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**INSTITUTUL CENTRAL DE FIZICĂ**

BUCUREȘTI-MĂGURELE

ROMANIA

CENTRAL INSTITUTE OF PHYSICS
INSTITUTE FOR PHYSICS AND NUCLEAR ENGINEERING
Bucharest, POB MG-6, ROMANIA

IFIN -
FT-194-1980 August

# Hartree - Fock - Bogoliubov Approximation for Finite Systems

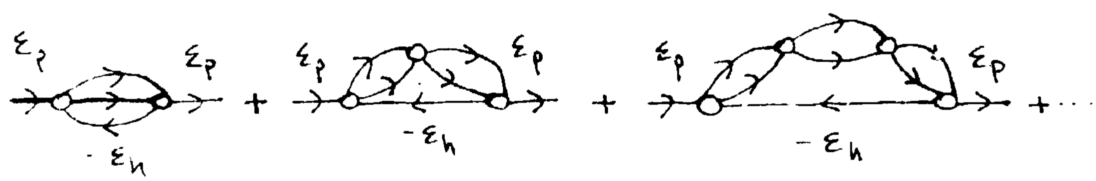
A.Bulgac

*Abstract :* The features of the spectrum of the Hartree-Fock-Bogoliubov equations are examined. Special attention is paid to the asymptotic behaviours of the single quasiparticle wave functions (s.qp.w.fs.), matter density distribution and density of the pair condensate. It is shown that, due to the coupling between hole and particle, the sufficiently deeply bound hole states acquire a width and consequently have to be treated as continuum states. The proper normalization of the s.qp w.fs. is discussed.

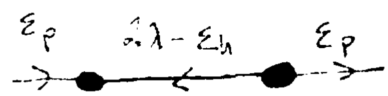
## Introduction

The Hartree-Fock-Bogoliubov (HFB) approximation was outlined more than twenty years ago[1] for infinite systems and almost immediately it was introduced in nuclear physics[2]. In the case of infinite systems the HFB procedure is well studied and the character of the wave-functions (w.f) is well understood. However, in the case of finite systems (nuclei) the things are not so clear. The HFB eqs. in the case of nuclei are meaningfull provided the boundary conditions for the single quasiparticle (s.qp.) w.fs. are correctly stated in such a way as to describe a genuine finite state. In fact the aim of the present paper is the correct formulation of the HFB approximation in the case of finite systems.

As it is well known, pairing correlations appear always whenever a pole (i.e. a bound state) is present in the two-body Green function of the many-body system[3]. In this case, higher corrections to the single particle (s.p.) Green function of the type



give rise to diagrams of the type

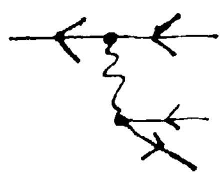


(1)

when only the contribution of the pole is taken into account[3]. Here $2\lambda$ stands for the energy of the two-particle bound state and $\varepsilon_p$ and $\varepsilon_h$ are the particle and hole energies respectively. The process represented by diagram (1) leads to a mixing between particle and hole states and therefore to a smearing of the Fermi surface[3-5]. This mixing has the special feature that a hole (particle) can transform into a particle (hole), due to the presence of the pair condensate, provided /their energies are related by the energy conservation law

$$\varepsilon_p + \varepsilon_h = 2\lambda$$

Whenever the energy of the hole state is less than $2\lambda$ the corresponding particle state to which the hole state is coupled lies in continuum. This situation resembles formally the case of an electron in a very strong field[6] or the case of a bound state embedded in continuum[7]. Consequently, deeply enough bound hole state becomes unstable with respect to the decay into a particle state by an interaction with the pair condensate. The same thing happens e.g. in the case of a deeply bound hole decaying into a less deeply bound hole with the excitation of a phonon, which can further decay by particle emission, i.e. the diagram



where the wavy line represents a phonon.
Therefore the physical situation is not new. New is the fact that due to the presence of the pair condensate some hole states acquire a width and therefore the s.qp.w.fs. are to be treated not anymore as corresponding to bound states, as usually people do, but as corresponding to continuum states. How to introduce this property of the s.qp.w.fs. into the HFB approximation for finite system and therefore how to define correctly the boundary conditions is the main goal of the present discussion.

## 2. HFB equations for finite systems

In this section the well known HFB eqs. will be derived for the sake of completeness. The stress will be made upon the correct definition or s.p.w.fs. so as to describe genuine finite systems i.e. systems with finite matter distribution. The forces between particles will not be specified except for some of their general properties, e.g. the finite range.

By analogy with the usual HF approximation, the HFB ground state w.f. $|0\rangle$ is defined as a vacuum for the Fermi quasiparticles[1,2,4,5]

$$\alpha_i |0\rangle = 0 , \qquad \langle 0|0\rangle = 1 , \tag{2}$$

where

$$\alpha_i = \int dx \{ u_i^*(x)\psi(x) + v_i(x)\psi^+(x) \} , \tag{3}$$

$$\alpha_i^+ = \int dx \{ v_i^*(x)\psi(x) + u_i(x)\psi^+(x) \} , \tag{3'}$$

and $\psi(x)$ and $\psi^+(x)$ stand for field operators for anihilation and creation of a particle with space-spin coordinates $x = \{\vec{r}, \sigma\}$ and obeving the usual anticomutation relations

$$\{ \psi(x), \psi^+(y) \} = \delta(x-y) , \tag{4}$$

$$\{ \psi(x), \psi(y) \} = \{ \psi^+(x), \psi^+(y) \} = 0 . \tag{4'}$$

Requiring for $\alpha_i$ and $\alpha_i^+$ to represent fermion operators, i.e.

$$\{ \alpha_i , \alpha_j^+ \} = \delta_{ij}$$

$$\{\alpha_i, \alpha_j\} = \{\alpha_i^+, \alpha_j^+\} = 0 \quad ,$$

one easily obtains the relations

$$\int \{u_i^*(x) u_j(x) + v_i(x) v_j^*(x)\} dx = \delta_{ij} \quad , \tag{5}$$

$$\int \{u_i^*(x) v_j(x) + v_i(x) u_j^*(x)\} dx = 0 \quad , \tag{5'}$$

$$\sum_i u_i(x) u_i^*(y) + v_i(x) v_i^*(y) = \delta(x-y) \quad , \tag{6}$$

$$\sum_i u_i(x) v_i(y) + v_i(x) u_i(y) = 0 \tag{6'}$$

The contrains (5), (5'), (6), (6') assure the unitary character of the transformations (3), (3').

The total energy of the many-body system and the mean number of particles are, respectively

$$\mathcal{E} = \langle 0 | \hat{H} | 0 \rangle$$

and

$$N = \langle 0 | \hat{N} | 0 \rangle$$

where $\hat{H}$ and $\hat{N} = \int dx\, \psi^+(x) \psi(x)$ stand for the hamiltonian and the number operator in the second quantization representation.

The mean values of the energy and particle number can be expressed through the densities

$$\rho(x,y) = \langle 0 | \psi^+(y) \psi(x) | 0 \rangle = \rho^*(y,x) = \sum_i v_i(x) v_i^*(y) \tag{7}$$

$$\phi(x,y) = \langle 0|\psi(y)\psi(x)|0\rangle = -\phi(y,x) = \sum_i v_i(x)u_i(y) \qquad (8)$$

in the following way

$$\mathcal{E} = \mathrm{Tr}(T\rho) + \frac{1}{2}\mathrm{Tr}(V\rho\rho)_a + \frac{1}{2}\mathrm{Tr}(V\phi\phi^*) \qquad (9)$$

$$N = \mathrm{Tr}(\rho) \qquad (10)$$

if only two-body interactions are present. T and V stand for the kinetic energy and the two-body interaction respectively. A short hand notation was used for the traces in rels.(9) and (1o).

The HFB eqs. for the two component s.qp.w.fs. $\{v_i(x), u_i^*(x)\}$ are deduced from the stationary condition of the total energy (9) under the subsidiary conditions (1o), (5) and (5 ). These equations are

$$\int h(x,y)v_i(y)\,dy - \lambda v_i(x) + \int \Delta(x,y)u_i^*(y)\,dy = E_i v_i(x) \qquad (11)$$

$$\int \Delta^{\dagger}(x,y)v_i(y)\,dy - \int h^*(x,y)u_i^*(y)\,dy + \lambda u_i^*(x) = E_i u_i^*(x) \qquad (12)$$

where

$$\lambda = \frac{\delta\mathcal{E}}{\delta N} < 0$$

is the chemical potential, and

$$h(x,y) = \frac{\delta\mathcal{E}}{\delta\rho(x,y)} = h^*(y,x)$$

and

$$\Delta(x,y) = \frac{\delta \mathcal{E}}{\delta \phi^*(x,y)} = - \Delta(y,x)$$

are the s.p. hamiltonian and pairing field respectively and $E_i$ stands for the s.qp. energies.

While effecting the summations in rels.(7) and (8) one must include only those solutions of the nonlinear system (11), (12) with $E_i < o$ which define the operators $\alpha_i$ . The solution with $E_i > o$ correspond to operators $\alpha_i^+$

Let us now analyse in more detail the eqs, in the case of a finite system. The problem which arises is the meaning of the normalization condition (5), namely should the right hand side of rel. (5) be a $\delta$ - function or a Kronecker symbol, as it is ussually treated[2,5,8,9] in complete analogy with the HF approximation.

A finite system is characterized by a finite matter distribution and therefore by a finite range of the s.p. field (if one disregards the Coulomb interaction). Naturally, being determined by the matter distribution, the pairing field must also have a finite range. (An infinite range of the pairing field can only occur if the considered system is unstable with respect to two particle decay.) The problem is what is the mechanism which leads to finite matter distribution when s.qp. states are not anymore bound as it was suggested in introduction.

When pairing is turned off (i.e. $\Delta \equiv 0$ ) the spectrum of eqs.(11), (12) looks like the one plotted in fig.1. The left hand side part corresponds to the spectrum of eq.(11) while the right hand side part to the spectrum corresponding to eq.(12).

When turning on pairing the two spectra mix together and the discrete states with energies outside the interval

$$\lambda < E < -\lambda \qquad (13)$$

will lie in continuum. Only the states with energy within the interval (13) will preserve the bound state character.

The continuum part of the spectrum for $E < \lambda$ and $E > \lambda$ can dissapear only if the s.p. potential is infinite and positive at infinity as in the case of the harmonic oscillator$^{+}$. There is no physical reason for this to be true in the case of finite systems.

By simply looking at fig.1 one can qualitatively understand why pairing correlations lead to a significant increase of the level density in the vicinity of the Fermi surface. Pairing correlations come into play generally when the chemical potential occurs within a shell as in fig.1. In such a case the total number of hole (the left hand side part of fig.1) and particle (the right hand side part of fig.1) states, which is practically equal to the number of s.qp. states with pairing included, is almost double the number of s.p. states in a simple HF approximation.

To see what happens with a bound state with an energy E when pairing is turned on we shall treat this case using the perturbation theory. For the sake of simplicity the nucleus will be assumed to be spherical and $\Delta \equiv \Delta^{*}$ . Also the spin and angular variables are assumed to be already separated from eqs.(11) and (12). From eq.(11) one easily obtains in the vicitity of a bound the relation

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+The discrete character of the spectrum for a limited number of states with $E<\lambda\,(E>-\lambda)$ can be also preserved if $[h,\Delta]=0$, a condition which does not hold in fact. E.g. this condition is fulfilled in the case of constant pairing approximation ( $\Delta \equiv$ const. ), an approximation which leads to an unphysical density distribution.

$$v_E(r) = \left(\frac{1}{E+\lambda-h}\Delta u_E\right)(r) \cong \varphi_0(r)\frac{\langle\varphi_0|\Delta|u_E\rangle}{E-E} = n^{1/2}(E)\varphi_0(r), \tag{14}$$

where

$$(h-\lambda-E)\varphi_0 = 0 \quad , \quad \langle\varphi_0|\varphi_0\rangle = 1 \quad .$$

The other component of the s.qp.w.f. becomes then

$$u_E(r) = C(E)u_{0E}(r) - n^{1/2}(E)\left(\frac{1}{-E+\lambda-h}\Delta\varphi_0\right)(r) \tag{15}$$

where

$$(h-\lambda+E)u_{0E} = 0 \quad , \quad \langle u_{0E}|u_{0E'}\rangle = \delta(E-E') \tag{16}$$

and O(E) is a normalization constant. From rels.(14) and (15) one obtains

$$n^{1/2}(E) = \frac{C(E)\langle\varphi_0|\Delta|u_{0E}\rangle}{E-E_0+\langle\varphi_0|\Delta\frac{1}{-E+\lambda-h}\Delta|\varphi_0\rangle}$$

In order to determine the normalization constant C(E) we shall use the representation of the Green function through regular a and iregular solutions of eq.(16)

$$G(r,r',-E+\lambda) = \left(\frac{1}{-E+\lambda-h}\right)(r,r') = \frac{2m}{\hbar^2}\frac{u_{0E}(r_<)\chi_E(r_>)}{W(u_{0E},\chi_E)}$$

where $W(u_{0E},\chi_E)$ stands for the Wronskian

$$W(u_{0E},\chi_E) = \frac{2m}{\pi\hbar^2}$$

The asymptotic behaviour for regular $u_{0E}$ and irregular $\chi_E$ solutions is

$$u_{0E}(r) \cong \sqrt{\frac{2m}{\hbar^2 \pi k}} \sin(kr + \delta_{0E}) ,$$

$$\chi_E(r) \cong \sqrt{\frac{2m}{\hbar^2 \pi k}} \cos(kr + \delta_{0E}) ,$$

where $k^2 = (-E+\lambda)\frac{2m}{\hbar^2}$. The asymptotic behaviour of the $u$-component is

$$u_E(r) \cong \sqrt{\frac{2m}{\hbar^2 \pi k}} \left[ C(E) \sin(kr + \delta_{0E}) + \pi\, n^{1/2}(E) \langle \varphi_0 | \Delta | u_{0E} \rangle \cos(kr + \delta_{0E}) \right]$$

and therefore

$$C^2(E) = \frac{\left[ E - E_0 + \langle \varphi_0 | \Delta \frac{1}{-E+\lambda-h} \Delta | \varphi_0 \rangle \right]^2}{\left[ E - E_0 + \langle \varphi_0 | \Delta \frac{1}{-E+\lambda-h} \Delta | \varphi_0 \rangle \right]^2 + \pi^2 |\langle \varphi_0 | \Delta | u_{0E} \rangle|^4} \qquad (17)$$

Consequently, the normalized solutions are

$$v_E(r) = n^{1/2}(E)\, \varphi_0(r) , \qquad (18)$$

$$u_E(r) = C(E)\, u_{0E}(r) - n^{1/2}(E) \left( \frac{1}{-E+\lambda-h} \Delta \varphi_0 \right)(r) , \qquad (19)$$

where

$$n(E) = \frac{1}{\pi} \frac{\frac{1}{2}\Gamma(E)}{(E - E_0 - \delta E)^2 + \frac{1}{4}\Gamma^2(E)} , \qquad (20)$$

$$\Gamma(E) = 2\pi\, |\langle \varphi_0 | \Delta | u_{0E} \rangle|^2 = 2\pi \frac{|\langle \varphi_0 | [h, \Delta] | u_{0E} \rangle|^2}{4E^2} ,$$

$$\delta E = - \langle \varphi_0 | \Delta \frac{1}{-E+\lambda-h} \Delta | \varphi_0 \rangle$$

and

$$\int_{-\infty}^{\infty} n(E)\, dE = 1 \qquad (21)$$

The matrix elements involved in the above relations can be calculated without any significant loss of accuracy for $E = E_o$.

One can remark, that due to the coupling with continuum, the bound state spreads over the whole spectrum (see rel.(21)). The quantity n(E) has to be interpreted as the occupation number probability.

The solution is formally equivalent to the solution of a coupled channel problem with a bound state embedded in the continuum[7] It displays a well defined resonant character with a width $\Gamma(E_o)$ and a shift $\bar{\delta}$ E. The case of a resonant state can be treated in a similar way.

Far from the resonance energy $E_o$ + E the amplitude of the $v$ - component is very small while the u - component is practically equal to the unperturbed solutions $u_{oE}$. In the vicinity of the resonance the amplitude of the $v$ - component rises significantly (see rels.(18), (20)) inside the potential well while the pha the u - component increases by $\pi$ . The phase of the u - component is

$$\delta_E = \delta_{oE} - \operatorname{arctg} \frac{\frac{1}{2}\Gamma(E)}{E - E_o - \delta E}$$

The so obtained solution is characterized by the fact that the $v$ - component is square integrable even though the s.qp.w.f. solution $\{v_E(x), u_E^*(x)\}$ represents a continuum state. The question is: does this feature remain true for the selfconsistent solution of the HFB eqs?

The density distribution $\rho$ satisfies rel.(10) (i.e. the diagonal part of $\rho$ is integrable) while the density of the pair condensate satisfies the condition

$$\int |\phi(x,y)|^2 = \mathrm{Tr}\left(\rho(1-\rho)\right) < N \qquad (22)$$

which can be easily deduced by means of rels.(5), (5'), (6), (6')[1,4,5,8]. Therefore one can expect that $\rho$ and $\phi$ have to fall down quickly enough outside the system. The density distribution $\rho$ determines the asymptotic behaviour of the s.p.selfconsistent potential U while the density $\phi$ defines the pairing potential $\Delta = \frac{1}{2} V \phi^*$

The nonlocality of these potentials is governed by the range of the two-body interaction V assumed finite.

We shall show next that we have

$$U \sim \rho \sim O\left(e^{-2\sqrt{\frac{2m|\lambda|}{\hbar^2}}\, r}\right) \qquad (23)$$

and

$$\Delta \sim \phi \sim O\left(e^{-\sqrt{2\frac{2m|\lambda|}{\hbar^2}}\, r}\right) \qquad (24)$$

when $\vec{R} = \frac{1}{2}(\vec{r}+\vec{r}\,')$ goes to infinity and $\vec{s} = \vec{r}-\vec{r}\,'$ remains finite (practically of the order of the range of V). (In the above rels. (23), (24) the symbol O means that the quantities in the left hand side part behave like the corresponding arguments of the O - function.) As one can remark the pairing field $\Delta$ has a longer tail than the s.p. selfconsistent field U.

Using (23) and (24) one can show using HFB eqs.(11), (12) that the $V$ - and $U$ - components behave asymptoticaly as

$$v(r) \sim e^{-\sqrt{\frac{2m}{\hbar^2}|E+\lambda|}\, r} , \qquad (25)$$

$$u(r) \sim e^{-\sqrt{\frac{2m}{\hbar^2}|E-\lambda|}\, r} , \qquad (26)$$

if

$$\lambda < E < 0 \qquad (27)$$

and

$$v(r) \sim e^{-\sqrt{\frac{2m}{\hbar^2}|2\lambda|}\, r} , \qquad (28)$$

$$u(r) \sim O(1) , \qquad (29)$$

if

$$E < \lambda \qquad (30)$$

Outside the potential well, for energies in the interval (27), the two eqs.(11), (12) decouple and the corresponding asymptotic behaviour of $v$ - and $u$ - components of the s.qp.w.f. is determined by the "energies" $\lambda$ + E and $\lambda$ - E, respectively. For energies in the interval (3o) the asymptotic behaviour of the $v$ - component is governed by the inomogeneous part of the eq.(11) (i.e by the term $\Delta u^*$) which can not be disregarded anymore as we did it for energies in the interval (27). On the other side, the term $\overset{+}{\Delta} v$ falls down exponentially and it does not influence the asymptotic behaviour of the component.

The asymptotic behaviour of the u - component is fully determined by the "energy" - E + $\lambda$ which is positive in the interval (3o).

Now, if one takes into account e definitions of densities $\rho$ and $\phi$ (rels.(7) and (8), respect ely) one can easily remark that the asymptotic behaviours of $v$ - and u - components (see rels.(25) - (30)) completely agree with the asymptotic behaviours (23) and (24). This means that the corresponding asymptotic behaviours are selfconsistent. It is physically natural that the asymptotic behaviour of the density is controlled by the chemical potential, i.e. by the energy of the less bound particle. The density of the pair condensate can be interpreted as the w.f. of a bound /state of two interacting particles in an external field with energy $2\lambda$ . Using HFB eqs.(11) and (12) and the definition (8) one can show that the density $\phi$ satisfies the equation[1']

$$\left(h(1) + h(2) + V(1,2) - 2\lambda\right) \phi(1,2) = \tag{31}$$

= other terms negligible outside the system.

From this eq. it follows that $\phi$ behaves at large distances like

$$e^{-\alpha r_1 - \beta r_2} \quad ,$$

where

$$-\left(\alpha^2 + \beta^2\right) = \frac{2m}{\hbar^2} 2\lambda \quad ,$$

which also agrees with rel.(24).

Properly speaking, this asymptote is correct only outside the range of V. But if one takes into account the fact that in the real case a system of two identical nucleons does not have bound states, the asymptote remains true everywhere outside the s.p. potential weel.

Summing up, in the normalization conditions (5) for the s.qp.w.fs. the right hand side part has to be interpreted as a Kronecker symbol if $\lambda < E < -\lambda$ and as a Dirac $\delta$ - function if $|E| > |\lambda|$ (as it is well known[1-5,8] the system (11), (12) has the property that if $\{v_i, u_i^*, E_i\}$ is a solution then $\{u_i, v_i^*, -E_i\}$ is also a solution.) Furthermore, for E < o the $v$- component of the s.qp.w.f. is always square integrable and its norm has to be interpreted as the occupation number probability. On the other hand the relation

$$1 - n_i = 1 - \int |v_i(x)|^2 dx = \int |u_i(x)|^2 dx$$

holds only for $\lambda < E < -\lambda$ and it does not hold if the energy E is outside this interval.

## 3. Some simple examples

This section is devoted to some simple examples of HFB eqs. which although unrealistic, prove to be instructive for an easier understanding of the problems which can arise when solving eqs.(11), (12).

### a) Constant pairing approximation $\Delta \equiv$ constant

This approximation is known also as BCS approximation[10]. The solution in this case is

$$v(r) = n_{BCS}^{1/2} \varphi(r) ,$$

$$u(r) = \sqrt{1 - n_{BCS}}\, \varphi(r) ,$$

where

$$(h-\varepsilon)\varphi = 0 \quad , \tag{32}$$

$$n_{BCS} = \frac{1}{2}\left(1 - \frac{\varepsilon-\lambda}{\sqrt{(\varepsilon-\lambda)^2 + \Delta^2}}\right)$$

and

$$E = -\sqrt{(\varepsilon-\lambda)^2 + \Delta^2}$$

Usually, the pairing field $\Delta \equiv$ const is taken different from zero in a limited energy band arround the Fermi surface. However, there is no recipe to determine this energy band in a unique way. It is obvious that the density $\rho$ can not be finite if solutions belonging to the continuum part of the spectrum of the eq.(32) are included (i.e., if pairing field is different from zero for such states). Furthermore, if one considers that $\Delta$ is acting for all energies, then the density $\phi$ is

$$\phi(x,y) = \sum_i \varphi_i(x)\varphi_i^*(y)\,\frac{\Delta}{2\sqrt{(\varepsilon_i-\lambda)^2+\Delta^2}} =$$

$$= \frac{\Delta}{2}\sum_i \varphi_i(x)\varphi_i^*(y)\left[\frac{1}{\varepsilon_i-\lambda} - \frac{1}{\sqrt{(\varepsilon_i-\lambda)^2+\Delta^2}}\right] +$$

$$+ \frac{\Delta}{2}\,G(x,y,\lambda)$$

where $G(x,y,\lambda)$ stands for the s.p. Green function of the eq.(32). As it is well known the Green function diverges like $|\vec{r}-\vec{r}\,'|^{-1}$ when $\vec{r}\to\vec{r}\,'$. Therefore, the density $\phi$ can not be defined in this case because of this divergency. One can remark that the divergency is not a logarithmic one as it is usually stated[3,10]. The point is that whenever the pairing field is local, as in this case, it correspond to a zero range two-body interaction. Then, as one can easily establish by analysing eq.(31), the density $\phi$ will always be singular for coinciding arguments. The conclusion is that the two-body interaction which leads to pairing must have a finite range (not zero) in order that the entire HFB procedure to be meaningfull.

<u>b) Square well single particle potential and square well pairing potential</u>

We assume that

$$\mathcal{U}(r) = -\mathcal{U}_0\, \theta(R-r) \quad , \quad \mathcal{U}_0 > 0 \quad ,$$

$$\Delta(r) = \Delta_0\, \theta(R-r) \quad ,$$

and look for solutions of eqs.(11), (12) with l = o.

For $r < R$ the solution reads

$$v_{in}(r) = C_+ \cos k_+ r + C_- \sin k_- r \quad ,$$

$$u_{in}(r) = \beta_+ C_+ \cos k_+ r + \beta_- C_- \sin k_- r \quad ,$$

$$k_\pm^2 = \mathcal{U}_0 + \lambda \pm \sqrt{E^2 - \Delta_0^2} \quad ,$$

where

$$\beta_\pm = \frac{E \mp \sqrt{E^2 - \Delta_0^2}}{\Delta_0} \quad , \quad \beta_+ \beta_- = 1$$

and $C_\pm$ stand for some constants which have to be determined from the matching of the interior with exterior solutions and normalization.

For $r > R$

$$v_{out}(r) = v_0 \, e^{-k_0 r} \quad , \quad \frac{\hbar^2}{2m} k_0^2 = -(E+\lambda) > 0 \quad ,$$

$$u_{out}(r) = u_0 \, e^{-k_0' r} \quad \text{if} \quad \frac{\hbar^2}{2m} k_0'^2 = -(-E+\lambda) > 0$$

or

$$u_{out}(r) = u_1 \sin k_0' r + u_2 \cos k_0' r$$

if

$$\frac{\hbar^2}{2m} k_0'^2 = -E+\lambda > 0$$

where $v_0$, $u_0$, $u_1$ and $u_2$ have to be determined from matching and normalization.

If $\lambda < E < -\lambda$ the spectrum is discrete and the energies have to be determined from the matching condition which reads

$$\frac{\beta_+ \left(k_- \cos k_- R + k_0 \sin k_- R\right)\left(k_+ \cos k_+ R + k_0' \sin k_+ R\right)}{\beta_- \left(k_- \cos k_- R + k_0' \sin k_- R\right)\left(k_+ \cos k_+ R + k_0 \sin k_+ R\right)} = 1 \; .$$

If $E < \lambda$, then one deals with a state lying in continuum. In contrast to the case of constant pairing one has now $[h, \Delta] \neq 0$ and the sufficiently bound hole states get a width.

Inside the potential well the two components of the s.qp.w.f. look like a superposition of two s.p.w.f. with energies equal approximatively to $E + \lambda + U_0$ and $-E + \lambda + U_0$, respectively (as a rule $\Delta_0$ is very small and it can be neglected in the determination of $k_\pm$ ).

In the vicinity of a bound hole state (the left hand side of the fig.1) $C_+$ has a zero and one gets the ratio

$$\frac{v_{in}(r)}{u_{in}(r)} \cong \beta_+ \cong \frac{2E}{\Delta_o} \cong \sqrt{\frac{n_{BCS}}{1-n_{BCS}}} \quad , \; n_{BCS} \lesssim 1$$

as in the BCS approximation. This relation holds only inside the potential well and it is not true outside the potential well.

Far from the resonance this ratio equals

$$\frac{u_{in}(r)}{v_{in}(r)} \cong \beta_+ \cong \frac{2E}{\Delta_o} \cong \sqrt{\frac{1-n_{BCS}}{n_{BCS}}} \quad , \; n_{BCS} \ll 1$$

Unlike the BCS approximation, the radial behaviours of $v$ - and u - components are not anymore identical.

The density $\rho$ has in this case the correct asymptotic behaviour but the density $\phi$ displays the same divergency as in the case of BCS approximation due to the local character of the pairing potential.

<u>c) Surface pairing</u> $\quad \Delta = \Delta_o \, \delta(r-R)$

This is another approximation which has been used in nuclear physics. The solution of eqs.(11), (12) is now

$$v(r) = \Delta_o \, G(r, R, E+\lambda) \, u(R) \quad ,$$

$$u(r) = n \, u_o(r) - \Delta_o^2 \, G(r, R, -E+\lambda) \, G(R, R, E+\lambda) \, u(R) \quad ,$$

where

$$u(R) = \frac{\kappa\, u_0(R)}{1 + \Delta_0^2\, G(R,R,-E+\lambda)\, G(R,R,E+\lambda)}$$

and $\kappa$ is a normalization constant and $u_0$ is given by the equation

$$(h-\lambda+E)\, u_0 = 0$$

($u_0$ has to be included only for $E < \lambda$ )

The density $\rho$ has a correct asymptotic behaviour but the same problems arise in conection with definition of the density $\Phi$ as before. The density of the pair condensate $\Phi$ is singular for $r = r^{\prime} = R$ (see eq.(31)).

Even though the analysed cases have little in common with the real selfconsistent solution of eqs.(11), (12) they help however in our opinion, to get a deeper insight into the structure of the HFB eqs. in the case of finite systems. Especially instructive in this sense is the role played by the nonlocality of the pairing field and consequently by the range of the two-body forces.

## 4. Conclusion

The HFB approximation in the case of finite systems were thoroughly examined when the two-body interaction between particles has a finite range. Special attention was paid to the asymptotic behaviour of the s.qp.w.fs. It was shown that the s.qp. states located in spectrum far enough from the Fermi surface have the character of continuum states. E.g. the deeply enough bound hole states get a width corresponding to the decay into a particle state and the pair condensate. This width has to be interpreted as a contribution to the imaginary part of the s p. optical potential.

Even though the largest part of the s.qp.w.fs. lie in continuum the distribution of the matter is nevertheless finite. The same thing holds also for the density of the pair condensate which is also finite.

These features of the general solutions of the HFB eqs. have to be included in any HFB calculations. To what extent the available HFB results[8,9] correspond to the real solution, this is a problem which needs further studies.

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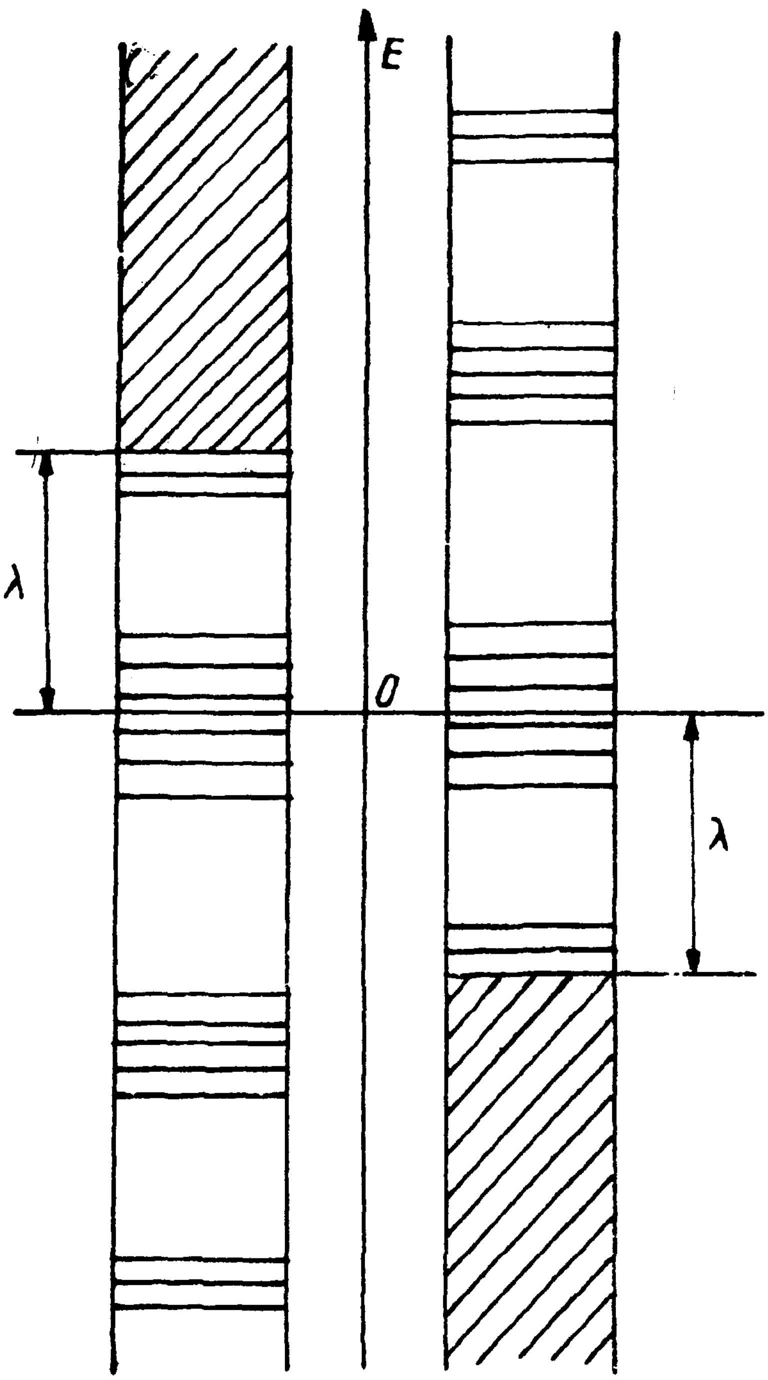


Fig.1. The spectra of eqs.(11) and (12) when pairing field is turned off ($\Delta \equiv o$). The left hand side part corresponds to the spectrum of eq.(11) while the right hand side part to eq.(12) respectively. The two spectra are symmetric about the point O. The hatched regions represent the continuum and the lines correspond to discrete states.

CENTRAL INSTITUTE OF PHYSICS
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