formulae about the harmonic maps is presented in the Appendix.

## 2. GENERAL PROPERTIES

In this section we shall derive some general properties of the solutions of the model described in the *Introduction* assuming that the four dimensional manifold M is compact without boundary.

The Euler - Lagrange equations corresponding to the Lagrangian (1.3) are

$$(2.1) \qquad R_{\mu\nu} + \frac{3}{4}\lambda^2 g_{\mu\nu} = - f\, \partial_\mu \Phi^a \partial_\nu \Phi^a$$

$$(2.2) \qquad \Delta \Phi^a \equiv \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g}\, g^{\mu\nu} \partial_\nu \Phi^a \right) = - \left( g^{\mu\nu} \partial_\mu \Phi^b \partial_\nu \Phi^b \right) \Phi^a$$

where the left hand side of Eq.(2.2) is the usual Laplace -

Beltrami operator defined on M. As it was observed in paper [2] for $\lambda^2 = 0$ the field Eqs. (2.1) and (2.2) are invariant under the simple rescaling.

$$g_{\mu\nu} \rightarrow c^2 \cdot g_{\mu\nu} \tag{2.3}$$

and consequently any solution of the metric field $g_{\mu\nu}$ is defined in this case up to a multiplicative constant.

It is important to observe that any finite action solution of Eq. (2.2) is a harmonic map between the manifold M with the metric $g_{\mu\nu}$ and an Euclidean sphere $S^{n-1}$ isometrically immersed in $\mathbb{R}^n$. In fact Eq. (2.2) is just Eq. (A.10) in which the field $\Phi$ is the composition of a map $\varphi : M \to S^{n-1}$ and of the Riemannian immersion $\psi : S^{n-1} \to \mathbb{R}^n$. In our case the energy density of the harmonic map $\Phi$ is (see Eqs. (A.5) (A.10)).

$$\begin{aligned} e(\Phi) &= e(\psi \circ \varphi) = e(\varphi) = \\ &= \frac{1}{2} g^{\mu\nu} h_{ab} \partial_\mu \Phi^a \partial_\nu \Phi^b = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^a \partial_\nu \Phi^a \end{aligned} \tag{2.4}$$

since $h_{ab} = \delta_{ab}$ on $\mathbb{R}^n$

Taking into account this observation, we can use the general properties of the harmonic maps to obtain some features of th solutions of the model

Eells and Sampson [13], using the general method initiated by Bochner [12] and the Ricci identities, were able to obtain the following equation satisfied by the energy density of an harmonic map $\varphi : M \to N$

(2.5) $$\Delta e(\varphi) = |\nabla(d\varphi)|^2 + Q(d\varphi)$$

with

(2.6) $$Q(d\varphi) = -R^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b h_{ab} - R'_{abcd}\partial_\mu\varphi^a\partial_\nu\varphi^b\partial_\lambda\varphi^c\partial_\sigma\varphi^d g^{\mu\lambda}g^{\nu\sigma}$$

where $R'_{abcd}$ is the Riemann - Christoffel curvature tensor on N (with the metric $h_{ab}$) and $R_{\mu\nu}$ is the Ricci tensor on M (with the metric $g_{\mu\nu}$). Assuming that the Riemannian manifold M is compact without boundary from Stokes' theorem we have

(2.7) $$\int_M \Delta e(\varphi)\sqrt{g}\, d^4x = 0$$

Applying Eq. (2.7) to Eq. (2.5) and observing that $|\nabla(d\varphi)|^2$ is non-negative we have

(2.8) $$\int_M Q(d\varphi)\sqrt{g}\, d^4x \leqslant 0$$

If $\nabla(d\varphi)=0$, which means that the map $\varphi$ is totally geodesic (see Appendix A), then in Eq. (2.8) we have equality. We note also that for constant energy density, Eq. (2.8) can be expressed simply as

(2.9) $$Q(d\varphi) \leqslant 0$$

We shall apply Eqs. (2.8) and (2.9) in order to obtain some information about the coupling constant f. In our case the field $\varphi$ takes values in an Euclidean sphere $N = S^{n-1}$ and using Eq.(2.1) we shall first evaluate $Q(d\varphi)$ for $\lambda^2=0$:

$$(2.10)\quad \begin{aligned} Q(d\varphi) &= \frac{1}{f} R^{\mu\nu} R_{\mu\nu} - \\ &-(h_{ac}h_{bd}-h_{ad}h_{bc})\partial_\mu\varphi^a\partial_\nu\varphi^b\partial_\lambda\varphi^c\partial_\sigma\varphi^d g^{\mu\lambda}g^{\nu\sigma} = \\ &= \frac{1}{f} R^{\mu\nu}R_{\mu\nu} - \frac{1}{f^2}\left[(R_{\mu\nu}g^{\mu\nu})^2 - R^{\mu\nu}R_{\mu\nu}\right] \\ &= \frac{1}{f^2}\left[|\rho|^2(1+f) - R^2\right] \end{aligned}$$

where

$$R = g^{\mu\nu}R_{\mu\nu}$$
$$|\rho|^2 = R^{\mu\nu}R_{\mu\nu}$$

We used explicitely that on a sphere (following the notations from Eisenhart [14]:

$$R^1_{abce} = K_0(h_{ac}h_{bd} - h_{ad}h_{bc})$$

with the sectional curvature $K_0 = 1$. Actually, Eq. (2.10) was deduced in slightly more general conditions, i.e. for a manifold N with constant sectional curvature .

Using Eq.s (2.8) and (2.10) we get

$$(2.11)\quad (1+f)\int_M |\rho|^2\sqrt{g}\, d^4x - \int_M R^2\sqrt{g}\, d^4x \leqslant 0$$

In order to obtain a bound on f, we shall use a general inequality for a m-dimensional Riemann manifold [15]

$$(2.12)\qquad m\,|\rho|^2 \geqslant R^2$$

The equality holds in the above equation if and only if $M$ is an Einstein space

(2.13) $$R_{\mu\nu} = \frac{R}{m} g_{\mu\nu} .$$

From Eqs. (2.11) and (2.12) we obtain for $m = 4$

(2.14) $$f \leqslant 3$$

assuming that the Ricci tensor $R_{\mu\nu} \neq 0$ (In fact if $R_{\mu\nu} = 0$ the model is trivial). We remark that the bound (2.14) was deduced for any dimensions of the final sphere $S^{n-1}$. On the other hand this bound is reached for an Einstein manifold $M$ and a totally geodesic map from that manifold to a sphere. That is exactly the instanton solution found in paper [2].

Now, if the cosmological constant is taken into ac - count for $Q(d\varphi)$ we obtain a more involved expression :

(2.15) $$Q(d\varphi) = \\ = \frac{1}{f^2}\left[-\frac{27}{4}\lambda^4 + \lambda^2 R\left(\frac{3}{4}f - \frac{9}{2}\right) + |s|^2(1+f) - R^2\right]$$

Of course, we can use condition (2.8), but a more transparent result is obtained using Eq. (2.9) valid for constant energy density. We get (for $f \geqslant -1$)

(2.16) $$f \leqslant 3 + \frac{9\lambda^2}{R}$$

and the equality holds also for totally geodesic maps and taking $M$ as Einstein space. For an Einstein space, $R < C$

is a sufficient condition to be compact [6, 16] . Hence $f$ decreases when one adds a cosmological term for a compact space with $R < 0$.

Now it is worth discussing about the solutions of Eqs. (2.1), (2.2) involving Einstein spaces. In general an harmonic map does not relate the metrics of the manifolds. A particular class of harmonic maps is represented by the minimal immersions (see Appendix A) for which

$$(2.17) \qquad g_{\mu\nu} = \partial_\mu \Phi^a \partial_\nu \Phi^a$$

and the energy density is $e(\Phi) = 2$. Since the model ( with $\lambda^2 = 0$ ) is invariant under the rescaling (2.3), Eq.( 2.17) can be relaxed requiring only a proportionality between left and right hand sides and consequently Eq. (2.1) will define an an Einstein manifold (2.13). For a compact Einstein manifold there are bounds on the constant of proportionality between $R_{\mu\nu}$ and $g_{\mu\nu}$ [5] . Using these bounds it is possible to deduce a lower bound (a non-positive value) for the coupling constant $f$ in terms of the Euler, signature characteristics and the volume of the manifold.

There is a wellknown construction of minimal immersion of a manifold $M$ into a sphere. Suppose that $\Phi$ realises a minimal immersion in a sphere $r \cdot S^{n-1}$ of radius $r$ in $R^n$. Then $\Phi$ satisfies [17, 18]

$$(2.18) \qquad \Delta \Phi^a = -\mu \Phi^a \qquad , a = 1, 2, \ldots, n$$

with $\mu = (\dim M)/r^2 = 4/r^2$. Wishing to obtain finally a sphere of radius 1 (Eq. (1.2)) we must rescale properly the metric

$$\tilde{g}_{\mu\nu} = \frac{1}{r^2} g_{\mu\nu} = \frac{\mu}{4} g_{\mu\nu} \tag{2.19}$$

such that Eq.(2.18) becomes

$$\tilde{\Delta} \Phi^a = -4 \Phi^a \tag{2.20}$$

From Eq.(2.1) with $\lambda^2 = 0$ we get

$$-4f = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} = \tilde{g}^{\mu\nu} R_{\mu\nu} = \frac{4}{\mu} g^{\mu\nu} R_{\mu\nu} = \frac{4}{\mu} R \tag{2.21}$$

so that

$$f = -\frac{R}{\mu} \tag{2.22}$$

which expresses the coupling constant $f$ in terms of the scalar curvature $R$ and an eigenvalue $\mu$ of the Laplace - Beltrami operator on the manifold M. The dimension of the sphere $S^{n-1}$ is related to the multiplicity (n) of that eigenvalue $\mu$ . As it was expected, $f$ is given in an invariant way, any rescaling of the metric being cancelled in the above ratio.

We mention that as a rule this construction can be applied to any Einstein compact, homogeneous manifold M since it admits a minimal immersion into some Euclidean sphere [19] , but in what follows we shall restrict ourselves to simple manifolds as spheres, tori ,complex projective spaces.

## 3. SIMPLE EXPLICIT SOLUTIONS

In this section we shall present a few sets of solutions of Eqs. (2.1), (2.2) for the field $\Phi$ and the corresponding metric field $g_{\mu\nu}$, beginning with those of finite action.

The first known example is the instanton solution of de Alfaro et al [2] for a sigma model with an internal $\mathcal{O}(5)$ invariance. The manifold M is chosen to be $S^4$ and the map $\Phi$ is the identity between two $S^4$ spheres. One can parametrize the $S^4$ sphere using the stereographic projection on $\mathbb{R}^4$.

$$\xi^\mu = \frac{2 b x^\mu}{b^2 + x^2} \quad , \mu = 1,2,3,4 \tag{3.1}$$
$$\xi^5 = \frac{b^2 - x^2}{b^2 + x^2}$$

and the field $\Phi$ being the identity map

$$\Phi^a = \xi^a \quad , \quad a = 1,2,\ldots 5 \quad . \tag{3.2}$$

We can choose the same metric on both $S^4$ spheres and the identity map realizes a minimal immersion. Due to rescaling freedom (2.3) we can put on the initial $S^4$ sphere a metric proportional to the final one

$$g_{\mu\nu} = c^2 \frac{4 b^2}{(b^2 + x^2)^2} \delta_{\mu\nu} \tag{3.3}$$

and a simple calculation gives $f = 3$. This is in agreement with our observation from previous section concerning the bound (2.14). Indeed the identity map (3.2) is a totally geodesic map and, with the metric (3.3), the $S^4$ sphere is an Einstein manifold.

If a cosmological term is taken into account then from (2.16) one gets

$$f = 3 + \frac{9\lambda^2}{R} \tag{3.4}$$

where R is the scalar curvature of the initial $S^4$ sphere. Of course the action is finite and the degree of the map is one (the topological characteristic of this instanton solution).

Having in mind that the eigenvalues of the Laplacian on $S^4$ are

$$\mu_k = k(k+3) \quad , \quad k \geq 1 \; , \; k \in \mathbb{Z} \tag{3.5}$$

with the multiplicites

$$m_k = \frac{(k+2)!}{k!\,3!}(2k+3) \tag{3.6}$$

we can consider minimal immersions of the sphere $S^4$ in $S^{m_k-1}$. The model will have an internal $\mathcal{O}(m_k)$ invariance and the coupling f will be given by (2.22) (for $\lambda^2 = 0$).

$$f = \frac{12}{k(k+3)} \tag{3.7}$$

being a positive rational number smaller than 3.

In what follows we shall study other finite action solutions trying to construct new minimal immersions between four dimensional manifolds M and various $S^{n-1}$ spheres.

A class of solutions will be of the form $M = M_1 \times M_2$ where $M_i$ is a sphere or a torus and $\varphi_i : M_i \to S_i^{p_i} \subset \mathbb{R}^{p_i+1}$ will be minimal isometric immersions. Then the field $\Phi$ will be constructed as a mapping $\Phi : M_1 \times M_2 \to S^{p_1+p_2+1} \subset \mathbb{R}^{p_1+p_2+2}$ (which corresponds to an internal $O(p_1+p_2+2)$ invariance ) setting [18]

$$\Phi(u,v) = (\cos\theta\ \varphi_1(u),\ \sin\theta\ \varphi_2(v)) \tag{3.8}$$

where $u$ and $v$ are points in $M_1$ and $M_2$ respectively and $\theta$ is a real constant. Since we shall restrict ourselves to final spheres of not too high dimensions we shall choose in our constructions the lowest possible $p_i$ - dimensions.

For example we can take $M = S^2 \times S^2$ , $\varphi_i$ being the identity maps. Denoting by $(\alpha, \beta)$ and $(\gamma, \delta)$ the usual angular coordinates on the corresponding $S^2$ spheres, $\Phi$ can be parametrized as

$$\begin{aligned}\Phi(\alpha,\beta,\gamma,\delta) = (&\cos\theta\cos\alpha\ , \cos\theta\sin\alpha\cos\beta, \cos\theta\sin\alpha\sin\beta, \\ &\sin\theta\cos\gamma, \sin\theta\sin\gamma\cos\delta, \sin\theta\sin\gamma\sin\delta)\end{aligned} \tag{3.9}$$

From Eq. (2.1) we have

$$\begin{aligned} -1 + \frac{3}{4}\lambda^2 c_1^2 &= -f\cos^2\theta \\ -1 + \frac{3}{4}\lambda^2 c_2^2 &= -f\sin^2\theta \end{aligned} \tag{3.10}$$

where $c_1^2$ and $c_2^2$ are two constants multiplying the standard metrics of the initial $S^2$ spheres. Using Eq. (2.2) we get

$$\sin^2\theta = \cos^2\theta = 1/2$$

$$c_1^2 = c_2^2 = c^2$$

so that

$$f = 2 - \frac{3}{2}\lambda^2 c^2 \tag{3.11}$$

In this example we can take $\lambda^2 = 0$ ( $f = 2$) since $M = S^2 \times S^2$ is an Einstein space and the field $\Phi$ realizes, up to a multiplicative constant, a minimal immersion.

The next example, is $M = S^3 \times S^1$ and $\Phi : M \to S^5$ choosing in Eq. (3.8) $\varphi_1 : S^3 \to S^3$ as the identity map and $\varphi_2 : S^1 \to S^1$ as a polynomial map $z \to z^n$. In a similar manner as above we get

$$f = \frac{2}{\cos^2\theta - 3\sin^2\theta}$$

$$\frac{3}{4}\lambda \quad \frac{2\sin^2\theta}{c^2(3\sin^2\theta - \cos^2\theta)}$$

which gives a relation between $f$ and $\lambda^2$

$$f = 2 - 3\lambda^2 c^2$$

In the above equations the constant $C^2$ is related to the constants $c_3^2$ and $C_1^2$ which can multiply the standard metrics on $S^3$ and $S^1$ respectively as follows

$$c_3^2 = 3c^2 \quad , \quad c_1^2 = n^2 c^2 \quad .$$

In this example $M = S^3 \times S^1$ is not an Einstein manifold so that $\lambda^2$ can not be put equal to zero. In fact we use a $\lambda^2 \neq 0$

just to compensate the vanishing of that part of the Ricci tensor which corresponds to the sphere $S^1$.

Another example is $M = T^2 \times S^2$ where $T^2$ is the flat torus $S^1 \times S^1$ and $\Phi$ is the map $\Phi : T^2 \times S^2 \to S^6$ realized as above with $\varphi_1 : T^2 \to S^3$ as a minimal immersion of a torus in $S^3$ [18] and $\varphi_2 : S^2 \to S^2$ as the identity map. Also this manifold is not Einstein and the cosmological term must be present. Denoting as above by $C^2$ the constant which can multiply the standard metrics on $T^2$ and $S^2$ we get

$$f = \frac{1}{\sin^2\theta - \cos^2\theta}$$

$$\frac{3}{4}\lambda^2 = \frac{\cos^2\theta}{c^2(\cos^2\theta - \sin^2\theta)}$$

with the obvious relation

$$f = 1 - \frac{3}{2}\lambda^2 c^2 \tag{3.13}$$

Many other examples can be constructed as above , involving final spheres of higher dimensions. For example, we can minimally immerse $S^1 \times S^1 \times S^1 \times S^1$ into $S^7$ which gives of course, a vanishing Ricci tensor and a $f \neq 0$ can be accomodated choosing an appropiate $\lambda^2 \neq 0$. A more interesting minimal immersion into $S^7$ is the Veronese mapping [7] $\Phi : CP^2 \to S^7$. The dimension 7 of the final sphere is the lowest possible dimension for minimal immersion of $CP^2$ into $S^n$ due to the fact that the multiplicity of the first eigenvalue of the Laplacian on $CP^2$ is 8 . The calculation of $f$ can be done directly or applying Eq. (2.22) getting $f = 2$. If we add a cosmological term we have

$$f = 2 - \frac{3}{8}\lambda^2 c^2 \tag{3.14}$$

where $C^2$ is a constant which can multiply the standard metric on $CP^2$

In all above examples the metric on the initial four dimensional manifold $M$ was obtained (up to a possible re - scaling) as the pull-back of the Euclidean metric of the final sphere $S^{n-1}$. If $M$ is an Einstein manifold we have solutions with and without the constant $\lambda^2$. But if the manifold $M$ is not an Einstein space we found solutions only for $\lambda^2 \neq 0$.

We note that the model supports the solution $\Phi^a =$ constant for which Eq. (2.2) is satisfied trivially. Then Eq. (2.1) becomes

$$R_{\mu\nu} + \frac{3}{4}\lambda^2 g_{\mu\nu} = 0 \tag{3.15}$$

which will be obviously satisfied by any four dimensional Einstein space. Of course for $\lambda^2 = 0$ any vacuum solutions is allowed.

Finally, we shall discuss a solution of the meron type for which the action is not finite. Such a solution can be realized by applying the manifold $M$ on a three dimensional sphere $S^3 \subset S^4$. The simplest procedure is to start with a conformally flat space $M$ :

$$g_{\mu\nu}(x) = e^{2\sigma(x^2)}\delta_{\mu\nu} \quad , \quad \mu,\nu = 1,2,3,4 \tag{3.16}$$

and to make a Riemannian harmonic submersion $\varphi$ $M \to S^3$

$$x_\mu \longrightarrow \frac{x_\mu}{|x|} \tag{3.17}$$

followed by the identity map $I : S^3 \to S^3$. Then the field will be defined as the composition of these two maps $\Phi = I \cdot \varphi$. We must remark that this composition of maps will satisfy formally Eq. (2.2) but it is not a proper harmonic map since the energy of the map is not finite.

In order to determine the function $\sigma(x^2)$ from Eq. (3.16) we shall use Eq. (2.1). For this purpose it is necessary to evaluate the Ricci tensor for a conformally flat space [14] :

$$\begin{aligned} R_{\mu\nu} = {} & \delta_{\mu\nu}\left[12\sigma' + 4x^2\sigma'' + 8x^2(\sigma')^2\right] + \\ & + x_\mu x_\nu\left[8\sigma'' - 8(\sigma')^2\right] \end{aligned} \tag{3.18}$$

where $\sigma' = \dfrac{d\sigma(x^2)}{dx^2}$, $\sigma'' = \dfrac{d^2\sigma(x^2)}{(dx^2)^2}$. From Eq. (2.1) we get

$$\begin{aligned} & 8\sigma'' - 8\sigma'^2 = \frac{f}{(x^2)^2} \\ & \frac{12\sigma'}{x^2} + 4\sigma'' + 8\sigma'^2 + \frac{3}{4x^2}\lambda^2 e^{2\sigma} = -\frac{f}{x^2} \end{aligned} \tag{3.19}$$

The solution of this system of equations will contain two arbitrary positive constants a, b with appropriate dimensions such that dim $g_{\mu\nu}$ = (length)$^{-2}$ :

$$g_{\mu\nu} = \frac{a\,(x^2)^{d}}{[(x^2)^{d+1} - b]^2}\,\delta_{\mu\nu}$$

(3.20)

$$f = 2 - \frac{a}{8b}\,\lambda^2$$

where

(3.21) $$d = -1 \pm [1 - f/2]^{1/2}$$

We note tha we mu t have $f \leqslant 2$ in order to obtain a r al metric For $f = 2$ we re over the meron sol tion f om [ ] with $\lambda^2 = 0$, a and b d n s.

## 4. CONCLUSIONS

This paper can be considered as an illustration of the utility of the harmonic maps in the study of some non - linear field equations which appears in physical theories. The harmonic maps turn out to be of real utility to understand better some of the nonlinearities that occur in Einstein ec ations for general relativity.

We were able to find some restrictions on the coupling between the metric tensor and the sigma field. Also we expressed this coupling for a class of solutions involving Einstein spaces in terms of geometrical quantities. Since these results do not depend on the dimension of the sphere in which the field takes values, we were tempted to consider higher $\mathcal{C}^{\sigma}(n)$ ($n \geqslant 5$) invariance models. We found that such models support complicated topologies for the Euclidean four dimensional space M.

S ch kind o spaces appears also in other related problems in physi like gravitational instantons, spacetime foam, odel , etc . The fact hat we und that ther are, in pr ciple other soluti th $f < 3$ means that the model is flexib so that there i a hope to ob other solu - tions with th coupling F in the ty ca r of high energy physics.

From this point of view the model see deserve attentio Of course the met used in this paper can be extended to other models for wh ch the field $\Phi$ takes values in a more complicated space [20] .

## Appendix A : ELEMENTS O HARMONIC MAPS

Let M N be s ooth man folds of dimensions m, n equiped wi Rieman ian me ics g, h respectively. Let $(x^1, \ldots x$ $(y^1, \ldots, y$ de te smoth local coordinates on .. N. In es ordinate we have

$$ds_M^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, \ldots, m) \tag{A.1}$$
$$ds_N^2 = h_{ab}\, dy^a dy^b \qquad (a, b = 1, \ldots, n)$$

and the usual Christoffel symbols of M and N will be denoted by ${}^M\Gamma^\gamma_{\alpha\beta}$ , ${}^N\Gamma^c_{ab}$ respectively.

Given a smooth map $\varphi : (M, g) \to (N, h)$ the first (the pull-back of the metric $h_{ab}$) and second fundamental forms of $\varphi$ are :

(A.2) $$(\varphi^* h)_{\alpha\beta} = \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta} h_{ab}$$

(A.3) $$(\nabla(d\varphi))^c_{\alpha\beta} = \frac{\partial^2 \varphi^c}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta} \frac{\partial \varphi^c}{\partial x^\gamma} + {}^N\Gamma^c_{ab} \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta}$$

where $\nabla$ is the covariant differentiation on M.

If the metrics g, h on M, N are related by $\varphi^* h = g$ the map $\varphi$ is a Riemannian immersion. A map with vanishing second fundamental form is said to be totally geodesic.

The energy of the map $\varphi$ is defined by the formula (a generalization of the classical integral of Dirichlet )

(A.4) $$E(\varphi) = \int_M e(\varphi)(x) \sqrt{g}\, d^m x$$

where the energy density is

(A.5) $$e(\varphi) = \frac{1}{2} \operatorname{Trace}(\varphi^* h)(x) = \frac{1}{2} g^{\alpha\beta} h_{ab} \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta}$$

The map $\varphi$ is called harmonic if it is an extremal of the energy integral $E(\varphi)$. The corresponding Euler - Lagrange equations are

(A.6) $$\begin{aligned} \tau(\varphi)^c &= g^{\alpha\beta} (\nabla(d\varphi))^c_{\alpha\beta} = \\ &= \Delta \varphi^c + {}^N\Gamma^c_{ab} \frac{\partial \varphi^a}{\partial x^\alpha} \frac{\partial \varphi^b}{\partial x^\beta} g^{\alpha\beta} = 0 \end{aligned}$$

where $\Delta$ is the Laplace-Beltrami operator and $\tau(\varphi)$ is called the tension field of $\varphi$ . If $\varphi$ is an harmonic map and an isometric immersion then $\varphi$ is called minimal immersion.

The minimal immersions are the extremals of the volume functional

(A.7) $$V(\varphi) = \int_M |\det(\varphi^* h)|^{1/2} d^m x$$

It is useful to consider the composition of two maps $\varphi : M \to M'$ and $\psi : M' \to N$ and to calculate the tension from Eq. (A.6)

(A.8) $$\tau(\psi \circ \varphi) = d\psi \cdot \tau(\varphi) + \text{Trace } \nabla d\psi(d\varphi, d\varphi)$$

or explicitely in coordinates :

$$g^{\alpha\beta}\left[\frac{\partial^2(\psi\circ\varphi)^a}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta}\frac{\partial(\psi\circ\varphi)^a}{\partial x^\gamma} + {}^N\Gamma^a_{bc}\frac{\partial(\psi\circ\varphi)^b}{\partial x^\alpha}\frac{\partial(\psi\circ\varphi)^c}{\partial x^\beta}\right] =$$

(A.9)
$$= g^{\alpha\beta}\frac{\partial\psi^a}{\partial\varphi^i}\left[\frac{\partial^2\varphi^i}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^i}{\partial x^\gamma} + {}^{M'}\Gamma^i_{jk}\frac{\partial\varphi^j}{\partial x^\alpha}\frac{\partial\varphi^k}{\partial x^\beta}\right] +$$

$$+ g^{\alpha\beta}\left[\frac{\partial^2\psi^a}{\partial\varphi^j \partial\varphi^k} - {}^{M'}\Gamma^i_{jk}\frac{\partial\psi^a}{\partial\varphi^i} + {}^N\Gamma^a_{bc}\frac{\partial\psi^b}{\partial\varphi^j}\frac{\partial\psi^c}{\partial\varphi^k}\right]\frac{\partial\varphi^j}{\partial x^\alpha}\frac{\partial\varphi^k}{\partial x^\beta}$$

We are interested to study maps between a manifold M and a sphere $S^{n-1}$ which is isometrically immersed in $\mathbb{R}^n$. In that case $\psi$ is the immersion $\psi : S^{n-1} \to \mathbb{R}^n$ and assuming that $\varphi : M \to S^{n-1}$ is harmonic we obtain from Eqs. (A.6),(A.8) and (A.9) :

$$\Delta(\psi\circ\varphi)^a = g^{\alpha\beta}\left[\frac{\partial^2\psi^a}{\partial\varphi^j\partial\varphi^k} - {}^{M'}\Gamma^i_{jk}\frac{\partial\psi^a}{\partial\varphi^i}\right]\frac{\partial\varphi^j}{\partial x^\alpha}\frac{\partial\varphi^k}{\partial x^\beta} = \tag{A.10}$$

$$= -(\psi\circ\varphi)^a g^{\alpha\beta} h_{jk}\frac{\partial\varphi^j}{\partial x^\alpha}\frac{\partial\varphi^k}{\partial x^\beta} =$$

$$= -2e(\varphi)\cdot(\psi\circ\varphi)^a = -2e(\psi\circ\varphi)\cdot(\psi\circ\varphi)^a$$

Moreover it is possible to prove that $\varphi : M \quad S^{n-1}$ is harmonic if and only if Eq. (A.7) holds.

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CENTRAL INSTITUTE OF PHYSICS
Documentation Office
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