RO8101631 V

COMITETUL DE STAT PENTRU ENERGIA NUCLEARĂ

INSTITUTUL CENTRAL DE FIZICĂ

BUCUREŞTI-MĂGURELE

ROMANIA

CENTRAL INSTITUTE OF PHYSICS

INSTITUTE FOR PHYSICS AND NUCLEAR ENGINEERING

Bucharest, P.O.B.5206, ROMANIA

IFIN -
FT-185-1979 November

A rigorous phenomenological analysis of the $\pi\pi$ scattering lengths

I.CAPRINI, P.DIȚĂ and M.SĂRARU

Abstract : *The constraining power of the present experimental data, combined with the general theoretical knowledge about $\pi\pi$ scattering, upon the scattering lengths of this process, is investigated by means of a rigorous functional method. We take as input the experimental phase shifts and make no hypotheses about the high energy behaviour of the amplitudes, using only absolute bounds derived from axiomatic field theory and exact consequences of crossing symmetry. In the simplest application of the method, involving only the $\pi^0\pi^0$ S-wave, we explored numerically a number of values proposed by various authors for the scattering lengths a_0 and a_2 and found that no one appears to be especially favoured.*

1. INTRODUCTION

The low energy behaviour of $\pi\pi$ scattering has been lately the subject of intense study, both experimental and theoretical. Particularly, theoretical predictions for the S-wave scattering lengths are available for a long time, either in the form of absolute bounds derived from axiomatic field theory /1/, or computed in specific models. Among these, the most popular are the current algebra values for a_0 and a_2 proposed by Weinberg /2/. On the other hand, the improved accuracy of the new experimental data on $\pi\pi$ scattering /3/ has stimulated in the last time a considerable amount of phenomenological studies of the problem. In order to extract from the data the values at threshold, i.e. at a point not accessible experimentally one has to resort to some theoretical tools, the most usual being various kinds of dispersion relations or Roy equations /4/. The results of all such analyses are in agreement in recognizing that the S and P wave scattering lenghts seem to be small numbers (i.e. below 1, expressed in pion mass units). However, beside this general consensus, the situation is still very contradictory as concerns the precise values of these parameters /4/. The results seem to depend very strongly both on the details of the experimental data and on the theoretical method.

A rather common opinion among phenomenologists is that the S-wave scattering lengths a_0 (I=0) and a_2 (I=2) have to be confined in the (a_0, a_2) plane within a small region surrounding a curve known as "universal curve" /4/.

The description of this region has been obtained firstly by Morgan and Shaw /5/ by means of a forward dispersion relation, a subsequent refinement of the method based on optimal dispersion relations /6/ retaining the gross features of the previous region but slightly enlarging it. An alternative derivation of the "universal band", based on Roy equations was performed by Basdevant et al. /7/. However, the conclusions of a very recent analysis of the Roy system /8/ are in a certain disagreement with the above results. Using in principle the same input as in ref. /7/, these authors found that the two $\ell = 0$ scattering lengths are not so strongly correlated, in the sense that there exist many pairs of values for a_0, a_2, situated largely outside the small band around the universal curve /7/ for which the Roy system has solutions. This means probably that the strong confinement of the currently accepted range for a_0, a_2 is the effect of some limited parametrizations. We mention that some inconsistencies related to the universal curve have been noticed also by other authors, for instance in connection with the P wave scattering length a_1 /6/.

In view of this situation, we appreciate that a rigorous phenomenological investigation, as much as possible model-independent, of the $\pi\pi$ scattering lengths, is necessary at present. In this work we attempted to perform such an investigation. We studied, by means of a new, rigorous method the constraining power of the present experimental information, combined with the general theoretical knowledge of $\pi\pi$ scattering, upon the low energy features of this process. Stated otherwise, the problem was to know whether the experimental data, used in the most unbiased theoretical context, actually favour one of the many values proposed up to now for the $\pi\pi$ scattering lengths.

For treating this problem we applied a new method, which may be considered as an alternative to the Roy equations and which is suited for taking into account the present theoretical and experimental information without any additional hypotheses. Indeed, while in the application of the Roy equation, a specific Regge formula for the amplitude is usually adopted for evaluating the driving term (see for instance /8/), we make here no assumption about the high energy behaviour, relying only upon the absolute bounds derived from first principles. Crossing symmetry is taken into account by means of the integral relations of Roskies for the partial waves. In this way, we are able to introduce in the formalism only a finite number of partial waves, for which experimental phase shifts are available, thus avoiding any assumptions about the higher, unknown, partial waves.

Working in a more general context, than for instance ref. /8/, we expect to check in our work the conclusions exposed there. This alternative treatment is therefore useful for evaluating the weight of various hypotheses in narrowing the range of the allowed values for the scattering lengths.

The paper is organized as follows: in Sect. 2 we treat, for simplicity only the $\pi^0\pi^0$ S-wave amplitude, describing the simplest version of the method together with the experimental and theoretical input. The numerical results of this analysis are discussed in Sect. 3. In Sect. 4 we present the general method, which correlates the information about various partial waves. The paper contains an Appendix where the duality theorem for convex sets applied in the text is briefly exposed.

2. Description of the method

We present in this section, for simplicity, the case of a single partial wave, which we choose for convenience to be the S-wave $f^{oo}(s)$ of the $\pi^o\pi^o$ scattering. It is given by the expression:

$$f(s) = f^{oo}(s) = \frac{1}{3}\left(\frac{s}{s-4m^2}\right)^{1/2}\left[\frac{\eta_o^{(0)} e^{2i\delta_o^{(0)}}-1}{2i} + \frac{\eta_o^{(2)} e^{2i\delta_o^{(2)}}-1}{2i}\right], \quad (2.1)$$

in terms of the $l = 0$ phase shifts $\delta_o^{(I)}$ and elasticities $\eta_o^{(I)}$. The $\pi^o\pi^o$ scattering length is therefore

$$a^{oo} \equiv f(s=4m^2) = \frac{1}{3}\left(a_o + 2\,a_2\right) , \quad (2.2)$$

where we wrote as usual $a_I = a_o^{(I)}$. As we mentioned above, we shall make no assumptions about the high energy behaviour of $f(s)$, keeping ourselves in the most rigorous theoretical frame. To this end we recall that $f(s)$ is a real analytic function inside the circle /10/

$$\left| s - 26\,m^2 \right| \leqslant 26\,m^2 \quad (2.3)$$

cut along the real axis between $4\,m^2$ and $52\,m^2$. This domain is entirely contained in the complex region of the s - plane where an absolute bound on $|f(s)|$ can be derived from first principles /11/, so we can write the inequality

$$|f(s)| \leq M(s) \qquad \text{for} |s-26\ m^2| = 26\ m^2 , \qquad (2.4)$$

where $M(s)$ is a known function /11/ computed explicitely (for details of the present calculation see /10, 15/).

A remark about the analiticity domain considered is now in order: the circle (2.3), choosed by us in the practical application has the advantage that the conformal mapping required for bringing the problem to a standard formulation is particularly simple (see below). On the other hand, as it will become immediately clear, this domain is rather restrictive as concerns the complete use of the experimental information. Indeed, we can take into account the experimental phase shifts along the physical region for $15\ m^2 \leq s \leq 52\ m^2$, which expressed in c.m. energy goes up to about 1 GeV. This means that we make use only partly of the experimental data, available at present up to 2.1 GeV /3/. The remedy to this shortcoming is simple: we must consider instead of (2.3) another domain, still inside the complex region where absolute bounds can be derived /11/ and which extends to larger energies around the physical region, the only complication being a more sophisticated conformal mapping than the formula (2.8) given below This modified domain is required also for the proper treatment of the higher partial waves (see next section), since they are very small inside the domain (2.3) and become relevant only at higher energies.

In the present application we adopted for simplicity the domain (2.3) and, using the data of Protopopescu et al. /3/ we constructed for $15\ m^2 \leq s \leq 52\ m^2$ the experimental amplitude $f_{exp}(s)$ and the error channel $\varepsilon_{exp}(s)$. This gives for the amplitude the condition

$$\left| f(s) - f_{exp}(s) \right| \leq \varepsilon_{exp}(s) , \quad 15 m^2 \leq s \leq 52 m^2 . \qquad (2.5)$$

At this point it is worth to mention that, by allowing the amplitude to vary around the data within the error channel, we adopted the most unbiased attitude concerning the experimental information. This will prevent the results from being, as in many previous studies, too strongly dependent on the details of the data and therefore we expect that our conclusions will be stable against the changes in the input. Equations (2.4) and (2.5) represent the constraints upon the amplitude, taken as input in our analysis. The problem is to investigate in what extent these constraints restrict the behaviour of the partial wave along the unexplored region from threshold up to 15 m^2. The features of this low energy behaviour are expressed in terms of the scattering lengths and the effective range parameters by means of a power series expansion of the form /6/

$$\delta_0^{(I)}(s) = \operatorname{arctg} \left\{ \left(\frac{s-4m^2}{s} \right)^{1/2} \left[a_I + 2\alpha_I (s-4m^2) \right] \right\} , \quad I = 0,2. \qquad (2.6)$$

Unlike the usual phenomenological determination of the scattering lengths, which assume the validity of (2.6) above the c.m. energy 550 MeV and extract the parameters by fitting the experimental phase shifts, we apply this formula only to the unknown threshold region, up to 15 m^2. The only assumption we make is that, given the values of the scattering lengths a_I , the effective range parameters α_I are determined by reproducing with (2.6) the first experimental phase shifts. We can therefore construct, by means of (2.6) and (2.1) the low energy amplitude $\mathcal{F}(a, s)$, which depends parametrically only on a_0 and a_2 .

The problem is to establish the consistency of this form with the available information contained in equations (2.4) and (2.5). To this end, let us cor ider the maximum of the difference $|f(s) - \mathcal{P}(a,s)|$ along $4\,m^2 \le s \le 15\,m^2$, f being a function which obeys the above constraints. For a definite pair of values (a_0, a_2), i.e. for a fixed $\mathcal{P}(a,s)$, this quantity will obviously depend on $f(s)$. Let us further take the infimum of all these numbers with respect to the admissible analytic functions f, i.e. let us compute

$$\varepsilon^{\circ}(a_0, a_2) = \inf_{f} \max_{4m^2 \le s \le 15m^2} \left| f(s) - \mathcal{P}(a,s) \right| . \qquad (2.7)$$

If a good choice, compatible with (2.4) and (2.5), for the pair (a_0, a_2) is made, this would manifest itself in the possibility of a good approximation of the low energy amplitude $\mathcal{P}(a,s)$ through an admissible $f(s)$. The parameter ε° will have therefore a reasonable low value. If, on the contrary, a pair inconsistent with the input information is taken, it will surely yield a comparatively higher value for ε°, reflecting the inability of the functions obeying (2.4) and (2.5) to pass, along the low energy region, close enough to $\mathcal{P}(a,s)$.

The parameter ε° defined in (2.7) seems therefore to be a reasonable criterion for measuring the "goodness" of various pairs of (a_0, a_2) and for discriminating among them. From the dependence of ε° upon (a_0, a_2) we can infer whether some pairs are indeed strongly favoured by the data, these corresponding to marked minima in ε°. On the other hand, a resulting picture revealing a weak variation of ε° with the values (a_0, a_2), at least when these are located in some region in the (a_0, a_2) plane indicates that all these values are equally favoured. As we shall show in the next section, our numerical analysis gave support to the last alternative.

The effective calculation of (2.6) amounts to solving a functional extremum problem. We shall first bring the conditions to a standard form, by making the conformal transformation /11/

$$z(s) = \frac{12\, s + i\left[13\,(s-4m^2)(11\,s+52\,m^2)\right]^{1/2}}{12\, s - i\left[13\,(s-4m^2)(11\,s+52\,m^2)\right]^{1/2}} \,. \qquad (2.8)$$

This maps the domain (2.3) of the s - plane into the unit disk $|z| \leq 1$, such that the cut $4\,m^2 \leq s \leq 52\,m^2$ becomes the right semicircle $\zeta = e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$, the frontier of the disk (2.3) the left semicircle $\pi/2 \leq \theta \leq 3\pi/2$ and the unphysical region $0 \leq s \leq 4m^2$ the diameter $-1 \leq x \leq 1$.

In this variable one can show /12/ that ε^0 is given by the unique solution of the algebraic equation

$$\mu(\varepsilon) \equiv \inf_{f \in H^\infty} \| f - h\, G \|_\infty = 1 \,, \qquad (2.9)$$

where H^∞ is the Hardy space of bounded analytic functions /13/, and the L^∞ norm was denoted as usually.

In this relation the function h is defined as /13/

$$h(\theta) = \begin{cases} \mathcal{P}(a, s(\theta)) & , \quad 0 \le \theta \le \theta_1 = \theta(15m^2) \\ f_{exp}(s(\theta)) & , \quad \theta_1 \le \theta \le \pi/2 \\ 0 & , \quad \pi/2 < \theta \le \pi \end{cases} , \tag{2.10}$$

and the dependence on the parameter ε is brought by the outer function

$$G(z) = \exp\left\{ -\frac{1}{\pi} \int_0^{\pi} Re \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln \varepsilon(\theta) \, d\theta \right\} ,$$

where

$$\varepsilon(\theta) = \begin{cases} \varepsilon \ (const) & , \quad 0 \le \theta \le \theta_1 \\ \varepsilon_{exp}(s(\theta)) & , \quad \theta_1 < \theta \le \pi/2 \\ M(s(\theta)) & , \quad \pi/2 < \theta \le \pi \end{cases}$$

The solution u of the minimum H^∞-norm problem ... is well-known /12, 13/: it is given by the norm of the Hankel matrix

$$H = (c_{i+j-1})_{i,j=1,2,...} ,$$

constructed from the negative-frequency Fourier coefficients

$$c_n = \frac{1}{\pi} \int_0^{\pi} Re\left(h(\theta) \, G(\theta) e^{in\theta} \right) d\theta , \quad n = 1, 2, ...$$

The numerical treatment of the above problem was performed by means of a complex set of programs, described in /14/. For the computation of the Fourier coefficients [illegible] /14/ we applied an improved method /15/, based on approximations by spline functions.

3. Results

Using the method and the experimental information discussed in the previous section, we computed the parameter χ^2 for a number of pairs (a_0, a_2). We explored particularly the usually accepted range in the plane (a_0, a_2) [illegible] in the neighbourhood of the "universal curve" [illegible] in the Introduction. Some of the results [illegible] are presented in Table I, where we indicate also [illegible] the corresponding value was taken. As a first remark, we [illegible] that we are in agreement with the commonly [illegible] values for the [illegible] wave scattering lengths [illegible] - [illegible] may be seen from Table I, the [illegible] $a_0 =$ [illegible] $a_2 =$ [illegible] χ^2 [illegible] by an order of magnitude [illegible] the other [illegible], this being, in view of the arguments [illegible], a definite [illegible] within the generally accepted range of (a_0, a_2) [illegible] χ^2 [illegible] a_0, a_2 [illegible] was [illegible] the [illegible] "universal curve" [illegible] Table I, [illegible] we are in agreement with [illegible].

The results exposed above were obtained in the simplest application of the method, which involved only a single partial wave. It is clear that the information, both experimental and theoretical, on the higher partial waves, yields additional constraints upon the S-waves and is expected to narrow the range of the allowed values for a_0, a_2. Particularly, the influence of the specific features of the P-wave, such as the value of the scattering length a_1 and the presence of the ρ meson is of much theoretical interest and the answers to this problem are still very different. In the next section we present the general formulation of the method which allows to treat rigorously the correlation between several partial waves.

4. The general method

We shall consider a finite number of $\pi\pi$ partial waves $f_i(z)$, $i = 1, \dots, n$, the index i taking into account both the isospin and the orbital momentum indices. We assume for convenience that the conformal mapping discussed in Section 2 was already performed, so that in the variable z the analyticity domain of f_i is the unit disk $|z|<1$. The functions f_i obey a set of linear integral relations along the unphysical region $0 \leq s \leq 4m^2$ (which in the variable z is the diameter $-1 \leq x \leq 1$). These relations can be written in the general form

$$\sum_{i=1}^{n} \int_{-1}^{1} \varphi_{ji}(x)\, f_i(x)\, dx = \beta_j \quad , j=1,\dots p \, , \; p > n \, , \qquad (4.1)$$

where the functions φ_{ji} are polynomials in s, multiplied by the Jacobian $|ds/dx|$ of the conformal transformation. The relations (4.1) which express crossing symmetry, may be considered as an alternative to the Roy system of equations, for establishing correlations among a finite number of partial waves. It is important to point out here that we are able to treat rigorously these constraints together with the information available upon separate partial waves. As it was discussed in detail in Section 2, this information consists from the experimental data $f_{i,exp}(\zeta)$ with the errors $\varepsilon_{i,exp}(\zeta)$, known for $\zeta = e^{i\theta}$, $0 \leq \theta_1 \leq \pi/2$ and the absolute bounds $M_i(s)$ /11,21/ given along a suitable curve in the complex s-plane, which is supposed to be brought onto the left semicircle in the ζ-plane, $\zeta = e^{i\theta}$, $\pi/2 < \theta \leq \pi$. As concerns the low energy region, which corresponds to $\zeta = e^{i\theta}$, $0 \leq \theta < \theta_1$, we assume along it an effective range parametrization $\mathcal{P}_i(a_i, \zeta)$, defined in terms of the scattering lengths a_i. As in Sect. 2, we shall measure the consistency of the low energy behaviour with the above experimental and theoretical input, by the minimal deviation ε_i^0 of the "admissible" partial waves from the explicit parametrizations $\mathcal{P}_i(a_i, \zeta)$. Therefore, the above conditions can be put together in the form

$$|f_i(\zeta) - h_i(\zeta)| \leq \varepsilon_i(\zeta), \quad i=1,\dots n, \quad |\zeta|=1, \qquad (4.2)$$

where the functions $h_i(\zeta)$ and $\varepsilon_i(\zeta)$ are defined as in (2.10) and (2.12) respectively.

The relations (4.1) and (4.2) represent a set of constraints upon the analytic functions $f_i(z)$, in terms of the input, consisting from the functions h_i, $\varepsilon_{i,exp}$, φ_{ji} and the constants ε_i. Now, it may happen that for a certain, fixed input the above condition are too stringent, so that there are no analytic functions f_i obeying them. This remark is fundamental for the philosophy of our method: namely, from the above input we have the freedom to vary the scattering lengths a_i (which enter $h_i(s)$) and the deviations ε_i along the low energy region (which appear in $\varepsilon_i(s)$). The problem amounts then to establishing the limitations that must be imposed to a_i and ε_i , such that the relations (4.1) and (4.2) admit solutions. In this way we can find minimal deviations ε_i^0 , for given scattering lengths a_i or we can decide which are the allowed values of a_i , if some phenomenologically estimated errors ε_i are taken as input.

As we shall show below , the treatment of this problem leads to a minimum norm problem for a convex set of vector-valued analytic functions. First, we notice that the relations (4.1) and (4.2) can be written in the equivalent form:

$$\| f_i - h_i G_i \|_\infty \leq 1 , \quad i = 1, 2, \dots m, \tag{4.3a}$$

$$\sum_{i=1}^{m} \int_{-1}^{1} \varphi_{ji}(x) f_i(x) G_i^{-1}(x) dx = \beta_j , \; j = 1 \dots , p, \; p \geq m, \tag{4.3b}$$

where $G_i(s)$ are outer functions defined as in (3.11). When the functions $f_i(s)$ are the partial waves the constants β_i entering the right hand side of (4.3b) are equal to zero and the convex set of vector valued functions becomes a subspace. However, we shall treat in the following the general case, at least one of $\beta_i^* \neq 0$, since the problem takes this form when f_i satisfy other constraints at unphysical energies or when one works with the S - matrix elements $S_i = 1 - \left(\frac{4m^2 - s}{s}\right)^{1/2} f_i$ /16/. The study of the consistency conditions for the input i.e. of the conditions in which the relations (4.3) admit solutions is, due to the crossing conditions (4.3b), much more complicated than the previous problem (2.9). In ref. /16/ we investigated in detail, in connection with the derivation of absolute bounds on the partical waves at unphysical points, the case of a single function, subjected to a number of integral constraints. For completeness we shall treat here the general vectorial case (4.3). In the present section we shall only formulate the problem in a convenient way and we shall give the final result, i.e. the description of the range of allowed values for the free parameters of our approach. The duality theorem applied for solving the problem and some remarks concerning the proof are given in the Appendix.

The starting point of the analysis is the remark that, due to the dependence of the left hand sides of (4.3a) upon the parameters $\mathcal{E}_i$, the largest range of allowed values for these parameters will be obtained, as in Sect. 2, by taking the infimum of the norms (4.3a) with respect to the functions f_i , restricted by the additional conditions (4.3b).

As it is shown in /16/, for solving this problem it is convenient to embed it from the initial H^∞ space into the H^2-Hilbert space. Therefore, the conditions (4.3a) are shown to be exactly equivalent to the set of inequalities

$$\sup_{\|g_i\|_2^2 \leq 1} \inf_{f_i} \| f_i \, g_i - h_i \, G_i \, g_i \|_2^2 \leq 1, \quad i = 1, 2, \ldots n, \qquad (4.4)$$

where, as explained in /16/, g_i are outer functions from the unit sphere of H^2 /13/, and the extrema over f_i are constrained by the relations (4.3b). This problem was treated in /16/ separately for each amplitude f_i , by splitting the in-

tegral conditions (4.3b) in constant terms containing only a simple function, a final optimization upon these constants subjected to a set of linear relations being finally performed.
In what follows we shall give alternatively a compact vectorial solution, based on the remark that the conditions (4.4) can be obviously expressed in the equivalent form

$$\sup_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{m} \alpha_i = 1}} \; \sup_{\|g_i\|_2^2 \leq 1} \; \inf_{f_i} \; \sum_{i=1}^{m} \alpha_i \, \| f_i g_i - h_i G_i g_i \|_2^2 \leq 1 ,$$

which can be verified immediately. In this way, we obtain after some evident transformations of functions, that the required domain of consistency is described analytically by the inequality

$$\sup_{\sum_{i=1}^{m} \|g_i\|_2^2 \leq 1} \; \inf_{f \in \mathcal{K}} \; \sum_{i=1}^{m} \| f_i - g_i h_i G_i \|_2^2 \leq 1 , \qquad (4.5a)$$

where by $\mathcal{K}$ we have denoted the convex set

$$\mathcal{K} = \left\{ f_i \in H^2 \;\middle|\; \sum_{i=1}^{m} \int_{-1}^{1} \varphi_{ji}(x) f_i(x) g_i^{-1}(x) G_i^{-1}(x) dx = \beta_j ,\; j = 1, \ldots p \right\}. \qquad (4.5b)$$

By applying the duality theorem formulated in the Appendix, the nontrivial part of the above extremum problem, i.e. the infimum over f_i can be explicitely solved. In what follows we shall give directly the solution , the actual proof involving only minor modifications with respect to the scalar case treated in /16/.

In order to write down in a compact form the final result, we have to introduce some notations. We define first the coefficients

$$c_{kl}^{j} = \int_{-1}^{1} x^{j-1} \varphi_{kl}(x) g_l^{-1}(x) G_l^{-1}(x) dx , \quad l = 1, \ldots, m ; \; k = 1, \ldots p ; \; j = 1, 2, \ldots$$

If the relations (4.1) are linear independent we can construct the $p \times p$ matrix

$$A = \begin{pmatrix} c_{11}^{1} & \cdots & c_{11}^{l_1} & c_{12}^{1} & \cdots & c_{12}^{l_2} & \cdots & c_{1m}^{l_m} \\ \vdots & & \vdots & & & & & \\ c_{p1}^{1} & \cdots & c_{p1}^{l_1} & c_{p2}^{1} & \cdots & c_{p2}^{l_2} & \cdots & c_{pm}^{l_m} \end{pmatrix} \qquad (4.7)$$

such that

$$\Delta = \det A \neq 0 .$$

In (4.7) $l_1, \ldots l_m$ are nonnegative indices satisfying $\sum_{q=1}^{m} l_q = p$. We denote now by $A^{(l)}$ the matrix obtained by deleting from A the column l, and by A_{lm} the determinant of the matrix obtained by deleting from A the row l and the column m. Let us define further the $p \times p$ matrix X as

$$X_{ij} = \delta_{ij} + \frac{(-1)^{i+j}}{\Delta^2} \sum_{l,m=1}^{p} A_{li} A_{mj} I_{lm} , \quad i,j = 1, \ldots p, \qquad (4.8)$$

where the numbers I_{lm} are given by the integrals

$$I_{lm} = \sum_{q=1}^{m} \int_{-1}^{1} dx \int_{-1}^{1} dy \, \frac{\varphi_{lq}(x)\, \varphi_{mq}(x)\, x^{l_q} y^{l_q}}{g_q(x)\, g_q(y)\, G_q(x)\, G_q(y)\, (1 - xy)} , \qquad (4.9)$$

with the convention that if $l_q = 0$ the corresponding term in the above sum is missing. For what follows we need the eigenvalues λ_k and, the eigenvectors v^k of the matrix X, and also the new matrix U defined as

$$U_{ij} = v^i_j \quad , \; i,j = 1,\dots p. \tag{4.10}$$

Let us introduce further the functions

$$Q^q_j(s) = \frac{1}{s^j} + \frac{(-1)^{j+1}}{\Delta} \det \begin{vmatrix} \int_{-1}^{1} dx\, x^{l_q} \varphi_{1q}(x)/G_q(x) g_q(x)(s-x) & \\ \vdots & A^{(j)} \\ \int_{-1}^{1} dx\, x^{l_q} \varphi_{pq}(x)/G_q(x) g_q(x)(s-x) & \end{vmatrix} \tag{4.11}$$

in terms of which we compute the coefficients

$$D^q_j = \frac{1}{2\pi i} \oint_{|s|=1} ds\, g_q(s) h_q(s) G_q(s) Q^q_j(s)\,, \; q = 1,\dots m\,, \; j = 1,\dots, l_q\,,$$

which we arrange in the row $R = (R_j)_{j=1}^{p}$ as follows

$$R = (D^1_1, \dots D^1_{l_1}, D^2_1, \dots D^2_{l_2}, \dots, D^m_1, \dots D^m_{l_m}). \tag{4.12}$$

If we define now the numbers

$$\eta_k = \sum_{l=1}^{p} U_{kl} \left[R_l + \frac{(-1)^l}{\Delta} \det \begin{vmatrix} \beta_1 & \\ \vdots & A^{(l)} \\ \beta_p & \end{vmatrix} \right], \; k = 1,\dots p, \tag{4.13}$$

we can write down explicitely the infimum over $\{f_i\}$ in (4.5a), so that this inequality becomes

$$\sup_{\sum_{i=1}^{m} \|g_i\|_2^2 \le 1} \left[\sum_{i=1}^{m} \|(g_i h_i G_i)_-\|_2^2 + \sum_{k=1}^{p} \eta_k^2/\lambda_k \right] \le 1. \tag{4.14}$$

In this relation, by $(g_i h_i G_i)_-$ we have denoted the nonanalytic projection of the corresponding complex function defined on the boundary of the unit disk, and hence

$$\| (g_\ell h_\ell G_\ell)_- \|_2^2 = \sum_{m=1}^{\infty} \left[\frac{1}{\pi} \int_0^{\pi} \mathrm{Re} \left(h_\ell g_\ell G_\ell e^{in\theta} \right) d\theta \right]^2 .$$

The inequality (4.14) is our final result describing the compatibility conditions for the relations (4.1) and (4.2). In practice, we have to recall that the expression in the left hand side of (4.14) depends on the parameters free in our approach, i.e. the scattering lengths a_i and the low energy deviations ε_i. The inequality (4.14) describes therefore the range of the allowed values for these parameters. The only practical complication of the above description consists in the maximization upon the outer functions g_i, which remained in (4.14). However, it is useful to recall that, with every choice of an admissible g_i, we have through (4.14) an approximation from outside of the required domain. Moreover, with some useful guess of g_i, we can, as discussed in /16/, approach rather closely the exact solution.

APPENDIX

We applied the following duality theorem /22/: let x_1 be a point in a normed vector space X and let d denote its distance from the convex set $\mathcal{K}$; then

$$d = \inf_{x \in \mathcal{K}} \| x - x_1 \| = \max_{\|x^*\| \leq 1} \left[\langle x_1, x^* \rangle - \sup_{x \in \mathcal{K}} \langle x, x^* \rangle \right] \tag{A1}$$

where $x^* \in X^*$, X^* being the dual of X and the maximum on the right is achieved by some $x_0^* \in X^*$.

In our problem, the space X consists from the set of vector-valued complex functions $h = \{h_i\}_{i=1}^{m}$, defined on $\zeta = e^{i\theta}$, $-\pi < \theta \leq \pi$, such that $\| h \|_2^2 < \infty$, where the norm is

$$\| h \|_2^2 \equiv \sum_{i=1}^{m} | h_i |^2. \tag{A2}$$

Particularly, from (4.5a) the element x_1 is seen to be the vector function with components equal to $g_i h_i G_i$, $i = 1, \dots m$. As for the convex $\mathcal{K}$, it is the subset of X consisting from the vector-valued analytic functions $f = \{f_i\}_{i=1}^{m}$ with the components restricted by the linear relations (4.5b).

The description of the dual space X^* is given by Riesz theorem /13,22/. In our particular case X^* is identical with X, so that the supremum to be evaluated in (A1) can be written in the form

$$\max_{\sum_{i=1}^{m} \|F_i\|_2^2 \le 1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \sum_{i=1}^{m} F_i h_i g_i G_i - \sup_{f \in \mathcal{K}} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \sum_{i=1}^{m} F_i f_i \right].$$

The evaluation of this maximum, which led to the relation (4.14) is a simple generalization of the technique described in /16/.

TABLE I

a_0	a_2	$a_{00} = \frac{1}{3}(a_0 + 2a_2)$	$\varepsilon^0 \cdot 10^3$	References
0.35	- 0.1	0.05	2.19	
0.16	- 0.045	0.023	2.08	* Weinberg (1966) /2/
- 0.1	- 0.2	- 0.166	2.2	Basdevant (1972) /7/
- 0.5	0.2	- 0.033	1.72	Pisut (1968) /17/
- 0.3	0.2	0.033	1.75	"
- 0.013	- 0.015	- 0.0143	2.02	"
0.24	- 0.06	0.04	2.12	* Franklin (1975)/18/
0.8	0.1	0.33	2.09	Fulko (1967)/19/
0.1	- 0.1	- 0.033	2.13	
2.0	1.0	1.33	15.2	

REFERENCES

/1/. Bonnier, B. and Vinh Mau, R.: Phys.Rev. 165, 1928 (1968)
Lopes, C. and Mennessier, G.: Nucl.Phys. B118, 428 (1977)

/2/. Weinberg, S.: Phys.Rev.Lett. 17, 616 (1966)

/3/. Protopopescu, S.D. et al.: Phys.Rev. D7, 1279 (1973)
Hyams, B.et al.: Nucl.Phys. B100, 205 (1975)
Estabrook, P. et al.: Nucl.Phys. B133, 490 (1978)

/4/. Martin, B.R., Morgan, D. and Shaw, G.: Pion-pion interactions in particle physics, Acad.Press. 1976

/5/. Morgan, D. and Shaw, G.: Phys.Rev. D2, 520 (1970)

/6/. Ciulli, S., Treleani, D. and Verzegnassi, C.: Nuovo Cim. 40A, 24 (1977)

/7/. Basdevant, J.L., Froggatt, C.D. and Petersen, J.L.: Phys. Lett. 41B, 173 (1972); Nucl.Phys. 72B, 413 (1974)

/8/. Heemskerk, A.C. and Pool, T.P.: Nuovo Cim. 49A, 393 (1979)

/9/. Srinivasan, V. et al.: Phys.Rev. D12, 681 (1975)

/10/. Caprini, I. and Diţă, P.: Topics in theoretical physics, vol.1, page 45, Bucharest 1978

/11/. Bonnier, B.: Nucl.Phys. B95, 98 (1975)

/12/. Ciulli, S., Pomponiu, C. and Sabba-Stefănescu, I.: Phys. Reports 17C, 133 (1975)

/13/. Duren, P.: Theory of H^p -spaces, Acad.Press, N.Y. and London, 1970
Krein, M.G. and Nudelman, A.A.: The problem of Markov moments and extremal problems, Moskow, Nauka, 1973 (in Russian)

/14/. Caprini, I. et al.: preprint CERN TH-2268, 1977, to appear in Comp.Phys.Comm.

/15/. Pomponiu, C. and Săraru, M.: Comp.Phys.Comm. 16, 93 (1978)

/16/. Caprini, I. and Diţă, P.: preprint ICEFIZ FT - 178, June, 1979, Bucharest

/17/. Pisut, J.: Nucl.Phys. B8, 159 (1968)

/18/. Franklin, J.: Phys.Rev. D11, 513 (1975)

/19/. Fulco, J.R. and Wang, D.Y.: Phys.Rev.Lett. 19, 1399 (1967)

/20/. Roskies, R.: Nuovo Cim. 65A, 467 (1970)

/21/. Bonnier, B. and Donohue, J.T.: Nucl.Phys. B134, 351 (1978)

/22/. Luenberger, D.G.: Optimization by vector space methods, John Wiley and Sons, N.Y. 1968, p. 136

CENTRAL INSTITUTE OF PHYSICS
Documentation Office
Bucharest, P.O.B. 5206
ROMANIA