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Constraints on low energy Compton scattering amplitudes

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Abstract : *We derive the constraints and correlations of fairly general type for Compton scattering amplitudes at energies below photoproduction threshold and fixed momentum transfer, following from (an upper bound on) the corresponding differential cross section above photoproduction threshold. The derivation involves the solution of an extremal problem in a certain space of vector - valued analytic functions.*

1. Introduction

During the last several years efforts have been made towards a rigorous phenomenology of Compton scattering /1/. This task is considerably more difficult than e.g. that referring to the pion electromagnetic form factor. The source of these difficulties lies in the facts that Compton scattering is described by more than one amplitude and that these amplitudes depend on two variables. The mathematical problems involved could therefore be formulated as extremal problems for vector-valued analytic functions of two variables. Considerable simplification, although accompanied by a reduction of scope, is achieved if one of the variables is fixed and only analyticity in the other is exploited: one arrives thereby at extremal problems for vector-valued analytic functions of one variable, which may be treated sometimes in considerable detail.

In the present paper we solve one of the problems of this type, namely we derive the constraints implied on (finite sets of) threshold and low energy values of Compton scattering amplitudes by a known value (or only upper bound) of the cross section above the pion photoproduction threshold. Thereby we extend the results obtained previously /1-3/ and provide for our results general and yet simple proofs.

We describe Compton scattering kinematically by the two variables $\nu = s - u$, t, where t is the squared four-momentum transfer between the photons, s - the invariant squared mass of the scattering particles (photon + nucleon), and u is given by $s + u + t = 2m^2$ (m is the nucleon mass). The invariant differential cross section with respect to t, $\frac{d\sigma(\nu,t)}{dt}$, is a positive quadratic expression of the amplitudes $A_i(\nu,t)$, with a given hermitian kinematic matrix $M(\nu,t)$ /1/,

$$\frac{d\sigma}{dt} = \sum_{i,j=1}^{n} A_i^*(\nu,t)\, M_{ij}(\nu,t)\, A_j(\nu,t) \quad . \tag{1.1}$$

We treat the amplitudes $A_i(\nu,t)$ as analytic functions of ν and consider t as a fixed real (negative) parameter. The domain of analyticity of $A_i(\nu,t)$ depends on t : it is the complex ν-plane with cuts from $-\infty$ to $-\nu_0$ and from ν_0 to $+\infty$, $\nu_0 = 2((m+\mu)^2-m^2)+t$ (μ = pion mass), except poles at $\nu = \pm\nu_B$, $\nu_B = -t$. The residues of $A_i(\nu,t)$ at $\nu = \pm\nu_B$ depend on the static electromagnetic constants of the scattered particle (nucleon) and are therefore quantities of interest. On the other hand it appears that the cuts are completely physical only for

$$-\frac{\left[(m+\mu)^2-m^2\right]^2}{(m+\mu)^2} \le t \le 0 \quad . \tag{1.2}$$

Since in our problem we are interested from the mathematical point of view in the knowledge of the right hand side of (1.1) on the whole cut, we have to restrict t to the interval (1.2), so as to provide it by measured values $\rho(\nu,t)$ of $\frac{d\sigma}{dt}$ at $\nu_0 \le \nu < \infty$.

The amplitudes $A_i(\nu,t)$ obey symmetry relations: $A_i(-\nu,t) = \varepsilon_i A_i(\nu,t)$, with ε_i equal to +1 or −1, depending on the amplitude. This allows to restrict our considerations to $\mathrm{Re}\,\nu \geq 0$. With the functions $\alpha_i(\nu) = \frac{1+\varepsilon_i}{2} + \frac{\nu}{\nu_0+(\nu_0^2-\nu^2)^{\frac{1}{2}}}\,\frac{1-\varepsilon_i}{2}$, equal to 1 for $\varepsilon_i = 1$ and to $\frac{\nu}{\nu_0+(\nu_0^2-\nu^2)^{\frac{1}{2}}}$ for $\varepsilon_i = -1$, we factorize $A_i(\nu,t) = \alpha_i(\nu)\, B_i(\nu^2,t)$ in order to get analytic functions of $\xi = \nu^2$ in the ξ-plane cut from $\xi_0 = \nu_0^2$ to $+\infty$, with a pole at $\xi_B = \nu_B^2$. A conformal mapping

$$z = \frac{(\xi_0-\xi_B)^{\frac{1}{2}} - (\xi_0-\xi)^{\frac{1}{2}}}{(\xi_0-\xi_B)^{\frac{1}{2}} + (\xi_0-\xi)^{\frac{1}{2}}} \tag{1.3}$$

brings this domain on the open unit disk in the z-plane and $\xi = \xi_B$ to $z=0$. The functions $C_i(z,t) = B_i(\xi,t)$ are thus analytic in $|z|<1$, except poles at $z=0$. Their behaviour at these poles is known; we denote it by $C_i(z,t) = \frac{r_i}{z}$ + terms regular at $z=0$. Therefore we obtain finally functions $\widetilde{w}_i(z,t) = \beta_i(z)\, C_i(z,t)$ analytic in the unit disk $|z|<1$ ($\beta_i(z) = z^{p_i}$, $p_i=1$ if $r_i \neq 0$ and $p_i=0$ if $r_i=0$), and exprecs the information coming from the measurement of $\frac{d\sigma}{dt}$ as a constraint for $\widetilde{w}_i(z,t)$ on the unit circle

$$\sum_{i,j=1}^{n} \widetilde{w}_i^{*}(\tau,t)\, p_{ij}(\tau,t)\, \widetilde{w}_j(\tau,t) \leq \sigma(\tau,t) \tag{1.4}$$

in terms of $\sigma(\tau,t) = \beta(\nu,t) > 0$ and the n-dimensional hermitian matrix

$$p(\tau,t) = \{p_{ij}(\tau,t)\}_{i,j=1}^{n}, \quad p_{ij}(\tau,t) = \beta_i^{*}\alpha_i^{*} M_{ij}\, \alpha_j \beta_j \quad .$$

Since the functions $A_i(\nu,t)$ are real analytic: $A_i^{*}(\nu,t) = A_i(\nu^{*},t)$, and the performed factorizations preserve reality, this property is shared by $\widetilde{w}_i(z,t)$. The matrix $p(\tau,t)$ and $\sigma(\tau,t)$ obey $p^{*}(\tau,t) = p(\tau^{*},t)$, $\sigma(\tau,t) = \sigma(\tau^{*},t)$, respectively. Therefore we may bring the information for the amplitudes from measurement of at least an upper bound on the cross section at the energies above the pion photoproduction threshold and fixed four-momentum transfer (in the range (1.2)) to the form

$$\sum_{i,j=1}^{n} w_i^{*}(\tau,t)\, p_{ij}(\tau,t)\, w_j(\tau,t) \leq 1 \quad , \tag{1.5}$$

where

$$w_i(z,t) = \widetilde{w}_i(z,t)\, q^{-1}(z) \qquad \left(w_i^{*}(z,t) = w_i(z^{*},t) \right), \tag{1.6}$$

$$g(z) = \exp\left(\frac{1}{2\pi}\int_0^{\pi}\frac{1-z^2}{1+z^2-2z\cos\theta}\ln\sigma(\tau,t)\,d\theta\right) . \qquad (1.7)$$

From inequality (1.5) we derive constraints on the values of $\frac{d^k}{dz^k}w(z,t)\Big|_{z=0}$, $k= 0,\ldots,m_i-1$ ($m_i \geqslant 1$), $i = 1,\ldots,n$, since they express the low energy expansion of the cross section (at t=o) and are related to the electromagnetic structure constants of the scattered particle. Further, we investigate the compatibility between (known or assumed) values of these structure constants and values of amplitudes (or combinations (functions) thereof, as e.g. the cross section) at various energies below photoproduction threshold, $(t^2-4m^2t)^{\frac{1}{2}} \leqslant \nu \leqslant \nu_0$, and at fixed momentum transfer.

The mathematical problem which arises is thus : to derive from (1.5) the constraints implied for the values of the functions $w_i(z) \equiv w_i(z,t)$ and of their derivatives up to certain orders in a finite number of points on the real diameter of the open unit disk. This problem would be considerably simplified if $p(\tau) \equiv p(\tau,t)$ were the unit matrix ($p_{ij}(\tau)= \delta_{ij}$). We are not so fortunate, but we use the fact that a positive-definite hermitian matrix allows a convenient analytic factorization, in order to bring our problem back to one of this form. This factorization is here an entirely kinematic problem.

The structure of the paper is as follows : In Sect.2 we state the necessary mathematical preliminaries for the solution of the reduced problem, with $p(\tau)= I$ (the unit matrix). This solution is given in Sect.3, with examples added in Sect.4, via an interpolation problem for vector-valued bounded analytic functions. The factorization and the reduction of our problem to that solved in Sect.3 is discussed in Sect.5 . Finally, Sect.6 gives several additional comments.

2. Preliminaries

Our mathematical objects will be complex functions $f(\zeta)=\{f_i(\zeta)\}_{i=1}^{n}$, defined on the unit circle $\zeta = e^{i\theta}$, $-\pi \le \theta < \pi$, with values in the n-dimensional complex space $\mathbb{C}^n$ considered with Euclidean norm $\|f(\zeta)\| = \left(\sum_{i=1}^{n} |f_i(\zeta)|^2\right)^{\frac{1}{2}}$, and measurable with respect to the (normalized) Lebesgue measure $d\mu(\theta) = \frac{1}{2\pi} d\theta$. The set of all vector-functions obeying $\|f\|_p < \infty$ ($1 \le p \le \infty$), where

$$\|f\|_p = \left(\int_{-\pi}^{\pi} \|f(\zeta)\|^p d\mu(\theta) \right)^{\frac{1}{p}} \quad , \quad 1 \le p < \infty \quad , \tag{2.1}$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{-\pi \le \theta < \pi} \|f(\zeta)\| \quad ,$$

are Banach spaces /4/ denoted by $\mathbb{L}^p$. These spaces satisfy the duality relation /4/: the space of all continuous linear functionals $\phi(f)$ on $\mathbb{L}^p$ ($1 \le p < \infty$) is isometrically isomorphic to the space $\mathbb{L}^q$,where $\frac{1}{p} + \frac{1}{q} = 1$ ($q = \infty$ for $p=1$). This is expressed by the fact that $\phi(f)$ has the form

$$\phi(f) = \int_{-\pi}^{\pi} \langle f(\zeta), \varphi(\zeta) \rangle \, d\mu(\theta) \quad , f \in \mathbb{L}^p \; , \; \varphi \in \mathbb{L}^q \; , \tag{2.2}$$

$$\sup_{\|f\|_p \le 1} |\phi(f)| = \|\varphi\|_q \quad ,$$

where $\langle \varphi(\zeta), f(\zeta) \rangle = \sum_{i=1}^{n} \varphi_i(\zeta) f_i(\zeta)$. One could use as well the representation $\phi(f) = \int_{-\pi}^{\pi} (\varphi(\zeta), f(\zeta)) d\mu(\theta)$, where $(\varphi(\zeta), f(\zeta)) = \langle \varphi^*(\zeta), f(\zeta) \rangle$, since the complex conjugation is an isometric isomorphism in $\mathbb{L}^q$, $1 \le q \le \infty$.

The closed subspaces of functions $h(\zeta) \in \mathbb{L}^p$ with vanishing negative Fourier coefficients are isometrically isomorphic to the spaces $\mathbb{H}^p$ of vector-valued functions $h(z) = \{h_i(z)\}_{i=1}^{n}$ analytic in

$|z|<1$ and obeying

$$\|h\|_p = \sup_{0<r<1} \left(\int_{-\pi}^{\pi} \|h(r\tau)\|^p d\mu(\theta) \right)^{\frac{1}{p}} < \infty \quad , \quad 1 \le p < \infty \; ,$$

$$\|h\|_\infty = \sup_{|z|<1} \|h(z)\| < \infty \qquad . \tag{2.3}$$

These spaces may therefore be identified, as in the scalar case, in terms of $h(z)$ and their boundary values $h(\tau)$.

The duality relation for a closed subspace of a Banach space is not as simple /5-6/. We consider the annihilator $\mathbb{H}_0^q$ of $\mathbb{H}^p$, i.e. the subspace of elements $\varphi(\tau)$ in $\mathbb{L}^q$ such that

$$\int_{-\pi}^{\pi} \langle \varphi(\tau), h(\tau) \rangle \, d\mu(\theta) = 0 \tag{2.4}$$

for any $h(z) \in \mathbb{H}^p$. It is the set of all functions $\psi(z) = z\,\varphi(z)$, $\varphi(z) \in \mathbb{H}^q$. Then the duality relation for a functional $\phi \in \mathbb{L}^q$, in which we are interested, may be formulated /5-6/ as

$$\sup_{h \in \mathbb{H}^p, \|h\|_p \le 1} |\phi(h)| = \min_{g \in \mathbb{H}^q} \|\varphi + \tau g\|_q \quad , \; 1 \le p < \infty \; . \tag{2.5}$$

Because of the isometry $\|\varphi + \tau g\|_q = \|\tau^{-1}\varphi + g\|_q$ this gives a relation which will prove convenient in the following :

$$\min_{g \in \mathbb{H}^q} \|\tau^{-1}\varphi + g\|_q = \sup_{h \in \mathbb{H}^p, \|h\|_p \le 1} |\phi(h)| \qquad . \tag{2.6}$$

Another useful relation is the equality between the norm of $\varphi(z) \in \mathbb{H}^\infty$ considered as an operator from the scalar H^2 to $\mathbb{H}^2$ and its norm $\|\varphi\|_\infty$ as vector in $\mathbb{H}^\infty$ /7/, i.e.

$$\|\varphi\|_\infty = \sup_{\|h\|_2 \le 1} \|\varphi h\|_2 \qquad . \tag{2.7}$$

This isometric representation of H^∞ has as a consequence the inequality

$$\inf_{\varphi\in\Sigma} \|\varphi\|_\infty \geq \sup_{\|h\|_2\leq 1} \inf_{\varphi\in\Sigma} \|\varphi h\|_2 \tag{2.3}$$

for any set $\Sigma \subset H^\infty$.

3. Interpolation

We consider now in H^∞ the subspace of functions of the form $V(z)\varphi(z)$, where $V(z)$ is a fixed inner matrix-valued analytic function from $\mathbb{C}^n$ to $\mathbb{C}^n$ /3-9/ ($V^+(z)V(z) \leq I$, $|z|<1$, and unitary for almost all values $\tau = e^{i\theta}$: $V^+(\tau)V(\tau) = I$), and $\varphi(z)$ is arbitrary in H^p . This subspace is denoted by VH^p . A simple situation corresponds to a multiple of the unit matrix I , given by a finite Blaschke product $\widetilde{B}(z) = \left(\frac{z-x_0}{1-x_0 z}\right)^{k_0+1} \cdots \left(\frac{z-x_m}{1-x_m z}\right)^{k_m+1}$, i.e. $V(z) = \widetilde{B}(z) I$. In the general situation of interest to us $V(z)$ will be a finite Blaschke-Potapov product /8/. The functions $u(z) \in H^p$ may be decomposed as

$$u(z) = \chi(z) + V(z)\,\varphi(z) , \tag{3.1}$$

where $\chi(z)$ belongs to the factor space H^p/VH^p . In the situation we consider the factor space is finite-dimensional and related to interpolation. The form of interpolation is especially simple when $V(z) = \widetilde{B}(z) I$, namely

$$u^{(j)}(x_i) = \chi^{(j)}(x_i) \quad , \quad j = 0,\ldots,k_i \;;\; i = 0,\ldots,m . \tag{3.2}$$

At the moment we do not yet specify $V(z)$: we take the point of view that an interpolation problem is defined by given functions $\chi(z)$ and $V(z)$.

Our problem described in the Introduction is related to a minimum norm interpolation problem in H^∞ : to find the set of all functions $\chi(z)$ in H^∞/VH^∞ such that $\min\limits_{\varphi\in H^\infty}\|\chi+V\varphi\|_\infty \leq 1$. The exact nature of this relation between these two problems will be specified when we shall reduce (1.5) to a form with unit matrix $p(z)$. At the moment we consider a fixed $\chi(z)$ and look for

$$\min_{\varphi\in H^\infty}\|\chi+V\varphi\|_\infty \quad . \tag{3.3}$$

Because of the isometry (in L^∞)

$$\|w\|_\infty = \|\chi+V\varphi\|_\infty = \|V^{-1}\chi+\varphi\|_\infty \tag{3.4}$$

this minimum is achieved, according to (2.6), and its value is equal to

$$\sup_{f\in H^1,\ \|f\|_1\leq 1}\left|\int_{-\pi}^{\pi}\langle f(z),\, z V^{-1}(z)\chi(z)\rangle\, d\mu(\theta)\right| \quad . \tag{3.5}$$

In order to compute this quantity we use the factorization

$$f(z) = g(z)\, h(z) \tag{3.6}$$

of any $f(z)\in H^1$ into a product with $h(z)\in H^2$ and $g(z)\in H^2$, and such that $\|f\|_1 = \|g\|_2^2 = \|h\|_2^2$. This factorization is analogous to the scalar case : we take the components $g(z)$ and $h(z)$ as

$$\begin{aligned} g(z) &= e^{-1}(z)\, f(z) \quad ,\\ h(z) &= e(z) \quad , \end{aligned} \tag{3.7}$$

where

$$e(z) = \exp\left(\int_{-\pi}^{\pi}\frac{z+z}{z-z}\,\ln\|f(z)\|^{\frac{1}{2}}\, d\mu(\theta)\right) . \tag{3.8}$$

Since the functions $g(z)$, $h(z)$ are correlated we generally increase the quantity (3.5) if we insert (3.6) into it and take the supremum over the unit spheres in $\mathbb{H}^2$ and H^2 :

$$\min_{\mathcal{U} \in \mathfrak{X} + \mathbb{H}^\infty} \|\mathcal{U}\|_\infty \leq \sup_{\substack{g \in \mathbb{H}^2, h \in H^2 \\ \|g\|_2 \leq 1, \|h\|_2 \leq 1}} \left| \int_{-\pi}^{\pi} \langle \bar{z} g(z) V^{-1}(z), \mathfrak{X}(z) h(z) \rangle \, d\mu(\theta) \right| \tag{3.9}$$

The integral in (3.9) is zero if $g(z) \in V^T \mathbb{H}^2$ and $h(z) \in BH^2$, where $V^T(z)$ is the transposed of $V(z)$ and $B(z)$ is the least scalar inner multiple of $V(z)$ /i.e.: $V(z)\Omega(z) = B(z) I$ ($\Omega(z)$ is an inner matrix-function); so it is enough to take $g(z) \in H^2 \ominus V^T H^2$, $h(z) \in H^2 \ominus BH^2$. But for such an $g(z)$ we have the relation

$$h^*(z) = \bar{z} g(z) V^{-1}(z) \tag{3.10}$$

where $h(z) \in H^2 \ominus V H^2$ (the proof of analyticity of $h(z)$ ($h(z) \in H^2$) rests on the relation $z V^T \mathbb{H}^2 \subset V^T \mathbb{H}^2$; orthogonality to $V\mathbb{H}^2$ is evident).

In this way we have brought the right hand side of (3.9) to the norm of a finite-dimensional operator $M : H^2 \ominus BH^2 \to H^2 \ominus VH^2$ the projection on the subspace $H^2 \ominus VH^2$ of the operator $\mathfrak{X}(z) : H^2 \to \mathbb{H}^2$. With suitable orthonormalized bases in $H^2 \ominus VH^2$ and $H^2 \ominus BH^2$ this becomes the norm of a matrix M of dimension $N \times N$, $N = \dim(H^2 \ominus BH^2)$, $N = \dim(H^2 \ominus VH^2)$,

$$\sup_{\substack{g \in H^2, h \in H^2 \\ \|g\|_2 \leq 1, \|h\|_2 \leq 1}} \left| \int_{-\pi}^{\pi} \langle \bar{z} g(z) V^{-1}(z), \mathfrak{X}(z) h(z) \rangle \, d\mu(\theta) \right| =$$

$$\sup_{\substack{h \in H^2 \ominus VH^2, h \in H^2 \ominus BH^2 \\ \|h\|_2 \leq 1, \|h\|_2 \leq 1}} \left| \int_{-\pi}^{\pi} (h(z), \mathfrak{X}(z) h(z)) \, d\mu(\theta) \right| = \|M\| . \tag{3.11}$$

Since we have so far only an inequality

$$\min_{\mu\in\chi+VH^\infty} \|\mu\|_\infty \le \|M\| \quad , \tag{3.12}$$

we use (2.3) in order to get an inequality of the opposite type,

$$\min_{\mu\in\chi+VH^\infty} \|\mu\|_\infty \geq \sup_{\|h\|_2\le 1} \inf_{\mu\in\chi+VH^\infty} \|\mu h\|_2 \quad . \tag{3.13}$$

Now

$$\inf_{\mu\in\chi+VH^\infty} \|\mu h\|_2 = \inf_{\varphi\in H^\infty} \|V^{-1}\chi h + \varphi h\|_2 \tag{3.14}$$

and $\varphi(z)h(z)\in H^2$; thus

$$\inf_{\mu\in\chi+VH^\infty} \|\mu h\|_2 \geq \inf_{f\in H^2} \|V^{-1}\chi h + f\|_2 \tag{3.15}$$

and, according to (2.5) with p=2,

$$\inf_{f\in H^2} \|V^{-1}\chi h + f\|_2 = \sup_{g\in H^2, \|g\|_2\le 1} \left|\int_{-\pi}^{\pi} \langle z g(z) V^{-1}(z), \chi(z) h(z)\rangle d\mu(\theta)\right| \tag{3.16}$$

Finally we have therefore

$$\min_{\mu\in\chi+VH^\infty} \|\mu\|_\infty \geq \sup_{\|h\|_2\le 1} \inf_{\mu\in\chi+VH^\infty} \|\mu h\|_2 \geq$$

$$\sup_{\substack{\tilde{h}\in H^2\ominus VH^2,\ h\in H^2\ominus BH^2 \\ \|\tilde{h}\|_2\le 1,\ \|h\|_2\le 1}} \left|\int_{-\pi}^{\pi} (\tilde{h}(z), \chi(z)h(z))\, d\mu(\theta)\right| = \|M\| \tag{3.17}$$

and, together with (3.12):

$$\min_{\mu\in\chi+VH^\infty} \|\mu\|_\infty = \|M\| \quad . \tag{3.18}$$

The set of all functions $\chi(z)\in H^\infty/VH^\infty$ for which there exists at least one function $\mu(z)\in H^\infty$ of the form (3.1) and of $\|\mu\|_\infty\le 1$ is thus given by those functions for which $\|M\| \le 1$. This is equivalent to saying that they are the functions for which

$$I - M^{+}M \geq 0 \qquad , \tag{3.19}$$

where I is the unit matrix in $H^2 \ominus BH^2$. This inequality is expressed numerically, in our situation, by a finite set of (algebraic) inequalities /1o/ involving interpolation data. In order to derive them, we have to compute the matrix M (up to unitary equivalence: $\|\mathbb{U} M U\| = \|M\|$ if $\mathbb{U}$, U are unitary transformations in $H^2 \ominus VH^2$ and $H^2 \ominus BH^2$, respectively) from the quadratic form

$$(f, \mathcal{X} h) = \int_{-\pi}^{\pi} (f(z), \mathcal{X}(z) h(z)) d\mu(t) \tag{3.2o}$$

with $f \in H^2 \ominus VH^2$, $h \in H^2 \ominus BH^2$, and $\mathcal{X} \in H^{\infty}/VH^{\infty}$ (it is, of course, allowed to insert in (3.2o) instead of $\mathcal{X}(z)$ any function $\mu(z) = \mathcal{X}(z) + V(z)\varphi(z)$, $\varphi(z) \in H^{\infty}$).

4. Computation of M

1. Example

The simplest situation is certainly that in which we ask only for the values $\frac{1}{k!}\mu^{(k)}(0) = c_k$, $k = 0, \dots, m-1$. In this case $V(z) = z^m I$, and $\mathcal{X}(z)$ may be chosen as

$$\mathcal{X}(z) = \sum_{k=0}^{m-1} c_k z^k \quad . \tag{4.1}$$

Also, because $B(z) = z^n$, we may span $H^2 \ominus BH^2$ ($\dim(H^2 \ominus BH^2) = n$) by the orthogonal basis z^i ($i = 0, \dots, n-1$) as

$$h(z) = \sum_{i=0}^{m-1} h_i z^i \quad , \tag{4.2}$$

and $H^2 \ominus VH^2$ (dim($H^2 \ominus VH^2$)=n×m) as

$$\ell(z) = \sum_{i=0}^{m-1} \ell_i z^i \quad . \tag{4.3}$$

Therefore

$$(\ell, \mathcal{X} h) = \sum_{\substack{i,j \\ i \geq j}}^{m-1} (\ell_i, c_{i-j} h_j) \tag{4.4}$$

and $\mathbb{M} = \mathbb{C}$, with

$$\mathbb{C} = \begin{pmatrix} c_0 & 0 & \cdots\cdots & 0 \\ c_1 & c_0 & & 0 \\ \vdots & & & \\ \vdots & & & \\ c_{m-1} & c_{m-2} & \cdots & c_0 \end{pmatrix} . \tag{4.5}$$

2. Example

Comparable to the situation considered before is that in which we are interested in the values $u(x_i) = u_i \;\; , i=1,\dots,p \;\; (x_i \neq x_j \;\; , |x_i|<1)$. Here we have $V(z) = B(z)I$, with $B(z) = \prod_{i=1}^{p} \frac{z-x_i}{1-\bar{x}_i z}$ and we may choose

$$\mathcal{X}(z) = \sum_{i=1}^{p} u_i \varphi_i(z) \quad , \tag{4.6}$$

where

$$\varphi_i(z) = \frac{B(z)}{z-x_i} \frac{1}{B'(x_i)} \quad , \quad \varphi_i(x_j) = \delta_{ij} \quad . \tag{4.7}$$

The spaces $H^2 \ominus BH^2$ (dim($H^2 \ominus BH^2$)= p) and $H^2 \ominus VH^2$ (dim($H^2 \ominus VH^2$)=$m \times p$) can be spanned by the functions

$$\begin{aligned} h(z) &= \sum_{i=1}^{p} h_i \, r_i(z) \quad , \\ \ell(z) &= \sum_{i=1}^{p} \ell_i \, r_i(z) \quad , \end{aligned} \tag{4.8}$$

where $r_i(z)$ are the orthonormalized rational functions (in H^2)

$$f_i(z) = \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_{i-1}}{1-x_{i-1} z} \frac{(1-x_i^2)^{\frac{1}{2}}}{1-x_i z} \quad . \tag{4.9}$$

We note that $\varphi_j(z) f_i(z)$ may be expanded

$$\varphi_j(z) f_i(z) = \sum_{k=1}^{p} \alpha_{kji} f_k(z) + \text{terms in } BH^2 \tag{4.10}$$

The coefficients α_{kji} are given in terms of the nonsingular matrix R , defined by

$$R_{\alpha\beta} = f_\beta(x_\alpha) \quad , \quad \alpha,\beta = 1,\ldots,p \quad , \tag{4.11}$$

with $R_{\alpha\beta} = 0$ for $\beta > \alpha$ and $\det R = \prod_{i=1}^{p} f_i(x_i) \neq 0$, i.e.

$$R = \begin{pmatrix} f_1(x_1) & 0 & \cdots & 0 \\ f_1(x_2) & f_2(x_2) & \cdots & 0 \\ \vdots & & & \\ f_1(x_p) & f_2(x_p) & \cdots & f_p(x_p) \end{pmatrix} \; ; \tag{4.12}$$

their form is

$$\alpha_{kji} = (R^{-1})_{kj} R_{ji} \quad . \tag{4.13}$$

Now we may bring

$$(h, \mathcal{X} h) = \sum_{i,j,k} (h_k, u_j h_i)(f_k, \varphi_j f_i) \quad , \tag{4.14}$$

due to the relation $(f_k, \varphi_j f_i) = \alpha_{kji}$ which follows from (4.10), to the form

$$\sum_{k,i=1}^{p} (h_k, M_{ki} h_i) \quad , \tag{4.15}$$

with

$$M_{ki} = \sum_{j=1}^{p} (R^{-1})_{kj} u_j R_{ji} \quad . \tag{4.16}$$

The matrix M is thus given by

$$M = R^{-1} \mathbb{C}_D R \tag{4.17}$$

where $\mathbb{C}_D$ is the diagonal block-matrix

$$\mathbb{C}_D = \begin{pmatrix} \mathcal{U}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{U}_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \mathcal{U}_p \end{pmatrix} \tag{4.18}$$

and $\mathbb{R}$ is the block-matrix

$$\mathbb{R} = \begin{pmatrix} \mathbb{t}_1(x_1) & 0 & \cdots & 0 \\ \mathbb{t}_1(x_2) & \mathbb{t}_2(x_2) & \cdots & 0 \\ \vdots & & & \\ \mathbb{t}_1(x_p) & \mathbb{t}_2(x_p) & \cdots & \mathbb{t}_p(x_p) \end{pmatrix}, \tag{4.19}$$

with $\mathbb{t}_i(x_j) = t_i(x_j) I_n$, a multiple of the n-dimensional unit matrix .

3. Example

A more general situation is that in which we are looking for the values of $\frac{1}{k!} \mathcal{U}^{(k)}(0) = \mathcal{C}_k$, $k = 0, \ldots, m-1$ and of $\mathcal{U}(x_j) = \mathcal{U}_j$, $j = 1, \ldots, p$ ($x_j \neq x_k$, $j \neq k$, $x_j \neq 0$). Now $V(z) = B(z) I$ with $B(z) = z^m \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_p}{1-x_p z}$ and $H^2 \ominus BH^2$, $H^2 \ominus V H^2$ are spanned by

$$\ell(z) = \sum_{i=0}^{m-1} h_i z^i + \sum_{j=1}^{p} g_j t_j(z) = \sum_{i=0}^{m-1+p} \tilde{h}_i \tilde{t}_i(z)$$

$$h(z) = \sum_{i=0}^{m-1} h_i z^i + \sum_{j=1}^{p} g_j t_j(z) = \sum_{i=0}^{m-1+p} \tilde{h}_i \tilde{t}_i(z) , \tag{4.20}$$

respectively, with

$$t_j(z) = z^m \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_{j-1}}{1-x_{j-1} z} \frac{(1-x_j^2)^{\frac{1}{2}}}{1-x_j z} , \tag{4.21}$$

and $\tilde{t}_i(z) = z^i$ ($i = 0, \ldots, m-1$), $\tilde{t}_i(z) = t_{i-m+1}(z)$ ($i = m, \ldots, m-1+p$).

The quantity (3.20) for this situation may be expressed as

$$(h, \mathcal{I} h) = \sum_{k,i=0}^{m-1+p} (\tilde{h}_k, M_{ki} \tilde{h}_i) , \tag{4.22}$$

with

$$M_{ki} = (\tilde{f}_k , \chi \tilde{f}_i) \quad . \tag{4.23}$$

The computation of (4.23) is equivalent to that of the expansion

$$\chi(z)\tilde{f}_i(z) = \sum_{i=0}^{m-1+p} M_{ki} \tilde{f}_k(z) \ + \text{terms in } VH^2 \quad . \tag{4.24}$$

Therefore we may conclude, from (4.24), due to the properties of $\tilde{f}_i(z)$, that

$$M = R^{-1} C_D R \tag{4.25}$$

where now

$$D = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \\ 0 & 0 & & 1 & 0 & 0 & \cdots & 0 \\ \hline 1 & x_1 & & x_1^{m-1} & f_1(x_1) & 0 & \cdots & 0 \\ 1 & x_2 & & x_2^{m-1} & f_1(x_2) & f_2(x_2) & \cdots & 0 \\ \vdots & & & & & & & \\ 1 & x_p & & x_p^{m-1} & f_1(x_p) & f_2(x_p) & \cdots & f_p(x_p) \end{array}\right), \quad C_D = \left(\begin{array}{cccc|cccc} c_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \\ c_{m-1} & c_{m-2} & \cdots & c_0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & u_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & u_2 & \cdots & 0 \\ \vdots & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & u_p \end{array}\right) \tag{4.26}$$

and $\mathcal{R}$ is obtained from R of (4.26) by considering, analogously [illegible], instead of each element R_{ij}, the element $R_{ij} I$.

5. Factorization

Now we come back to the analytic factorization of the matrix $p(\tau) = \{p_{ij}(\tau)\}_{i,j=1}^{n}$ which is (almost everywhere) positive definite on the unit circle. Classical results of Akutowicz and Wiener /12/ and of [illegible]anov /1[illegible]/ state that if

$$\int_{-\pi}^{\pi} \ln \det p(\tau)\, d\mu(\theta) > -\infty \tag{5.1}$$

and $p_{ij}(\tau) \in L^1$, then $p(\tau)$ can be factorized in the form

$$p(\tau) = m^+(\tau)\, m(\tau) \quad , \tag{5.2}$$

where $m(z) = \{m_{ij}(z)\}_{i,j=1}^n$ is an analytic matrix in $|z|<1$ and $m_{ij}(z) \in H^2$. Among all factorizations there is a distinguished one, which is , in fact, of interest to us. Namely, for analytic matrices there is a unique factorization (up to constant unitary matrices) into inner and outer factors /8-9/, analogous to the scalar case : $m(z) = \Theta(z)\, E(z)$,

$$m^+(\tau)\, m(\tau) = E^+(\tau)\, E(\tau) \quad . \tag{5.3}$$

The factorization we need is that given by the outer matrix $E(z)$,

$$p(\tau) = E^+(\tau)\, E(\tau) \quad , \tag{5.4}$$

for the following reasons : The spaces convenient to us are L^p , H^p, and the relation

$$h(z) = E(z)\, g(z) \tag{5.5}$$

establishes an isometric isomorphism between H^∞ ($h(z) \in H^\infty$) and the space $H^\infty(p)$ ($g(z) \in H^\infty(p)$), defined by the norm

$$\begin{aligned} \|W\|_\infty &= \operatorname*{ess\,sup}_{-\pi \le \theta < \pi} \|W(\tau)\| \quad , \\ \|W\| &= \Big(\sum_{i,j=1}^{n} W_i^+(\tau)\, p_{ij}(\tau)\, W_j(\tau) \Big)^{\frac{1}{2}} \quad , \end{aligned} \tag{5.6}$$

whose unit sphere we are given by our problem.

We have now to determine the images in H^∞ of certain subspaces of $H^\infty(p)$, given by (5.5). In order to do this we observe that in the decomposition

$$W(z) = \chi(z) + B(z)\, f(z) \tag{5.7}$$

of $W(z) \in H^\infty(p)$ the functions $\varphi(z)$ belong to $H^\infty(B^+pB)$ (we keep, nevertheless, for the subspace of functions in $H^\infty(p)$ of the form $B(z)\varphi(z)$, $\varphi(z) \in H^\infty(B^+pB)$, the notation $BH^\infty(p)$). From (5.5) we have as the image of $BH^\infty(p)$ the subspace $E(z)B(z)\varphi(z)$. Its explicit form follows from the (left) factorization of $E(z)B(z)$/8-9/

$$E(z)B(z) = V(z)D(z) , \tag{5.8}$$

where $D(z)$ is the outer matrix-function defined by

$$D^+(\tau)D(\tau) = B^+(\tau)p(\tau)B(\tau) \tag{5.9}$$

and thus gives an isometric isomorphism between $H^\infty(B^+pB)$ and H^∞, and $V(z)$ is an inner matrix-function. When $B(z)$ is a Blaschke product ($B_{ij}(z) = B_i(z)\delta_{ij}$, $B_i(z)$ = scalar Blaschke products), then $V(z)$ is a Blaschke-Potapov product of the same multiplicity as $B(z)$ ($\det B(z) = 0 \leftrightarrow \det V(z) = 0$). The factors $V(z)$, $D(z)$ are (essentially) uniquely determined and give as the image of the subspace $BH^\infty(p)$ of $H^\infty(p)$ the subspace VH^∞ in H^∞.

Now we can make the connection between the solution of the problem stated in the Introduction, which requires to determine

$$\inf_{W \in \chi + BH^\infty(p)} \|W\|_\infty \tag{5.10}$$

for given $B(z)$ and $\chi(z)$, and the problem solved in Sect.3. It is given by

$$\inf_{W \in \chi + BH^\infty(p)} \|W\|_\infty = \inf_{\varphi \in H^\infty} \|V^{-1}E\chi + \varphi\|_\infty \tag{5.11}$$

The precise form of the solution is therefore : the functions $\chi(z) \in H^\infty(p)/BH^\infty(p)$ such that there exists at least one $W(z) = \chi(z) + B(z)\varphi(z)$ of norm $\|W\|_\infty \leq 1$ are these for which

$$\|M\| = \sup_{\substack{\ell \in H^2 \ominus VH^2,\ h \in H^2 \ominus BH^2 \\ \|h\|_2 \leq 1,\ \|\ell\|_2 \leq 1}} \left| \int_{-\pi}^{\pi} \left(\ell(\theta), E(\theta)\mathcal{M}(\theta) h(\theta)\right) d\mu(\theta) \right| \leq 1 \qquad (5.12)$$

or equivalently

$$\|M\| = \sup_{\substack{\ell \in H^2 \ominus VH^2,\ h \in H^2 \ominus BH^2 \\ \|\ell\|_2 \leq 1,\ \|h\|_2 \leq 1}} \left| \int_{-\pi}^{\pi} \left(\ell(\theta), E(\theta) W(\theta) h(\theta)\right) d\mu(\theta) \right| \leq 1. \qquad (5.13)$$

Here BH^2 is defined by $\mathcal{B}(z)$, the least scalar inner multiple of $B(z)$ (and of $V(z)$, since if there is an inner matrix-function $\Omega(z)$ such that $B(z)\Omega(z) = \mathcal{B}(z) I$, then $V(z)\Lambda(z) = \mathcal{B}(z) I$ with $\Lambda(z) = D(z)\Omega(z) E^{-1}(z)$).

In Sect.4 we have given only examples of the form $B(z) = \mathcal{B}(z) I$. The reason for this is apparent from (5.8) and (5.9) : in this case $V(z) = B(z) = \mathcal{B}(z) I$ and we need not compute $V(z)$ in order to state the result. It is obtained simply by the substitution $\mathcal{M}(z) \to E(z)W(z)$ in the corresponding formulas of Sect.4 . In the more general situation of a diagonal, but not scalar, (Blaschke product) $B(z)$, which arises when we ask for different (interpolatory) information for the components $W_\lambda(z)$ of $W(z)$, this is no more true.

6. Comments

In the construction of the solution to our problem there is, from the constructive point of view, one step left unperformed : the factorization of $\rho(\theta)$ in outer matrix-valued functions. It is, however, to be noted, that this factorization is an entirely kinematic problem. But, since it is decided also by the domain of analyticity of A_λ, it is generally not solved by the matrices performing the transformation from an orthogonal set of amplitudes (e.g. helicity amplitudes) to A_{λ}.

From explicit examples it turns out that the number of independent (algebraic) inequalities giving the constraints (3.19) is smaller than the formal number from matrix theory /10/. The mechanism and degree of this reduction have not been investigated in this paper.

The computation of the set $D = \{w_1(0),\dots; w_n(0),\dots : I - M^+ M \geq 0\}$, as given here, implies a simple proof of the conjecture /2/ : $D = D_0$ (where D_0 is a domain suggested in /2/ as outer approximation, $D \subset D_0$, to D). This conjecture has been proved by Auberson and Mennessier /14/; a simple alternative to it /15/ is implied by the equality

$$\min_{u \in \chi + VH^\infty} \|u\|_\infty = \sup_{\|h\|_2 \leq 1} \inf_{u \in \chi + VH^\infty} \|uh\|_2 \tag{6.1}$$

which we obtain here from (3.12) and (3.17).

In order to present here at least a superficial insight into the phenomena related to nonscalar $B(z)$ we give the matrices $B(z)I$, $V(z)$, the subspace $H^2 \ominus VH^2$, and the matrix M for the simplest situation : $B(z) = \frac{z-\lambda}{1-\lambda z} P + Q$, $P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, $Q = I_n - P$. Here $B(z) = \frac{z-\lambda}{1-\lambda z}$ and $V(z) = U B(z) U^{-1} = \frac{z-\lambda}{1-\lambda z} P_\lambda + Q_\lambda$, where U is a constant unitary matrix, which is computed from $E(\lambda)$ /8/. The subspace $H^2 \ominus VH^2$ is spanned by

$$h(z) = \frac{(1-\lambda^2)^{\frac{1}{2}}}{1-\lambda z} P_\lambda a \quad , \quad a \in \mathbb{C}^n \quad ,$$

$$\|h\|_2^2 = (P_\lambda a, P_\lambda a) = (PU^{-1}a, PU^{-1}a) \tag{6.2}$$

and since $(P_\lambda a, E(\lambda)W(\alpha) h) = (PU^{-1}a, PU^{-1}E(\lambda)W(\lambda) h)$ the matrix M may be chosen as

$$M = PU^{-1}E(\lambda)W(\lambda) \, . \tag{6.3}$$

From (5.8) it follows that $\mathcal{P}U^{-1}E(\lambda)\mathcal{Q}=0$ and therefore

$$M = \mathcal{P}U^{-1}E(\lambda)\mathcal{P}W(\lambda) \quad , \tag{6.4}$$

a form which exhibits its dependence (as known) only on $\mathcal{P}W(\lambda)$.

The steps left in the explicit solution of the problem stated in the Introduction depend on the computation of $E(z)$ and of $V(z)$. That of $V(z)$ is, for a finite Blaschke product $B(z)$, essentially a problem of algebra /3/. After $V(z)$ is known, one can determine the subspace $H^2 \ominus VH^2$ by (orthogonal) projection from H^2 , as

$$f(z) = \int_{-\pi}^{\pi} P(z,\zeta)\, g(\zeta)\, d\mu(\theta) \quad , \tag{6.5}$$

$g(z) \in H^2$,where

$$P(z,\zeta) = K(z,\zeta) - V(z)K(z,\zeta)V^{+}(\zeta) \equiv K(z,\zeta) - Q(z,\zeta) \, , \tag{6.6}$$

with $K(z,\zeta) = \frac{1}{1-z\bar{\zeta}} I$ the reproducing kernel of H^2 (i.e. the kernel of the orthogonal projector from L^2 on H^2) and $Q(z,\zeta)$ the kernel of the (orthogonal) projector from L^2 on VH^2 (the reproducing kernel of VH^2). $P(z,\zeta)$ is, in fact, the reproducing kernel of $H^2 \ominus VH^2$ /16-17/. It can be easily computed from the general (canonical) form of a Blaschke-Potapov product $V(z)$.

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