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# Optimal sum rule inequalities for spin 1/2 Compton scattering

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ABSTRACT: *A formalism appropriate for model independent dispersion theoretic investigations of the (not necessarily forward) Compton scattering off spin 1/2 hadronic targets, which fully exploits the analyticity properties of the amplitudes (to lowest order in electromagnetism) in $\nu^2$ at fixed t ($\nu = \frac{1}{4}(s-u)$ s, t, u = Mandelstam variables), is developed. It relies on methods which are specific to boundary value problems for analytic matrix-valued functions. An analytic factorization of the positive definite hermitian matrix associated with the bilinear expression of the unpolarized differential cross section (u.d.c.s.) in terms of the Bardeen-Tung (B.T.) invariant amplitudes is explicitly obtained. For t in a specified portion of the physical region, six new amplitudes describing the process are thereby constructed which have the same good analyticity structure in $\nu^2$ as the (crossing symmetrized) B.T. amplitudes, while their connection with the usual helicity amplitudes is given by a matrix which is unitary on the unitarity cut. A bound on a certain integral over the u.d.c.s. above the first inelastic threshold, established in terms of the target's charge and anomalous magnetic moment, improves a previous weaker result, being now optimal under the information accepted as known.*

## I. INTRODUCTION

In this paper we present a model independent approach to hadron Compton scattering aimed to achieve a maximal exploitation of the analyticity properties (to the lowest order in electromagnetism) of the S-matrix elements at fixed momentum transfer. The techniques we shall describe are particularly useful in the more difficult case of the non-forward scattering, when the physically accessible objects, like differential cross sections, give information merely on certain bilinear forms of scattering amplitudes rather than directly on their absorptive parts. As previous investigations showed /1/,/2/,/3/, starting from the analyticity and crossing properties of the amplitudes as well as from the consequences of the gauge invariance requirements for the considered process, one can derive model independent sum rule inequalities which relate certain integrals over the unpolarized differential cross section above the first inelastic threshold to static characteristics of the target like its charge and anomalous magnetic moment. The methods employed in /2/,/3/ rely mainly on the principle of maximum for subharmonic functions. While the bound found in /3/ in the case of Compton scattering off spinless hadrons is optimal under the information accepted as known, the corresponding inequality obtained in the same reference for spin 1/2 hadrons does not share this property. Due to the relatively large number (six) of independent amplitudes describing the process in the spin 1/2 case, the problems encountered are much more delicate in spite of the kinematic nature of the complications. The question of getting an optimal version of the bound found in /3/ for the spin 1/2 (non-forward) case, as well as that of finding other optimal constraints, was intensively studied by powerful mathematical means in refs./4/ and solved in principle. As shown in /4/ one has now to resort to methods which are specific to boundary value problems for analytic matrix-valued functions. To get concrete results, one essential step remained, however, unperformed and that refers to the explicit analytic factorisation of the positive definite hermitian matrix associated with the bilinear expression of the unpolarized differential cross section in terms of the analytically well behaved invariant amplitudes of the process. The last problem is completely solved in this paper. In this way a new formalism for treating the spin 1/2 Compton scattering in a certain (limited) range of physical momentum transfer values is established, which looks promising for further model independent dispersion theoretic analyses. It is based on the construction of six new amplitudes for spin 1/2 Compton scattering, which have the same good analyticity and crossing symmetry properties as those of the known invariant amplitudes and, in addition, are connected with the usual (six independent) helicity amplitudes (lacking such a nice analyticity structure and crossing symmetry) by a matrix which is unitary on the unitarity cut. The differential cross section above the first inelastic threshold can thereby be written down essentially as a sum of moduli squared of amplitudes analytic inside the domain and consequently the desired constraints at the relevant interior points will come out straightforwardly through known theorems, being this time optimal, i.e. strongest under the specified conditions of the problem. An important constraint is worked out in detail here and yields a sum rule inequality which improves in an optimal way the weaker bound found for the spin 1/2 case in ref./3/. Our present result deserves further attention in what concerns experimental checks since all the analyticity requirements are now fully and properly taken into account.

This article is organized as follows: Section II is devoted to the needed preliminaries regarding notation, kinematics, phase conventions, analyticity structure of the amplitudes, and crossing relations. The concrete problem to be solved is appropriately formulated in Section III and its solution explicitly found in Sections IV and V. The results are summarized and discussed in Section VI.

## II. PRELIMINARY CONSIDERATIONS

The spin 1/2 Compton effect is specified by six independent amplitudes. We shall work here with the Bardeen-Tung invariant amplitudes $A_i$ (i=1,2,...,6) /5/ known to be free of kinematic singularities, zeros or constraints, and achieving therefore a complete separation of dynamics from kinematics in the consideration of the process.

In the barycentric system one has

$$t = -2q^2(1-\cos\theta) \quad , \tag{2.1}$$
$$q = (s-m^2)/2s^{1/2} \quad ,$$

where $q$ is the magnitude of the three-momentum, $\theta$ the scattering angle, $m$ the mass of the target and $s$ the square of the total energy. Besides the Mandelstam variables $s$, $t$, $u$ ( $s+t+u = 2m^2$ ), we shall also use the variable

$$\nu = \frac{1}{4}(s-u) \quad . \tag{2.2}$$

The amplitudes $A_{1,2,4,5}(\nu,t)$ are even while $A_{3,6}(\nu,t)$ are odd under $s$-$u$ crossing ($\nu \to -\nu$ ), so that $A_{1,2,4,5}$ and $\frac{1}{\nu}A_{3,6}$ can all be viewed as functions of $\nu^2$ and $t$. Except $s$-$u$ channel Born pole terms at $\nu^2 = \nu_B^2 = \frac{t^2}{16}$, they are real analytic functions ( $A^*(\nu^{2*},t) = A(\nu^2,t)$) in the complex $\nu^2$ plane cut along the real axis from $\nu_0^2$ to $\infty$ ; to the lowest order in the fine structure constant the value of $\nu_0$ is determined by the first inelastic threshold $s_0$

$$\nu_0 = \frac{1}{2}\left( s_0 - m^2 + \frac{t}{2}\right) \tag{2.3}$$

For proton Compton scattering $s_0$ is the single pion photoproduction threshold

$$s_0 = (m+\mu)^2 , \quad \nu_0 = \frac{1}{2}\mu(\mu+2m) + \frac{t}{4} , \qquad \mu = \text{pion mass} \tag{2.4}$$

The physical region of the process in the $\nu$ , $t$ variables is

$$t \leq 0 , \qquad \nu \geqslant \nu_{min} = \frac{1}{2}(-t)^{1/2}\left(m^2 - \frac{t}{4}\right)^{1/2} \tag{2.5}$$

If $t$ is restricted to physical values in the range

$$0 \geqslant t > -(s_0 - m^2)^2 / s_0 \tag{2.6}$$

( for the proton $\quad 0 \geqslant t > -\left[\mu(2m+\mu)/(m+\mu)\right]^2$ ) (2.6')

the following ordering holds

$$\frac{t^2}{16} = \nu_B^2 \leq \nu_{min}^2 \leq \nu_0^2 \tag{2.7}$$

such that the whole unitarity cut will lie within the physical region (Fig.1).

Removing the Born poles, we shall deal with the following (dimensionless) functions $\bar{A}_i(\nu^2,t)(i=1,2,\ldots,6)$:

$$\begin{aligned} \bar{A}_1(\nu^2,t) &\equiv 4(\nu^2 - \nu_B^2)\frac{1}{m} A_1(\nu,t) \quad , \\ \bar{A}_2(\nu^2,t) &\equiv 4(\nu^2 - \nu_B^2) \quad A_2(\nu,t) \quad , \end{aligned} \tag{2.8}$$

$$\bar{A}_3(\nu^2,t) \equiv 4(\nu^2-\nu_B^2)\,\frac{m}{\nu}\,A_3(\nu,t) \quad ,$$

$$\bar{A}_4(\nu^2,t) \equiv 4(\nu^2-\nu_B^2)\,A_4(\nu,t) \quad ,$$

$$\bar{A}_5(\nu^2,t) \equiv 4(\nu^2-\nu_B^2)\,m\,A_5(\nu,t) \quad ,$$

$$\bar{A}_6(\nu^2,t) \equiv 4(\nu^2-\nu_B^2)\,\frac{m^2}{\nu}\,A_6(\nu,t) \quad ,$$

which for $t$ in the interval given by Eq.(2.6) are analytic in the cut complex $\nu^2$ -plane, taking at the point $\nu^2 = \nu_B^2$ values completely specified in terms of $t$ and the target's charge and anomalous magnetic moment. For the proton

$$\begin{aligned}
\bar{A}_1(\nu_B^2,t) &= -4e^2\left[1-(2\varkappa+\varkappa^2)\frac{t}{8m^2}\right] ,\\
\bar{A}_2(\nu_B^2,t) &= -4e^2\left[1+\varkappa+(2\varkappa+\varkappa^2)\frac{t}{8m^2}\right] ,\\
\bar{A}_3(\nu_B^2,t) &= -2e^2(2\varkappa+\varkappa^2) ,\\
\bar{A}_4(\nu_B^2,t) &= -4e^2\left(1+\varkappa+\varkappa^2\frac{t}{8m^2}\right) ,\\
\bar{A}_5(\nu_B^2,t) &= 8e^2\varkappa ,\\
\bar{A}_6(\nu_B^2,t) &= 4e^2\varkappa^2 .
\end{aligned} \tag{2.9}$$

where

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \quad , \quad \varkappa \simeq 1.79$$

The connection between $\bar{A}_i(\nu^2,t)$ and the six independent helicity amplitudes $f_i(\nu,t)$ specifying also the process but lacking the good analyticity and crossing symmetry properties of the former amplitudes is given by a matrix $M_{ij}(\nu,t)$ (i,j=1,2,...,6) which is not analytic in the cut complex $\nu^2$ -plane. This 6x6 matrix connection

$$f(\nu,t) = M(\nu,t)\,\bar{A}(\nu^2,t) \tag{2.10}$$

$$\left( f_i(\nu,t) = \sum_j M_{ij}(\nu,t)\,\bar{A}_j(\nu^2,t)\ , \quad i,j = 1,2,\ldots,6 \right)$$

can in fact be written in terms of 3x3 matrices since $M(\nu,t)$ is actually of the type

$$M(\nu,t) = \begin{pmatrix} M^{(1)}(\nu,t) & 0 \\ 0 & M^{(2)}(\nu,t) \end{pmatrix}\ , \tag{2.11}$$

where the superscripts (1),(2) refer to photon helicity flip and non-flip amplitudes (indices i,j=1,2,3 and i,j=4,5,6 , respectively). The concrete form of $M^{(1)}(\nu,t)$, $M^{(2)}(\nu,t)$ in our normalizations and phase conventions can be read off from the following relations :

$$f_1(\nu,t) \equiv f_{-1,-\frac{1}{2};1,\frac{1}{2}}(\nu,t) = \frac{m\,(-t)^{1/2}}{16\left(\nu-\frac{t}{4}\right)\left(\nu^2-\nu_0^2\right)}\ \cdot$$

$$\cdot\left[-t\left(\nu-\frac{t}{4}+m^2\right)\bar{A}_1 + t\left(\nu-\frac{t}{4}\right)\bar{A}_2 - \frac{\nu}{m^2}\left(8\nu^2-2\nu t+m^2 t\right)\bar{A}_3\right],$$

$$f_2(\nu,t) \equiv \sqrt{2}\,f_{-1,\frac{1}{2};1,\frac{1}{2}}(\nu,t) = \frac{m^2(-t)\sqrt{2}\left(\nu^2-\nu_{min}^2\right)^{1/2}}{8\left(\nu-\frac{t}{4}\right)\left(\nu^2-\nu_0^2\right)}\left[\bar{A}_1+\frac{\nu}{m^2}\bar{A}_3\right],$$

$$f_3(\nu,t) \equiv f_{-1,\frac{1}{2};1,-\frac{1}{2}}(\nu,t) = \frac{m\,(-t)^{1/2}(-t)}{16\left(\nu-\frac{t}{4}\right)\left(\nu^2-\nu_0^2\right)}\ \cdot$$

$$\cdot\left[\left(\nu-\frac{t}{4}+m^2\right)\bar{A}_1 + \left(\nu-\frac{t}{4}\right)\bar{A}_2 + \nu\bar{A}_3\right]\ , \tag{2.12}$$

$$f_4(\nu,t) \equiv f_{1,\frac{1}{2};1,\frac{1}{2}}(\nu,t) = \frac{(\nu^2-\nu^2_{min})^{1/2}}{16(\nu-\frac{t}{4})(\nu^2-\nu_B^2)}\left[2(4\nu^2-2\nu t+\frac{t^2}{4}+m^2t)\bar{A}_4+\right.$$

$$\left.+4(\nu^2-\nu^2_{min})\bar{A}_5-\frac{\nu}{m^2}(4\nu^2-2\nu t+\frac{t^2}{4}-m^2t)\bar{A}_6\right],$$

$$f_5(\nu,t) \equiv \sqrt{2}\, f_{1,-\frac{1}{2};1,\frac{1}{2}}(\nu,t) = \frac{\sqrt{2}\,(-t)^{1/2}(\nu^2-\nu^2_{min})}{8(\nu-\frac{t}{4})(\nu^2-\nu_B^2)}\cdot$$

$$\cdot\left[-2m\bar{A}_4-\frac{1}{m}(\nu-\frac{t}{4}+m^2)\bar{A}_5-\frac{\nu}{m}\bar{A}_6\right],$$

$$f_6(\nu,t) \equiv f_{1,-\frac{1}{2};1,\frac{1}{2}}(\nu,t) = \frac{(\nu^2-\nu^2_{min})^{3/2}}{4(\nu-\frac{t}{4})(\nu^2-\nu_B^2)}\cdot$$

$$\cdot\left[2\bar{A}_4+\bar{A}_5+\frac{\nu}{m^2}\bar{A}_6\right]$$

The $s-u$ crossing relations for the helicity amplitudes are, in 6x6 matrix notation,

$$f(-\nu,t) = C(\nu^2,t)\, f(\nu,t) \qquad (2.13)$$

where

$$C(\nu^2,t) = \begin{pmatrix} C^{(1)}(\nu^2,t) & 0 \\ 0 & C^{(2)}(\nu^2,t) \end{pmatrix} \qquad (2.14)$$

and

$$C^{(k)}(\nu^2,t) = \begin{pmatrix} \frac{1}{2}(1+\cos\gamma^{(k)}) & -\frac{1}{\sqrt{2}}\sin\gamma^{(k)} & -\frac{1}{2}(1-\cos\gamma^{(k)}) \\ -\frac{1}{\sqrt{2}}\sin\gamma^{(k)} & -\cos\gamma^{(k)} & -\frac{1}{\sqrt{2}}\sin\gamma^{(k)} \\ -\frac{1}{2}(1-\cos\gamma^{(k)}) & -\frac{1}{\sqrt{2}}\sin\gamma^{(k)} & \frac{1}{2}(1+\cos\gamma^{(k)}) \end{pmatrix} \qquad (2.15)$$

$(k = 1, 2)$

$$\cos\gamma^{(1)} = \frac{\nu^2 - \nu_{min}^2 + \frac{m^2 t}{4}}{\nu^2 - \nu_0^2} \quad ,$$

$$\sin\gamma^{(1)} = \frac{m(-t)^{1/2}\,(\nu^2-\nu_{min}^2)^{1/2}}{\nu^2-\nu_0^2} \quad , \qquad (2.16)$$

$$\gamma^{(2)} = \gamma^{(1)} + \pi$$

The matrices $C^{(k)}(\nu^2,t)$ (k=1,2) are unitary (and, moreover, orthogonal) on the unitarity cut

$$C^{(k)\dagger}(\nu^2,t)\, C^{(k)}(\nu^2,t) = I \quad , \quad \nu^2 \geq \nu_0^2 \quad . \qquad (2.17)$$

The unpolarized centre of mass differential cross section is given by

$$128\pi^2 s \left(\frac{d\sigma}{d\Omega}\right)_{c.m.} \equiv \sigma(\nu^2,t) = f^+(\nu,t)\, f(\nu,t) = \sum_{i=1}^{6} |f_i(\nu,t)|^2 \qquad (2.18)$$

In further considerations it will be sometimes convenient to use the conformal mapping

$$\nu^2 = \frac{4\nu_0^2 z}{(1+z)^2} \quad , \quad z = \frac{\nu_0 - \sqrt{\nu_0^2 - \nu^2}}{\nu_0 + \sqrt{\nu_0^2 - \nu^2}} \tag{2.19}$$

which brings the cut complex $\nu^2$ -plane onto the interior of the unit disk of the new complex $z$ -plane such that the upper and lower borders of the unitarity cut in the $\nu^2$ -plane map onto the upper and lower semicircles ; the points $z = 1$ , $z = z_{min}$ , $z = z_0$ , $z = o$ , $z = -1$ correspond, respectively, to the points $\nu_0^2$ , $\nu_{min}^2$ , $\nu_0^2$ , $o$ , $\infty$ .

## III. FORMULATION OF THE PROBLEM

For $t$ in the region specified by Eqs.(2.6),(2.6') we shall consider the unpolarized differential cross section $\left(\frac{d\sigma}{d\Omega}\right)_{c.m.}$ as a known quantity along the whole unitarity cut (or, equivalently, we shall consider $\sigma(z') = \sigma(\nu^2(z'),t)$ as a given function along the unit circle $|z'| = 1$), and look for the constraints on the values of the amplitudes $\bar{A}_i$ at the interior point $z = z_0$ (corresponding to $\nu^2 = \nu_0^2$ ) which result as a consequence of the analyticity properties of $\bar{A}_i$ in the domain $|z| < 1$ . Omitting in this Section, to simplify notations, the parametric dependence on $t$ , we shall start, according to Eqs.(2.18), (2.10), from the boundary condition on the set of analytic functions $\bar{A}_i(z)$ $(i=1,2,\ldots,6)$ :

$$\sigma(z') = f^+ f = \bar{A}^+(z')\, P(z')\, \bar{A}(z') \, , \quad (|z'| = 1) \quad , \tag{3.1}$$

where the matrix

$$P(z') = M^+ M \tag{3.2}$$

is hermitian and positive definite on the circle $|z'| = 1$, and ask for the strongest restrictions which will follow on the domain of values which can be possibly taken by $\tilde{A}_i(z_0)$ $(i=1,2,\ldots,6)$, $|z_0|<1$.

If $P$ were the unit matrix, the procedure to be followed in order to derive bounds on some analytic functions $\varphi_i(z)$ $(i=1,2,\ldots)$ at an interior point $z_1$, $|z_1|<1$, starting from the condition

$$\sigma(z') = \varphi^+(z')\,\varphi(z') = \sum_i |\varphi_i(z')|^2 \quad , |z'| = 1 \tag{3.3}$$

would be simple (the solution of this problem has, in fact, already been used in some physical applications /2/,/3/,/6/). One firstly constructs through the known procedure a function $S(z)$, analytic, without zeros for $|z|<1$, and of modulus squared $|S(z)|^2$ equal to $\sigma(z')$ on the circle $z' = e^{i\theta}$, $\theta\in[-\pi,\pi]$:

$$S(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\sigma(z'=e^{i\theta})\,\frac{z'+z}{z'-z}\,d\theta\right) \tag{3.4}$$

Then the condition (3.3) can be written as

$$\sum_i \left|\frac{\varphi_i(z')}{S(z')}\right|^2 = 1 \quad , \qquad |z'| = 1 \tag{3.5}$$

which leads to

$$\sum_i \left|\frac{\varphi_i(z_1)}{S(z_1)}\right|^2 \leqslant 1 \qquad \text{for any } z_1 ,\ |z_1|<1 , \tag{3.6}$$

according to the principle of maximum for the subharmonic function $\sum_i|\varphi_i(z)/S(z)|^2$. This bound is optimal since it can be saturated with the functions $(\varphi_i(z)/S(z)) = a_i$, $\sum_i|a_i|^2 = 1$. More subtle bounds on the values of the functions $\varphi_i(z)$ and their derivatives at interior points which may change with the functions and the order of the derivatives, have been derived through more elaborate methods in refs./4/ (see also ref./7/).

In our concrete case we have, however, to solve the problem for a non-diagonal matrix $P$. General theorems /8/ ensure the existence of an analytic factorization for a hermitian matrix $P(z')$, positive definite (almost everywhere) on the unit circle $|z'| = 1$. Under general assumptions, fulfilled in our case, on which we shall not insist here (see refs./4/,/8/), such theorems state that $P(z')$ can be represented as a product

$$P(z') = N^{+}(z')\, N(z') \tag{3.7}$$

where $N(z)$ is a matrix, analytic and invertible ($\det N(z) \neq 0$) inside the unit disk ( $|z| < 1$ ), unique up to a constant unitary matrix to the left.

This is only an existence theorem and there are no practical means to construct explicitly $N(z)$ for a general given $P(z)$. Nevertheless, in the actual case of spin 1/2 hadron Compton scattering, i.e. for the particular $P = M^{+}M$ with $M$ given by Eqs.(2.10), we have succeeded to solve this problem of factorization. The next two Sections will be devoted to the detailed presentation of the derivation.

Once $N(z)$ is determined explicitly, the condition (3.1) can be rewritten entirely in terms of analytic objects

$$\sigma(z') = \bar{A}^{+}(z')\, N^{+}(z')\, N(z')\, \bar{A}(z') \; , \quad |z'| = 1 \tag{3.8}$$

Introducing new amplitudes $\varphi_i(z)$ $(i=1,2,\ldots,6)$, analytic in the unit disk

$$\varphi(z) = N(z)\, \bar{A}(z) \quad , \quad \bar{A}(z) = N^{-1}(z)\, \varphi(z) \tag{3.9}$$

Eq.(3.8) can be brought further to the form of Eq.(3.3) and so the non-diagonal problem has been reduced to the diagonal one. The (optimal) bound looked for will, therefore, be

$$\bar{A}^{+}(z_0)\, N^{+}(z_0)\, N(z_0)\, \bar{A}(z_0) = \sum_{i=1}^{6} \Big| \sum_{j=1}^{6} N_{ij}(z_0)\, \bar{A}_j(z_0) \Big|^2 \leq \big| S(z_0) \big|^2 \tag{3.10}$$

It relates through an inequality a known (according to Eqs.(2.9)) function of $t$ and of the target's charge and anomalous magnetic moment to a certain integral (Eq.(3.4)) over the unpolarized differential cross section above the first inelastic threshold.

Coming back to the variable $\nu^2$, we note that the amplitudes $\varphi_i(\nu^2,t)$, ($i=1,2,\dots,6$) are, in view of Eqs.(3.9), as good as $\bar{A}_i(\nu^2,t)$ from the point of view of their analyticity properties in $\nu^2$ (at $t$ fixed in the region specified by Eqs.(2.6),(2.6')), being at the same time unitarily connected (on the unitarity cut) with the helicity amplitudes $f_i(\nu,t)$:

$$\sigma(\nu^2,t) = f^{+}f = \varphi^{+}(\nu^2,t)\,\varphi(\nu^2,t) \qquad \text{for } \nu^2 > \nu_0^2 \tag{3.11}$$

IV. EXPLICIT CONSTRUCTION. PHOTON HELICITY FLIP CASE

As already mentioned in Section II, the relevant 6x6 matrices decouple in pairs of 3x3 matrices and so we shall deal separately with quantities referring to photon helicity flip and photon helicity non-flip subspaces, labelled (when needed) by superscripts (1) and (2), respectively. In this Section we are treating the former case.

$1^{\circ}$. The first step consists in diagonalizing the matrix $C^{(1)}(\nu^2,t)$ (Eqs.(2.15),(2.16),(2.17)) by a matrix $X^{(1)}(\nu^2,t)$ which is orthogonal on the unitarity cut

$$X^{(1)+}X^{(1)} = X^{(1)}X^{(1)+} = I \qquad \text{for } \nu^2 > \nu_0^2 \tag{4.1}$$

and has the following form

$$X^{(1)}(\nu,t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\varphi^{(1)} & -\sin\varphi^{(1)} & 0 \\ \sin\varphi^{(1)} & \cos\varphi^{(1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}, \tag{4.2}$$

where

$$\cos\varphi^{(1)} = \frac{(\nu^2-\nu_{min}^2)^{1/2}}{(\nu^2-\nu_0^2)^{1/2}}, \qquad \sin\varphi^{(1)} = \frac{m\,(-t)^{1/2}}{2\,(\nu^2-\nu_0^2)^{1/2}}. \tag{4.3}$$

One has

$$X^{(1)}C^{(1)}X^{(1)+} = \Gamma^{(1)} \tag{4.4}$$

with

$$\Gamma^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

The helicity amplitudes $f_i(\nu,t)$ (i=1,2,3) are transformed by the matrix $X^{(1)}$ into new "transversity" /9/ amplitudes $f_i'(\nu,t)$ (i=1,2,3)

$$\begin{pmatrix} f_1'(\nu,t) \\ f_2'(\nu,t) \\ f_3'(\nu,t) \end{pmatrix} \equiv f^{(1)\prime}(\nu,t) = X^{(1)}f^{(1)}(\nu,t) = X^{(1)} \begin{pmatrix} f_1(\nu,t) \\ f_2(\nu,t) \\ f_3(\nu,t) \end{pmatrix} \tag{4.6}$$

with well defined parity under s-u crossing ( $\nu \to -\nu$ ) :

$$f^{(1)'}(-\nu,t) = \Gamma^{(1)} f^{(1)'}(\nu,t) \quad . \tag{4.7}$$

For $\nu^2 > \nu_o^2$ , since $X^{(1)}$ is orthogonal, one has

$$f^{(1)+} f^{(1)} = f^{(1)'+} f^{(1)'} \tag{4.8}$$

The connection between $f_i'$ (i=1,2,3) and the analytic amplitudes $\bar{A}_i(\nu^2,t)$ (i=1,2,3) is given by

$$f_1^{(1)'}(\nu,t) = X^{(1)} M^{(1)} \bar{A}^{(1)} \, , \quad \bar{A}^{(1)}(\nu^2,t) \equiv \begin{pmatrix} \bar{A}_1(\nu^2,t) \\ \bar{A}_2(\nu^2,t) \\ \bar{A}_3(\nu^2,t) \end{pmatrix} \tag{4.9}$$

or, explicitly, by

$$f_1'(\nu,t) = \frac{m(-t)^{1/2}(\nu^2-\nu^2_{min})^{1/2}}{8\sqrt{2}\,(\nu^2-\nu_o^2)^{3/2}}\left[-t\bar{A}_1 - 4\frac{\nu^2}{m^2}\bar{A}_3\right] ,$$

$$f_2'(\nu,t) = \frac{m^2(-t)\,\nu}{16\sqrt{2}\,(\nu^2-\nu_o^2)^{3/2}}\left[4\bar{A}_1 + \frac{t}{m^2}\bar{A}_3\right] , \tag{4.10}$$

$$f_3'(\nu,t) = \frac{m\,(-t)^{1/2}}{8\sqrt{2}\,(\nu^2-\nu_o^2)}\left[t\bar{A}_2 - 4\frac{\nu^2}{m^2}\bar{A}_3\right] .$$

The matrix $X^{(1)}M^{(1)}$ is not analytic in the cut complex $\nu^2$ -plane but (and this is the remarkable fact which essentially enables us to solve the problem) all the non-analytic behaviour appears in Eqs.(4.10) factorized in front of the square brackets which, themselves, do have the required analyticity properties in $\nu^2$ .

2°. The next step is therefore the construction of three new ("regularized transversity") amplitudes $\tilde{f}_i(\nu^2,t)$ which (unlike $f_i'$) be analytic functions of $\nu^2$ in the complex $\nu^2$ -plane cut along the real axis from $\nu_0^2$ to $\infty$ and which on the cut have the same moduli as $f_i$ . This is easily achieved by finding for each of the factors $\varrho_i$ (i=1,2,3) multiplying the square brackets in Eqs.(4.10) a function $R_i(\nu^2,t)$ of $\nu^2$ , analytic in the considered domain, without zeros, and of the same modulus on the cut $[\nu_0^2, \infty)$ as the corresponding factor $\varrho_i$ itself. The procedure to be followed is just that which led to Eq.(3.4). In the $\nu^2$ - variable *one has*

$$R_i(\nu^2,t) = \exp\left| \frac{(\nu_0^2-\nu^2)^{1/2}}{\pi} \int_{\nu_0^2}^{\infty} \frac{\ln|\varrho_i(\nu'^2,t)|\, d\nu'^2}{(\nu'^2-\nu^2)(\nu'^2-\nu_0^2)^{1/2}} \right| , \qquad (4.11)$$

$$|R_i(\nu'^2,t)| = |\varrho_i(\nu'^2,t)| \qquad \textit{for} \quad \nu'^2 > \nu_0^2 \quad , i=1,2,3$$

Through simple calculations it is then seen that the new amplitudes $\tilde f_i(\nu^2,t)$ can be introduced in the following form :

$$\begin{pmatrix} \tilde f_1(\nu^2,t) \\ \tilde f_2(\nu^2,t) \\ \tilde f_3(\nu^2,t) \end{pmatrix} \equiv \tilde f^{(1)}(\nu^2,t) = \mathcal{N}^{(1)}(\nu^2,t)\, \bar A^{(1)}(\nu^2,t) \quad , \qquad (4.12)$$

where the matrix $\mathcal{N}_{ij}^{(1)}(\nu^2,t)$ (i,j=1,2,3) is

$$\mathcal{N}^{(1)}(\nu^2,t) = \frac{m(-t)^{1/2}}{\sqrt{2}\,[L_1(\nu^2,t)]^3} \begin{pmatrix} -\frac{t}{2} L_2(\nu^2,t) & 0 & -\frac{2\nu^2}{m^2} L_2(\nu^2,t) \\ 2m(-t)^{1/2}[\nu_0+(\nu_0^2-\nu^2)^{1/2}] & 0 & \frac{t(-t)^{1/2}}{2m}[\nu_0+(\nu_0^2-\nu^2)^{1/2}] \\ 0 & \frac{t}{2} L_1(\nu^2,t) & -\frac{2\nu^2}{m^2} L_1(\nu^2,t) \end{pmatrix} \qquad (4.13)$$

*The functions*

$$L_1 \equiv L_1(\nu^2,t) \equiv 2\left[(\nu_0^2-\nu_B^2)^{1/2} + (\nu_0^2-\nu^2)^{1/2}\right] ,$$

$$L_2 \equiv L_2(\nu^2,t) \equiv 2\left[(\nu_0^2-\nu_{min}^2)^{1/2} + (\nu_0^2-\nu^2)^{1/2}\right] , \tag{4.14}$$

are both non-vanishing in the cut complex $\nu^2$ -plane (i.e. in the domain which corresponds to the open unit disk by the conformal mapping (2.19)).

On the unitarity cut each of $\tilde{f}_i$ differs by construction from the corresponding $f_i'$ only by a phase and therefore

$$\tilde{f}^{(1)\dagger}\tilde{f}^{(1)} = f^{(1)'\dagger}f^{(1)'} = f^{(1)\dagger}f^{(1)} \qquad \text{for } \nu^2 > \nu_0^2 \tag{4.15}$$

So we have at hand some amplitudes $\tilde{f}_i(\nu^2,t)$ (i=1,2,3) which are analytic functions of $\nu^2$ in the cut complex plane and which are also connected with the original photon helicity flip amplitudes $f_i(\nu,t)$ by a matrix which is unitary on the unitarity cut. Unfortunately, there is one essential quality still lacking : the connection between $\tilde{f}_i(\nu^2,t)$ and $\bar{A}_i(\nu^2,t)$ (i=1,2,3) is not invertible. Looking at the determinant of the matrix $\mathcal{N}^{(1)}(\nu^2,t)$, one sees that as a function of $\nu^2$ it has a simple zero at the location of the Born poles

$$\det \mathcal{N}^{(1)}(\nu^2 = \nu_B^2 , t) = 0 \tag{4.16}$$

This zero is very unpleasant since the resulting bound would certainly lose its optimality, some existing information escaping consideration. To carry out our task to the end we have to go through the

$3^{0}$. last step of the derivation, devoted to the unitary factorization of this unwanted zero. This means that we would like to find instead of $\mathcal{N}^{(1)}(\nu^2,t)$ another matrix, $N^{(1)}(\nu^2,t)$ say, also analytic in the cut complex $\nu^2$ -plane, but such that det $N^{(1)}(\nu^2,t)$ be non-vanishing everywhere in the domain and nevertheless $N^{(1)}$ differ from $\mathcal{N}^{(1)}$ on the boundary of the domain (i.e. on the unitarity cut) only by a unitary matrix.

Returning to the $z$ -variable and working on the unit disk to simplify the discussion, we recall that for a scalar (i.e. not matrix-valued) function $g(z)$ , having a zero at $z = z_0$ , one can easily construct a new function $\bar{g}(z)$ , with the same modulus as $g(z)$ on the unit circle $|z| = 1$, but such that $\bar{g}(z) \neq 0$ at $z = z_0$ , by using the Blaschke factorization

$$\bar{g}(z) = \left( \frac{z - z_0}{1 - z_0^* z} \right)^{-1} g(z) \tag{4.17}$$

In our case, when one deals with matrix-valued functions, the generalisation of this procedure relies on the use of the Blaschke-Potapov factorization /10/. Let $m(z)$ be a matrix-valued function analytic in the unit disk $|z| < 1$ and $z_0$ ($|z_0| < 1$ ) be a simple zero of det $m(z)$. The numerical matrix $m(z_0)$ can then be represented according to known algebraic rules as a product

$$m(z_0) = U D V \tag{4.18}$$

where $U$ and $V$ are both unitary matrices while $D$ is diagonal. Since det $m(z_0) = 0$ , one of the diagonal elements of $D$ should be zero. Then, in our case, $D$ should be, say, of the form

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \tag{4.19}$$

Defining the diagonal matrix

$$b(z) = \begin{pmatrix} \frac{z - z_0}{1 - z_0^* z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.20}$$

which is unitary along the unit circle,

$$\mathcal{B}^{+}(z')\,\mathcal{B}(z') = \mathcal{B}(z')\,\mathcal{B}^{+}(z') = I \qquad \text{for } |z'| = 1 \qquad (4.21)$$

and analytic for $|z| < 1$ , one consequently introduces the Blaschke-Potapov matrix

$$B(z) \equiv U\mathcal{B}(z)V \qquad (4.22)$$

which is also unitary along the unit circle,

$$B^{+}(z')\,B(z') = B(z')\,B^{+}(z') = I \qquad \text{for } |z'| = 1 \qquad (4.23)$$

and invertible, excepting the pcint $z_0$ , everywhere in the unit disk. Then a theorem /10/ states that the new matrix $N(z)$ ,

$$N(z) \equiv B^{-1}(z)\; m(z) \qquad (4.24)$$

is analytic at $z = z_0$ . Now $\det N(z_0) \neq 0$ and on the unit circle $|z'| = 1$, $N(z')$ differs from $m(z')$ only by the unitary matrix $B^{-1}(z')$ .

In the following, coming back to the $\nu^2$ -variable, we shall apply the above prescriptions to the matrix $\mathcal{J}^{(1)}(\nu^2,t)$ in order to get rid in the desired way of the zero at $\nu^2 = \nu_B^2$ . We shall omit (actually non-trivial) algebraic details and display below only the essential results.

The Blaschke-Potapov matrix (Eq.(4.22)), for the particular case of the matrix $\mathcal{J}^{(1)}(\nu^2,t)$ we are interested in, has the form

$$B^{(1)}(\nu^2,t) = U^{(1)}\;\;\mathcal{B}^{(1)}(\nu^2,t) \qquad (4.25)$$

where

$$U^{(1)} = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.26}$$

$$\cos\alpha_1 = \frac{2m(-t)^{1/2}\left[\nu_0 + (\nu_0^2-\nu_B^2)^{1/2}\right]}{\lambda}$$

$$\sin\alpha_1 = \frac{t\left[(\nu_0^2-\nu_{min}^2)^{1/2} + (\nu_0^2-\nu_B^2)^{1/2}\right]}{\lambda} \tag{4.27}$$

$$\lambda = \left\{4m^2(-t)\left[\nu_0+(\nu_0^2-\nu_B^2)^{1/2}\right]^2 + t^2\left[(\nu_0^2-\nu_{min}^2)^{1/2}+(\nu_0^2-\nu_B^2)^{1/2}\right]^2\right\}^{1/2}$$

and

$$b^{(1)}(\nu^2,t) = \begin{pmatrix} b_B(\nu^2,t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.28}$$

with

$$b_B(\nu^2,t) = \frac{z-z_B}{1-z_B^* z} = \frac{(\nu_0^2-\nu_B^2)^{1/2} - (\nu_0^2-\nu^2)^{1/2}}{(\nu_0^2-\nu_B^2)^{1/2} + (\nu_0^2-\nu^2)^{1/2}} . \tag{4.29}$$

$B^{(1)}(\nu^2,t)$ is unitary on the unitarity cut.

The matrix $N^{(1)}(\nu^2,t)$, analytic and invertible everywhere in the cut complex $\nu^2$ -plane according to the previously mentioned rules (Eq.(4.24)), is given by

$$N^{(1)}(\nu^2,t) = B^{(1)-1}(\nu^2,t)\,\mathcal{N}^{(1)}(\nu^2,t) . \tag{4.30}$$

The above noted properties of $N^{(1)}(\nu^2,t)$ can be checked through straightforward calculations which finally yield

$$
N^{(1)}(\nu^2,t) = \begin{pmatrix} N_{11}^{(1)} & N_{12}^{(1)} & N_{13}^{(1)} \\ N_{21}^{(1)} & N_{22}^{(1)} & N_{23}^{(1)} \\ N_{31}^{(1)} & N_{32}^{(1)} & N_{33}^{(1)} \end{pmatrix} ,
$$

$$
N_{11}^{(1)}(\nu^2,t) = -2^{-1/2}\, \lambda^{-1}\, m^2 t^2 (L_1)^{-2} \left[\nu_0 - (\nu_0^2 - \nu_{min}^2)^{1/2}\right] ,
$$

$$
N_{12}^{(1)}(\nu^2,t) = 0 ,
$$

$$
N_{13}^{(1)}(\nu^2,t) = -2^{-1/2}\lambda^{-1}\, t\, (L_1)^{-2}\left\{-L_1 L_2\left[\nu_0 + (\nu_0^2 - \nu^2)^{1/2}\right] + \frac{t^2}{4}\left[\nu_0 - (\nu_0^2 - \nu_{min}^2)^{1/2}\right]\right\} , \quad (4.31)
$$

$$
N_{21}^{(1)}(\nu^2,t) = 2^{-1/2}\, m\, (-t)^{1/2} (L_1)^{-3}\left\{\frac{t}{2} L_2 \sin\alpha_1 + 2m(-t)^{1/2} \cdot \right.
$$

$$
\left. \cdot \left[\nu_0 + (\nu_0^2 - \nu^2)^{1/2}\right] \cos\alpha_1 \right\} ,
$$

$$
N_{22}^{(1)}(\nu^2,t) = 0 ,
$$

$$
N_{23}^{(1)}(\nu^2,t) = 2^{-3/2} m^{-1} (-t)^{1/2} (L_1)^{-3}\left\{4\nu^2 L_2 \sin\alpha_1 + mt(-t)^{1/2} \cdot \right.
$$

$$
\left. \cdot \left[\nu_0 + (\nu_0^2 - \nu^2)^{1/2}\right] \cos\alpha_1 \right\} ,
$$

$$
N_{31}^{(1)}(\nu^2,t) = 0 ,
$$

$$
N_{32}^{(1)}(\nu^2,t) = 2^{-3/2}\, m t\, (-t)^{1/2} (L_1)^{-2} ,
$$

$$
N_{33}^{(1)}(\nu^2,t) = -2^{1/2}\, m^{-1} (-t)^{1/2}\, \nu^2 (L_1)^{-2}
$$

The functions $L_1$ , $L_2$ have been defined in Eqs.(4.14) and $\sin\alpha_1$, $\cos\alpha_1$ are given by Eqs.(4.27).

At this point half of our task is completed. We have explicitly found three new ("fully regularized transversity") amplitudes

$$\varphi^{(1)}(\nu^2,t) = \begin{pmatrix} \varphi_1(\nu^2,t) \\ \varphi_2(\nu^2,t) \\ \varphi_3(\nu^2,t) \end{pmatrix} \tag{4.32}$$

defined as

$$\varphi^{(1)}(\nu^2,t) = N^{(1)}(\nu^2,t)\,\bar{A}^{(1)}(\nu^2,t) \tag{4.33}$$

such that for $t$ in the range given by Eqs.(2.6),(2.6') their connection with $\bar{A}_i$ (i=1,2,3) is analytic and invertible everywhere in the cut complex $\nu^2$-plane, while on the unitarity cut they are unitarily connected to the photon helicity flip amplitudes $f_i(\nu,t)$ (i=1,2,3) ;

$$\varphi^{(1)+}\varphi^{(1)} = \tilde{f}^{(1)+}\tilde{f}^{(1)} = f^{(1)'}f^{(1)'} = f^{(1)+}f^{(1)} \qquad \text{for } \nu^2>\nu_o^2 \;. \tag{4.34}$$

V. EXPLICIT CONSTRUCTION. PHOTON HELICITY NON-FLIP CASE

We have to pass now through the same main steps of the derivation as in Section IV. The diagonalization of the crossing matrix $C^{(2)}(\nu^2,t)$ (Eq.(2.15)) is performed with

$$X^{(2)}(\nu,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\varphi^{(2)} & -\sin\varphi^{(2)} & 0 \\ \sin\varphi^{(2)} & \cos\varphi^{(2)} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{5.1}$$

where

$$\varphi^{(2)} = \varphi^{(1)} + \frac{3\pi}{2} \tag{5.2}$$

and $\varphi^{(1)}$ is defined through Eqs.(4.3). So one has

$$X^{(2)+} X^{(2)} = X^{(2)} X^{(2)+} = I \qquad \text{for } \nu^2 > \nu_0^2 , \tag{5.3}$$

$$X^{(2)} C^{(2)} X^{(2)+} = \Gamma^{(2)} , \tag{5.4}$$

$$\Gamma^{(2)} - \Gamma^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{5.5}$$

The helicity amplitudes

$$f^{(2)}(\nu,t) \equiv \begin{pmatrix} f_4(\nu,t) \\ f_5(\nu,t) \\ f_6(\nu,t) \end{pmatrix} \tag{5.6}$$

go into new ("transversity") amplitudes

$$\begin{pmatrix} f_4'(\nu,t) \\ f_5'(\nu,t) \\ f_6'(\nu,t) \end{pmatrix} \equiv f^{(2)'}(\nu,t) = X^{(2)}(\nu,t)\, f^{(2)}(\nu,t) \tag{5.7}$$

even or odd under $s-u$ crossing

$$f^{(2)'}(-\nu,t) = \Gamma^{(2)} f^{(2)'}(\nu,t) . \tag{5.8}$$

On the unitarity cut one has

$$f^{(2)'+} f^{(2)'} = f^{(2)+} f^{(2)} , \qquad (\nu^2 > \nu_0^2) \tag{5.9}$$

The relationship between $f_i'$ (i=4,5,6) and the analytic objects

$$\bar{A}^{(2)}(\nu^2,t) = \begin{pmatrix} \bar{A}_4(\nu^2,t) \\ \bar{A}_5(\nu^2,t) \\ \bar{A}_6(\nu^2,t) \end{pmatrix} \tag{5.10}$$

is

$$f^{(2)'}(\nu,t) = X^{(2)}(\nu,t)\, M^{(2)}(\nu,t)\, \bar{A}^{(2)}(\nu^2,t) \qquad (5.11)$$

where

$$X^{(2)}(\nu,t)\, M^{(2)}(\nu,t) = \frac{(\nu^2-\nu^2_{min})^{1/2}}{16\sqrt{2}\,(\nu^2-\nu^2_{0})^{3/2}} \cdot$$

$$\begin{pmatrix} -2mt\,(-t)^{1/2} & -\frac{4(-t)^{1/2}}{m}(\nu^2-\nu^2_{min}) & -\frac{4(-t)^{1/2}}{m}\nu^2 \\ -16\nu\,(\nu^2-\nu^2_{min})^{1/2} & -8\nu\,(\nu^2-\nu^2_{min})^{1/2} & -\frac{2t}{m^2}\nu\,(\nu^2-\nu^2_{min})^{1/2} \\ -4t\,(\nu^2-\nu^2_0)^{1/2} & 0 & -\frac{8}{m^2}\nu^2(\nu^2-\nu^2_0)^{1/2} \end{pmatrix} \qquad (5.12)$$

**As in the previous Section new ("regularized transversity") amplitudes $\tilde{f}_i(\nu^2,t)$ (i=4,5,6) which are this time analytic in the cut complex $\nu^2$-plane can be defined through an analogous procedure of regularization in such a way that on the cut they differ from $f_i'$ (i=4,5,6) only by phases :**

$$\tilde{f}^{(2)}(\nu^2,t) \equiv \begin{pmatrix} \tilde{f}_4(\nu^2,t) \\ \tilde{f}_5(\nu^2,t) \\ \tilde{f}_6(\nu^2,t) \end{pmatrix} = \mathcal{N}^{(2)}(\nu^2,t)\, \bar{A}^{(2)}(\nu^2,t) \qquad (5.13)$$

$$\mathcal{S}^{(2)}(\nu^2,t) = \frac{L_2(\nu^2,t)}{4\sqrt{2}\,[L_1(\nu^2,t)]^3} \cdot$$

$$\cdot \begin{pmatrix} -2mt(-t)^{1/2} & -\frac{4(-t)^{1/2}}{m}(\nu^2-\nu^2_{min}) & -\frac{4(-t)^{1/2}}{m}\nu^2 \\ -8L(\nu^2,t) & -4L(\nu^2,t) & -\frac{t}{m^2}L(\nu^2,t) \\ -2tL_1(\nu^2,t) & 0 & -4\frac{\nu^2}{m^2}L_1(\nu^2,t) \end{pmatrix} \quad (5.14)$$

$L_1$ , $L_2$ were defined in Eqs.(4.14) and

$$L \equiv L(\nu^2,t) = 2\left[\nu_0 + (\nu_0^2-\nu^2)^{1/2}\right]\left[(\nu_0^2-\nu^2_{min})^{1/2} + (\nu_0^2-\nu^2)^{1/2}\right] \quad (5.15)$$

is also analytic and nonvanishing in the analyticity domain in which we are working. On the unitarity cut one has again

$$\tilde{f}^{(2)\dagger}\tilde{f}^{(2)} = f^{(2)'\dagger}f^{(2)'} = f^{(2)\dagger}f^{(2)}, \qquad (\nu^2>\nu_0^2) \,. \quad (5.16)$$

The transformation matrix $\mathcal{S}^{(2)}(\nu^2,t)$ is analytic but not (everywhere) invertible in the cut complex $\nu^2$ -plane. There is a little additional complication now compared to the case treated in Section IV, determined by the fact that the determinant of $\mathcal{S}^{(2)}$ has this time two (simple) zeros, one at $\nu^2 = \nu_B^2$ , as in the spin flip case already considered, and another one at $\nu^2 = \nu^2_{min}$ :

$$\det \mathcal{S}^{(2)}(\nu^2,t) = \frac{32\,(-t)^{1/2}}{m^2\sqrt{2}} L(\nu^2,t)L_2(\nu^2,t)\left[L_1(\nu^2,t)\right]^{-2}(\nu^2-\nu^2_{min})(\nu^2-\nu_B^2) \,. \quad (5.17)$$

Both these zeros can be unitarily removed, one after the other, through the same Blaschke-Potapov factorization procedure. We firstly consider the zero at $\nu^2 = \nu^2_{min}$ and find the first Blaschke-Potapov factor. It is

$$B^{(2)}_I(\nu^2,t) = U^{(2)}_I \, b^{(2)}_I(\nu^2,t) \tag{5.18}$$

where

$$U^{(2)}_I = \begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 & 0 \\ 0 & 0 & 1 \\ \sin\alpha_2 & \cos\alpha_2 & 0 \end{pmatrix} \tag{5.19}$$

$$\cos\alpha_2 = \frac{\bar{L}_1}{(\bar{L}_1^2 - m^2 t)^{1/2}} , \qquad \sin\alpha_2 = \frac{-m(-t)^{1/2}}{(\bar{L}_1^2 - m^2 t)^{1/2}} , \tag{5.20}$$

$$\bar{L}_1 \equiv L_1(\nu^2 = \nu^2_{min}, t) = 2\left[(\nu_0^2 - \nu_0^2)^{1/2} + (\nu_0^2 - \nu^2_{min})^{1/2}\right], \tag{5.21}$$

and

$$b^{(2)}_I(\nu^2,t) = \begin{pmatrix} b_I(\nu^2,t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{5.22}$$

with

$$b_I(\nu^2,t) = \frac{z - z_{min}}{1 - z^*_{min} z} = \frac{(\nu_0^2 - \nu^2_{min})^{1/2} - (\nu_0^2 - \nu^2)^{1/2}}{(\nu_0^2 - \nu^2_{min})^{1/2} + (\nu_0^2 - \nu^2)^{1/2}} . \tag{5.23}$$

$B^{(2)}_I(\nu^2,t)$ is unitary on the unitarity cut. The matrix

$$\mathcal{S}^{(2)}_I(\nu^2,t) \equiv B^{(2)-1}_I(\nu^2,t)\, \mathcal{S}^{(2)}(\nu^2,t) \tag{5.24}$$

is analytic everywhere in the cut complex $\nu^2$ -plane, invertible now at $\nu^2 = \nu^2_{min}$ , but still not invertible at $\nu^2 = \nu^2_B$ :

$$\det \mathcal{S}^{(2)}_I(\nu^2,t) = \frac{\det \mathcal{S}^{(2)}(\nu^2,t)}{\det B^{(2)}_I(\nu^2,t)} = - \frac{\det \mathcal{S}^{(2)}(\nu^2,t)}{\ell_I(\nu^2,t)} , \qquad (5.25)$$

$$\det \mathcal{S}^{(2)}_I(\nu^2 = \nu^2_{min} , t) \neq 0 \quad , \quad \det \mathcal{S}^{(2)}_I(\nu^2 = \nu^2_B , t) = 0$$

Direct calculations lead to the following explicit expression of the matrix $\mathcal{S}^{(2)}_I(\nu^2,t)$ :

$$\mathcal{S}^{(2)}_I(\nu^2,t) = \frac{L_2}{4\sqrt{2}(L_1)^3} \mathcal{S}(\nu^2,t)\, \mathcal{P} , \qquad (5.26)$$

$$\mathcal{S}(\nu^2,t) = \begin{pmatrix} 2t \sin\alpha_2 L_2 & ; & -\frac{2(-t)^{1/2}}{m}\cos\alpha_2 (L_2)^2 & , & \frac{4}{m^2}\sin\alpha_2(\nu^2-\nu^2_B) L_2 \\ 2t[m(-t)^{1/2}\sin\alpha_2 - L_1\cos\alpha_2] & ; & \frac{8(-t)^{1/2}}{m}(\nu^2-\nu^2_{min})\sin\alpha_2 & ; & \frac{4}{m^2}(\nu^2-\nu^2_B)[m(-t)^{1/2}\sin\alpha_2 - L_1\cos\alpha_2] \\ -8L & ; & -8L & ; & 0 \end{pmatrix} \qquad (5.27)$$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & \frac{t}{8m^2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5.28)$$

The $\nu^2$ -independent matrix $\mathcal{P}$ ($\det\mathcal{P} = \frac{1}{2}$ ) has been factorized only for convenience, to facilitate the practical construction of the next Blaschke-Potapov factor associated with the zero at $\nu^2 = \nu^2_B$ . The unitary removal of this last zero is accomplished again according to the general rules summarized in Section IV. One starts with the matrix

$\mathcal{S}(\nu,t)$ evaluated at $\nu^2 = \nu_\Theta^2$

$$\mathcal{S}(\nu^2 = \nu_\Theta^2, t) \equiv \mathcal{S} \equiv \begin{pmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \mathcal{S}_{21} & \mathcal{S}_{22} & \mathcal{S}_{23} \\ \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} 2t \sin\alpha_2 \, \bar{L}_1 & -\frac{2(-t)^{1/2}}{m} \cos\alpha_2 \, (\bar{L}_1)^2 & 0 \\ 2t\left[m(-t)^{1/2}\sin\alpha_2 - 4(\nu_0^2-\nu_\Theta^2)^{1/2}\cos\alpha_2\right] & 2mt(-t)^{1/2}\sin\alpha_2 & 0 \\ -8\bar{L}_1\left[\nu_0 + (\nu_0^2-\nu_\Theta^2)^{1/2}\right] & -8\bar{L}_1\left[\nu_0 + (\nu_0^2-\nu_\Theta^2)^{1/2}\right] & 0 \end{pmatrix} \qquad (5.29)$$

and looks for two unitary matrices $U$ , $V$ such that (see Eq.(4.18))

$$U^+ \mathcal{S} V^+ = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad (5.30)$$

One of the $\lambda_i$ (i=1,2,3) must be equal to zero. Taking $U$ of the form

$$\begin{pmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and representing $V$ similarly ( with some other angles), proceeding then through direct identification and exploiting the already mentioned existing arbitraryness (up to a constant unitary matrix to the left) in our problem of analytic factorization, one finally gets

$$B_{II}^{(2)}(\nu,t) = U_{II}^{(2)} \mathcal{B}_{II}^{(2)}(\nu,t) \qquad (5.31)$$

where

$$U_{\underline{\mathrm{II}}}^{(2)} = \begin{pmatrix} \cos\delta_1 & -\sin\delta_1 & 0 \\ \sin\delta_1 & \cos\delta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} , \tag{5.32}$$

$$\cos\delta_1 = \frac{s_{11} - s_{12}}{\left[ (s_{21} - s_{22})^2 + (s_{11} - s_{12})^2 \right]^{1/2}} = \cos\alpha_2 , \tag{5.33}$$

$$\sin\delta_1 = \frac{s_{21} - s_{22}}{\left[ (s_{21} - s_{22})^2 + (s_{11} - s_{12})^2 \right]^{1/2}} = -\sin\alpha_2 ,$$

$$\begin{aligned} \cos\theta &= (s_{11} s_{22} - s_{21} s_{12}) \left\{ s_{32}^2 \left[ (s_{21} - s_{22})^2 + (s_{11} - s_{12})^2 \right] + (s_{11} s_{22} - s_{21} s_{12})^2 \right\}^{-1/2} = \\ &= -\frac{t}{4} \left\{ 2\nu_0 \left[ \nu_0 + (\nu_0^2 - \nu_B^2)^{1/2} \right] \right\}^{-1/2} , \end{aligned} \tag{5.34}$$

$$\sin\theta = - \left[ \nu_0 + (\nu_0^2 - \nu_B^2)^{1/2} \right] \left\{ 2\nu_0 \left[ \nu_0 + (\nu_0^2 - \nu_B^2)^{1/2} \right] \right\}^{-1/2} .$$

and

$$b_{\underline{\mathrm{II}}}^{(2)}(\nu^2, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_B(\nu^2, t) \end{pmatrix} . \tag{5.35}$$

The angle $\alpha_2$ has been defined in Eqs.(5.2o) and the quantity $b_B(\nu^2, t)$ in Eq.(4.29). The (Blaschke-Potapov) matrix $B_{\underline{\mathrm{II}}}^{(2)}(\nu^2, t)$ is unitary on the unitarity cut ( $\nu^2 > \nu_0^2$ ) by construction and

$$\det B_{\mathrm{II}}^{(2)}(\nu^2,t) = b_0(\nu^2,t) \tag{5.36}$$

On the basis of the same mathematical arguments /10/ the matrix

$$B_{\mathrm{II}}^{(2)-1}(\nu^2,t)\, \mathcal{S}(\nu^2,t)$$

will be analytic everywhere in the cut complex $\nu^2$ -plane, and, according to Eq.(5.26), so will be the matrix

$$N^{(2)}(\nu^2,t) \equiv B_{\mathrm{II}}^{(2)-1}(\nu^2,t)\, \mathcal{S}_{\mathrm{I}}^{(2)}(\nu^2,t) = B_{\mathrm{II}}^{(2)-1}(\nu^2,t)\, B_{\mathrm{I}}^{(2)-1}(\nu^2,t)\, \mathcal{S}^{(2)}(\nu^2,t). \tag{5.37}$$

Moreover, $N^{(2)}(\nu^2,t)$ is now invertible everywhere in the cut complex $\nu^2$ - plane,

$$\det N^{(2)}(\nu^2,t) \neq 0 \tag{5.38}$$

and it differs from the original matrix $\mathcal{S}^{(2)}(\nu^2,t)$ only by a matrix factor which is unitary on the unitarity cut. All these conclusions can as a matter of fact, be again checked through straightforward algebraic calculations. Their result is

$$N^{(2)}(\nu^2,t) = \frac{L_2}{4\sqrt{2}\,(L_1)^3}\, N'(\nu^2,t)\, \mathcal{G} \tag{5.39}$$

with

$$N'(\nu^2,t) = \begin{pmatrix} N'_{11} & N'_{12} & N'_{13} \\ N'_{21} & N'_{22} & N'_{23} \\ N'_{31} & N'_{32} & N'_{33} \end{pmatrix},$$

$$N'_{11}(\nu^2,t) = 2t \sin\alpha_2 \left[ (L_1+L_2)\cos\alpha_2 - m\,(-t)^{1/2} \sin\alpha_2 \right],$$

$$N'_{12}(\nu^2,t) = -\frac{2(-t)^{1/2}}{m} \left[ (L_2)\cos^2\alpha_2 + 4(\nu^2-\nu^2_{min}) \sin^2\alpha_2 \right], \tag{5.40}$$

$$N'_{13}\ (\nu^2,t) = \frac{4\sin\alpha_2}{m^2}\ (\nu^2-\nu_B^2)\left[(L_1+L_2)\cos\alpha_2 - m\,(-t)^{1/2}\sin\alpha_2\right],$$

$$N'_{21}\ (\nu^2,t) = 2t\cos\theta\left[L_2\sin^2\alpha_2 - L_1\cos^2\alpha_2 + m\,(-t)^{1/2}\sin\alpha_2\cos\alpha_2\right] - 8L\sin\theta,$$

$$N'_{22}\ (\nu^2,t) = \frac{8(-t)^{1/2}}{m}\ \cos\theta\sin\alpha_2\cos\alpha_2\left[\nu^2-\nu^2_{min}-\tfrac{1}{4}(L_2)^2\right] - 8l\sin\theta\quad,$$

$$N'_{23}\ (\nu^2,t) = \frac{4\cos\theta}{m^2}\ (\nu^2-\nu_B^2)\left[L_2\sin^2\alpha_2 - L_1\cos^2\alpha_2 + m\,(-t)^{1/2}\sin\alpha_2\cos\alpha_2\right],$$

$$N'_{31}\ (\nu^2,t) = -\frac{4L_1\cos\theta}{(\nu_0^2-\nu_B^2)^{1/2}}\left\{2\nu_0\left[(\nu_0^2-\nu^2_{min})^{1/2}-(\nu_0^2-\nu_B^2)^{1/2}\right] - (\nu_0^2-\nu_B^2)^{1/2}L_1\right\},$$

$$N'_{32}\ (\nu^2,t) = -\frac{4\nu_0 L_1 L_2\cos\theta}{(\nu_0^2-\nu_B^2)^{1/2}}\quad,$$

$$N'_{33}\ (\nu^2,t) = \frac{1}{m^2}\,(L_1)^2\sin\theta\left[-L_2\sin^2\alpha_2 - m\,(-t)^{1/2}\sin\alpha_2\cos\alpha_2 + L_1\cos^2\alpha_2\right].$$

**$L_1$, $L_2$ are defined in Eqs.(4.14), $L$ in Eq.(5.15), $\bar{L}_1$ in Eq.(5.21), $\alpha_2$ through Eqs.(5.20), and $\theta$ through Eqs.(5.34).**

**Having found the matrix $N^{(2)}$, three new ("fully regularized transversity") amplitudes**

$$\varphi^{(2)}(\nu^2,t) = \begin{pmatrix} \varphi_4(\nu^2,t) \\ \varphi_5(\nu^2,t) \\ \varphi_6(\nu^2,t) \end{pmatrix} \qquad (5.41)$$

can be introduced

$$\varphi^{(2)}(\nu^2,t) = N^{(2)}(\nu^2,t)\,\bar{A}^{(2)}(\nu^2,t) \tag{5.42}$$

such that for $t$ in the specified region (Eqs.(2.6),(2.6')) they 1°.- are connected to $\bar{A}_4$, $\bar{A}_5$, $\bar{A}_6$ by a matrix which is analytic and invertible everywhere in the cut complex $\nu^2$-plane and 2°.- are related, at the same time, to the usual photon helicity non-flip amplitudes $f_{4,5,6}(\nu,t)$ by a matrix which is unitary on the unitarity cut. Indeed, according to Eqs.(5.37),(5.13),

$$\varphi^{(2)}(\nu^2,t) = B_{\mathrm{II}}^{(2)-1}(\nu^2,t)\,B_{\mathrm{I}}^{(2)-1}(\nu^2,t)\,\tilde{f}^{(2)}(\nu^2,t) \tag{5.43}$$

and hence, referring to Eqs.(5.16),

$$\varphi^{(2)+}\varphi^{(2)} = \tilde{f}^{(2)+}\tilde{f}^{(2)} = f^{(2)'+}f^{(2)'} = f^{(2)+}f^{(2)} \quad \text{for} \quad \nu^2>\nu_0^2 \tag{5.44}$$

since the Blaschke-Potapov matrices $B_{\mathrm{I}}^{(2)}$, $B_{\mathrm{II}}^{(2)}$ are unitary on the cut by construction.

So the derivation of the required set of new amplitudes for spin 1/2 Compton scattering is now completed.

## VI. SUMMARY OF THE RESULTS, DISCUSSION AND CONCLUSIONS

The concrete problem of analytic factorization we have been faced with, formulated in Section III, was completely solved by explicit construction in Sections IV,V. We have at our disposal six new amplitudes $\varphi_i(\nu^2,t)$ (i=1,2,...,6)

$$\varphi(\nu^2,t) = \begin{pmatrix} \varphi^{(1)}(\nu^2,t) \\ \varphi^{(2)}(\nu^2,t) \end{pmatrix} \tag{6.1}$$

related to the Bardeen-Tung amplitudes $\bar{A}_i(\nu^2,t)$, i=1,2,...,6 by

$$\varphi(\nu^2,t) = N(\nu^2,t)\,\bar{A}(\nu^2,t) = \begin{pmatrix} N^{(1)}(\nu^2,t) & 0 \\ 0 & N^{(2)}(\nu^2,t) \end{pmatrix} \begin{pmatrix} \bar{A}^{(1)}(\nu^2,t) \\ \bar{A}^{(2)}(\nu^2,t) \end{pmatrix} \qquad (6.2)$$

where the 3x3 matrices $N^{(1)}$, $N^{(2)}$, analytic and invertible in the cut complex $\nu^2$-plane, are given by Eqs.(4.31) and Eqs.(5.39),(5.40), respectively. Referring to Eq.(2.18), on the unitarity cut one has

$$\sum_{i=1}^{6} |\varphi_i(\nu'^2,t)|^2 \equiv \varphi^+(\nu'^2,t)\,\varphi(\nu'^2,t) = f^+(\nu',t)\,f(\nu',t) =$$
$$= 128\,\pi^2\left(m^2+2\nu'-\frac{t}{2}\right)\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} \equiv |S(\nu'^2,t)|^2\,,\quad \nu'^2>\nu_0^2 \qquad (6.3)$$

The function $S(\nu^2,t)$, analytic and without zeros in the considered domain, is constructed as in Eq.(3.4) :

$$S(\nu^2,t) = \exp\left\{\frac{(\nu_0^2-\nu^2)^{1/2}}{2\pi}\int_{\nu_0^2}^{\infty}\frac{\ln\left[128\,\pi^2(m^2+2\nu'-\frac{t}{2})\left(\frac{d\sigma}{d\Omega}\right)_{c.m.}(\nu',t)\right]d\nu'^2}{(\nu'^2-\nu^2)\,(\nu'^2-\nu_0^2)^{1/2}}\right\}\,. \qquad (6.4)$$

According to relations (3.5) and (3.10) the bound looked for is

$$\varphi^+(\nu^2=\nu_B^2,t)\;\varphi(\nu^2=\nu_B^2,t) \leqslant |S(\nu^2=\nu_B^2,t)|^2 \qquad (6.5)$$

or

$$\bar{A}^+(\nu_B^2,t)\,N^+(\nu_B^2,t)\,N(\nu_B^2,t)\,\bar{A}(\nu_B^2,t) \leqslant$$
$$\leqslant \exp\left\{\frac{(\nu_0^2-\nu_B^2)^{1/2}}{\pi}\int_{\nu_0^2}^{\infty}\frac{\ln\left[128\,\pi^2(m^2+2\nu-\frac{t}{2})\left(\frac{d\sigma}{d\Omega}\right)_{c.m.}(\nu,t)\right]d\nu^2}{(\nu^2-\nu_B^2)\,(\nu^2-\nu_0^2)^{1/2}}\right\} \qquad (6.6)$$

The left hand side of this inequality is a known function of $t$ and of the target's charge and anomalous magnetic moment ( $\bar{A}_i(\nu_B^2,t)$ ($i=1,2,\dots,6$) are given, for instance, by Eqs.(2.9)). This bound (which holds for $t$ in the range noted in Eqs.(2.6),(2.6')) is optimal since the information we have started with has been exhaustively taken into account and the technique of the derivation was so devised that no losses occur on the way. The matrix $N(\nu^2,t)$ found by us is unique up to an irrelevant constant (with respect to $\nu^2$ ) unitary matrix to the left, which, of course, does not affect in any way inequality (6.6). If instead of the cross section $\frac{d\sigma}{d\Omega}$ some upper bound on it is introduced, the sum rule inequality (6.6) will obviously remain valid.

Inequality (6.6) improves the corresponding result previously derived in Section IV of ref./3/. At $t = 0$ (and going to the laboratory frame) it becomes

$$1+\frac{\nu_0^2}{m^4}x^4 \leq \exp\left\{ \frac{\nu_0}{\pi}\int_{\nu_0^2}^{\infty} \frac{\ln\left[\frac{m^2}{\alpha^2}\left(\frac{d\sigma}{d\Omega}\right)_{lab.}(\nu,t=0)\right]d\nu^2}{\nu^2(\nu^2-\nu_0^2)^{1/2}}\right\}, \tag{6.7}$$

$$\nu_0 = \mu\left(m+\frac{\mu}{2}\right), \qquad \alpha \simeq \frac{1}{137}$$

i.e. one recovers the (optimal) bound already found in the easier case of forward proton Compton scattering in ref./2/.

The matrix $N(\nu^2,t)$ being explicitly obtained, all the special techniques developed in refs./4/ in order to find other rigorous constraints e.g. on sets of low energy expansion parameters or other objects which might be of interest in Compton scattering, become effective and lucrative.

In view of the existing renewed interest in the phenomenological dispersion theoretic consideration of the proton Compton effect /11/, and especially in connection with the difficulties encountered when trying to evaluate reliably the influence of the annihilation channel exchanges, the formalism presented here, or some of its adaptations may lead to new insights into the physics of the considered process by means of actually not so complicated, but hitherto less used, mathematical procedures.

A comment on the required high energy asymptotic behaviour of the amplitudes may be in order. As already emphasized in ref./3/, unlike the usual phenomenological dispersion approaches, in the context of the framework presented here we do not have to assume numbers of subtractions, a polynomial boundedness at high $\nu$ , fixed $t$ being in fact sufficient. Indeed, in the modulus representation whereby one constructs an analytic function starting from its modulus on the cut rather than from its imaginary part, much weaker assumptions at infinity are needed, the main requirements now referring to the knowledge of the interior zeros. The model independence of our results is essentially achieved by majorizing the unknown factors containing the zeros of the amplitudes and this is the point where customarily the equality signs are lost and replaced by inequality signs. In the case of non-forward scattering this seems, however, to be the only way available if one wishes to avoid the usual more or less reliable approximations.

We shall not dwell much on the here untouched question of actally testing sum rules like the one given by inequality (6.6), but only mention that if until now such attempts might have been discouraged not only by problems concerning the needed experimental data but also by the possible weakness of the bounds on account of insufficient mathematical treatment, now this latter reason of pessimism is removed.

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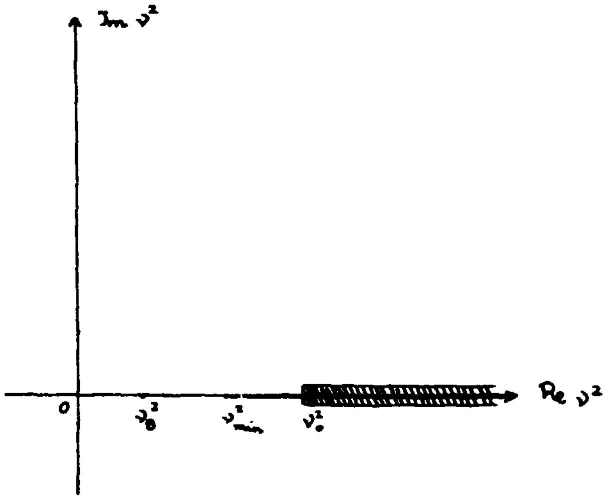


Fig. 1 : Domain of analyticity and physical region of the Compton scattering amplitudes in the complex $\nu^2$ -plane .