BNL-28629

Comments on the Integrability of the Loop-Space Chiral Equations

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*This talk was presented at the XXth International Conference on High Energy Physics, University of Wisconsin-Madison, July 17-23, 1980.

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INTRODUCTION

The path ordered phase factor $\Phi_{ab} \equiv P \exp\int_a^b dx_\mu A_\mu(x)$ along a loop (or part of a loop) has many desirable features. It is the minimal necessary object[1] describing a gauge theory, has simple gauge transformation properties[2] and is the spin-like quantity for lattice gauge theory.[3] Thus it is natural, may even be essential, that we formulate gauge theory in the loop space. Last year (1979), it has been realized that Yang-Mills equations give chiral-like equations in the loop space.[4,5] This beautiful realization immediately raised the hope that the loop-space chiral equations may also be a total integrable system, just like the ordinary-space chiral equations,[6] and thus lead to the full solutions of the Yang-Mills equations. However here we want to discuss some intricate properties of the integrability conditions of the loop space chiral equations, which do not have their correspondence in the ordinary chiral equations.

In part I we shall briefly demonstrate how the ordinary space chiral equations provide the existence conditions for the infinite number of conserved currents and how these currents are related to the so-called inverse-scattering equations, whose integrability is provided by the original chiral equations. In part II we briefly introduce the loop-space chiral equations and then discuss the integrability conditions of the non-local currents in two possible different situations. In the first case, the "generating" functions are functionals of the loop alone. We show that the integrability conditions are not satisfied and higher order conserved non-local currents do not exist. In the second case, the "generating" functions are functionals of the loop as well as a parameter the integrability conditions at a restricted point of the parameter are satisfied, however there is an infinite fold of arbitrariness.[9] It indicates that additional guiding principles are needed in addition to the original loop-space chiral equation in order to uniquely determine the infinite conserved non-local currents as functionals of the loop and the parameter.

I. CHIRAL FIELDS IN TWO DIMENSIONS

The chiral fields are simply described by the following equations,

$$A_\mu(x) \equiv g^{-1}(x)\partial_\mu g(x), \qquad \text{curvatureless,} \qquad (1.1)$$

$$\partial_\mu A_\mu(x) = 0 \quad , \qquad \text{continuity-like,} \qquad (1.2)$$

which follows from a Lagrangian density $\mathcal{L} = \mathrm{Tr}\,\partial_\mu g^{-1}(x)\partial_\mu g(x)$. Following the procedure of Brezin et al.,[6] we consider $A_\mu(x)$ being the first conserved current.

$$V_\mu^{(1)}(x) \equiv A_\mu(x) = \varepsilon_{\mu\alpha}\partial_\alpha \chi^{(1)}(x). \tag{1.3}$$

We shall see that $\chi^{(1)}(x)$ is needed to construct higher current. So we re-express eq. (1.3) as

$$\partial_\mu \chi^{(1)}(x) = -\varepsilon_{\mu\beta}A_\beta(x). \tag{1.3'}$$

Then integrability of $\chi^{(1)}(x)$ is

$$\partial_\nu\partial_\mu \chi^{(1)}(x) - \partial_\mu\partial_\nu \chi^{(1)}(x) = 0. \tag{1.4}$$

From eq. (1.3), we have

$$-\varepsilon_{\mu\beta}\partial_\nu A_\beta(x) + \varepsilon_{\nu\alpha}\partial_\mu A_\alpha(x) = 0, \tag{1.5}$$

$$\text{for } \mu=\nu=1,\ -\partial_1 A_2(x) + \partial_1 A_2(x) = 0, \tag{1.5a}$$

$$\text{for } \mu=1,\ \nu=2,\ -\partial_2 A_2 - \partial_1 A_1 = 0. \tag{1.5b}$$

Equation (1.5a) is automatically true and eq. (1.5b) is just eq. (1.2). Therefore $\partial_\mu A_\mu = 0$ is the necessary and sufficient condition for the integrability of $\chi^{(1)}(x)$.

Now we construct the second current,

$$V_\mu^{(2)} \equiv \mathcal{D}_\mu \chi^{(1)}, \qquad \text{where } \mathcal{D}_\mu \equiv \partial_\mu + A_\mu(x). \tag{1.6}$$

Using eqs. (1.1) and (1.2), we can easily show $\partial_\mu V_\mu^{(2)} = 0$. Similarly we can construct higher currents

$$V_\mu^{(n)} = \varepsilon_{\mu\nu}\partial_\nu \chi^{(n)} = \mathcal{D}_\mu \chi^{(n-1)}. \tag{1.7}$$

Again eqs. (1.1) and (1.2) give $\partial_\mu V_\mu^{(n)} = 0$. We shall call these $\chi^{(n)}$'s the "generating" function.

Next we shall demonstrate the connection of these infinite conserved currents with the inverse-scattering (or the linear) equations. Multiplying eq. (1.7) by L^n and sum over all n,

$$\sum_{n=1}^{\infty} \varepsilon_{\mu\nu}\partial_\nu L^n \chi^{(n)} = L\sum_{n=1}^{\infty} \mathcal{D}_\mu L^{n-1}\chi^{(n-1)}, \tag{1.8}$$

where L is an arbitrary constant, eq. (1.8) can be rewritten as, due to $\chi^{(0)} = 1$,

$$\varepsilon_{\mu\nu}\partial_\nu \sum_{n=0}^{\infty} L^n \chi^{(n)}(x) = L\mathcal{D}_\mu \sum_{n=0}^{\infty} L^n \chi^{(n)}(x).$$

Defining $\phi(x,L) \equiv \sum_{n=0} L^n \chi^{(n)}(x)$, (1.9)

we obtain $\varepsilon_{\mu\nu}\partial_\nu\phi = L\mathcal{D}_\mu\phi$, or $[\partial_\mu - A_\mu - L^{-1}\varepsilon_{\mu\nu}\partial_\nu]\phi(x,L) = 0$, (1.10)

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Defining $\phi(x,L) \equiv \sum_{n=0} L^n \chi^{(n)}(x)$, (1.9)

we obtain $\varepsilon_{\mu\nu}\partial_\nu\phi = L\mathcal{D}_\mu\phi$, or $[\partial_\mu - A_\mu - L^{-1}\varepsilon_{\mu\nu}\partial_\nu]\phi(x,L) = 0$, (1.10) which is the inverse-scattering equations[6] for the chiral equations.

Now we discuss the integrability conditions of ϕ. From eq. (1.10) it is easy to obtain

$$\partial_\mu \phi(x,L) = -\frac{1}{1+L^2}[A_\mu(x) + L\epsilon_{\mu\alpha}A_\alpha]\phi(x,L). \qquad (1.11)$$

Differentiating eq. (1.11) once more, and using eq. (1.11), it follows

$$\partial_\nu \partial_\mu \phi(x,L) = -\frac{1}{1+L^2}[\partial_\nu A_\mu(x) + L\epsilon_{\mu\alpha}\partial_\nu A_\alpha(x)]\phi(x,L) + \frac{1}{(1+L^2)^2}[A_\mu(x) + L\epsilon_{\mu\alpha}A_\alpha(x)][A_\nu(x) + L\epsilon_{\nu\beta}A_\beta(x)]\phi(x,L). \qquad (1.12)$$

The integrability conditions of $\phi(x,L)$, $\partial_\mu\partial_\nu\phi(x,L) - \partial_\nu\partial_\mu\phi(x,L) = 0$, for arbitrary L, gives the L^0 term, $\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] = 0$; the L^1 term, $-\epsilon_{\mu\alpha}\partial_\nu A_\alpha + \epsilon_{\nu\beta}\partial_\mu A_\beta + [\epsilon_{\mu\alpha}A_\alpha, A_\nu] + [A_\mu, \epsilon_{\nu\beta}A_\beta] = 0$; the L^2 term, $-\partial_\nu A_\mu + \partial_\mu A_\nu + [\epsilon_{\mu\alpha}A_\alpha, \epsilon_{\nu\beta}A_\beta] = 0$; the L^3 term, $\epsilon_{\mu\alpha}\partial_\nu A_\alpha - \epsilon_{\nu\beta}\partial_\nu A_\alpha = 0$. They imply simply $\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] = 0$, $\partial_\mu A_\mu = 0$. Therefore we see that the original equations of motion provide the integrability of $\phi(x,L)$ and also provide the conditions for constructing infinite number of conserved currents.

II. CHIRAL FIELDS IN LOOP SPACE

Let us consider the phase factor along the loop $\ell = x^\mu(s)$ as shown in Fig. 1a,

$$\Phi_{02S10} = \psi(\ell) = P\exp(i\oint A_\mu dx_\mu). \qquad (2.1)$$

The functional differentiation of the loop phase factor is defined as the change in $\psi(\ell)$, as ℓ changes to ℓ', which is infinitesimally deformed from ℓ at s (Fig. 1b),

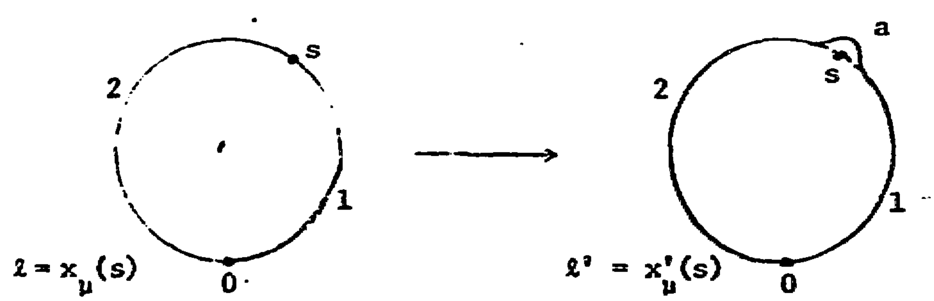

Figure 1a

Figure 1b

$$\frac{\delta\psi(\ell)}{\delta x^\mu(s)} = \frac{\psi(\ell')-\psi(\ell)}{ds\,dx^\mu(x)} = \Phi_{02s} f_{\lambda\mu}[x(s)]\frac{dx_\lambda(x)}{ds}\Phi_{s10}. \qquad (2.2)$$

It is just the parallel transported "normal flux" per unit area that went through the little area a shown in Fig. 1b. Define the loop space gauge potential

$$\mathscr{F}_\mu(\ell,s) \equiv \psi(\ell)^{-1} \frac{\delta\psi(\ell)}{\delta x^\mu(s)} = \phi_{01s} f_{\lambda\mu}[x(s)]\dot{x}_\lambda(s)\phi_{s10}. \qquad (2.3)$$

Functionally differentiating it again, and after some work, one can show[4,5]

$$\frac{\delta\mathscr{F}_\mu(\ell,s)}{\delta x^\nu(s')} - \frac{\delta\mathscr{F}_\nu(\ell,s')}{\delta x^\mu(s)} + [\mathscr{F}_\mu(\ell,s),\mathscr{F}_\nu(\ell,s')] = 0, \qquad (2.4)$$

which just gives the projected Bianchi identity:

$$(\mathscr{D}_\mu f_{\nu\lambda}[x(s)])\dot{x}_\lambda + (\mathscr{D}_\nu f_{\lambda\mu}[x(s)])\dot{x}_\lambda + (\mathscr{D}_\lambda f_{\mu\nu}[x(s)])\dot{x}_\lambda = 0, \qquad (2.4a)$$

and

$$\frac{\delta\mathscr{F}_\mu(\ell,s)}{\delta x_\mu(s)} = 0, \qquad (2.5)$$

which gives the projected Yang-Mills equation

$$\mathscr{D}_\mu f_{\mu\nu}[x(s)]\dot{x}_\nu(s) = 0. \qquad (2.5a)$$

The geometric meaning of the loop space equations (2.4), (2.5) is that the loop phase factor arrived from an initial loop to a given final loop is independent of the different volumes swapped out by the intermediate loops if the Bianchi identity and the sourceless Yang-Mills equations are satisfied.

Again eq. (2.5) is like a continuity equation so we try to follow the same procedure as for the chiral fields and identify the first current, here we specify in two dimension though the conclusion is general,

$$V^{(1)}_\mu(\ell,s) \equiv \mathscr{F}_\mu(\ell,s) = \varepsilon_{\mu\nu} \frac{\delta\chi^{(1)}}{\delta x^\nu(s)}. \qquad (2.6)$$

This satisfies eq. (2.5), but the question is whether eq. (2.5) provides the sufficient conditions for the integration of χ from eq. (2.6). We shall discuss separately the following two possible cases.

Case (1): $\chi^{(1)}$ is a functional of the loop above, i.e. Eq. (2.6) reads

$$\mathscr{F}_\mu(\ell,s) = \varepsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s)}. \qquad (2.6)'$$

Just as the finite dimensional case, the integrability conditions of $\chi^{(1)}(\ell)$ is

$$\frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s')\delta x^\mu(s)} - \frac{\delta\chi^{(1)}(\ell)}{\delta x^\mu(s)\delta x^\nu(s')} = 0. \qquad (2.7)$$

From $\frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s)} = -\varepsilon_{\nu\mu}\mathscr{F}_\mu(\ell,s)$, eq. (2.7) gives

$$\varepsilon_{\mu\alpha} \frac{\delta\mathscr{F}_\alpha(\ell,s)}{\delta x^\nu(s')} = \varepsilon_{\nu\alpha} \frac{\delta\mathscr{F}_\alpha(\ell,s')}{\delta x^\mu(s)}, \qquad (2.8)$$

$$\text{for } \mu = \nu = 1,\ \frac{\delta \mathcal{F}_2(\ell,s)}{\delta x^1(s')} = \frac{\delta \mathcal{F}_2(\ell,s')}{\delta x^1(s)}\ , \tag{2.8a}$$

which is false, unless $s' \to s$; for $\mu = 1$, $\nu = 2$,

$$\frac{\delta \mathcal{F}_2(\ell,s)}{\delta x^2(s')} = -\frac{\delta \mathcal{F}_1(\ell,s')}{\delta x^1(s)}\ , \tag{2.8b}$$

which become eq. (2.5) only in the limit $s' \to s$. Therefore we see that higher conserved currents cannot be constructed by this procedure. In ref. (7) we demonstrate this point by solving the 2-dimension Yang-Mills equations explicitly.

Since the higher conserved currents do not exist, we cannot construct the inverse-scattering equation following the procedure given in Section I; however, by analog to eq. (1.10), we can construct one,

$$\left(\frac{\delta}{\delta x^\mu(s)} + \mathcal{F}_\mu(\ell,s) - \gamma\epsilon_{\mu\nu}\frac{\delta}{\delta x^\nu(s)}\right)\phi(\ell,\gamma) = 0. \tag{2.9}$$

What are the conditions for the integration of $\phi(\ell,\gamma)$? Equation (2.9) can be rewritten as

$$\frac{\delta}{\delta x_\mu(s)}\phi(\ell,\gamma) = -\frac{1}{1+\gamma^2}[\mathcal{F}_\mu(\ell,s) + \gamma\epsilon_{\mu\alpha}\mathcal{F}_\alpha(\ell,s)]\phi(\ell,\gamma). \tag{2.9a}$$

Requiring, for arbitrary γ,

$$\frac{\delta^2\phi(\ell,\gamma)}{\delta x^\mu(s)\,\delta x^\nu(s')} - \frac{\delta^2\phi(\ell,\gamma)}{\delta x^\nu(s')\,\delta x^\mu(s)} = 0, \tag{2.10}$$

one obtains the following conditions for arbitrary s, and s'

$$\frac{1}{\gamma^3}\ \text{term:}\quad \epsilon_{\mu\alpha}\frac{\delta \mathcal{F}_\alpha(\ell,s)}{\delta x^\nu(s')} - \epsilon_{\nu\beta}\frac{\delta \mathcal{F}_\beta(\ell,s')}{\delta x^\mu(s)} = 0 \tag{2.10}$$

$$\mu=1,\ \nu=1,\ \frac{\delta \mathcal{F}_2(\ell,s)}{\delta x^1(s')} - \frac{\delta \mathcal{F}_2(\ell,s')}{\delta x^1(s)} = 0, \tag{2.10a}$$

$$\mu=1,\ \nu=2,\ \frac{\delta \mathcal{F}_2(\ell,s)}{\delta x^2(s')} + \frac{\delta \mathcal{F}_1(\ell,s')}{\delta x^1(s)} = 0; \tag{2.10b}$$

$$\frac{1}{\gamma^2}\ \text{term:}\quad -\frac{\delta \mathcal{F}_\mu(\ell,s')}{\delta x^\nu(s)} + \frac{\delta \mathcal{F}_\nu(\ell,s)}{\delta x^\mu(s')} + [\epsilon_{\mu\alpha}\mathcal{F}_\alpha(\ell,s'),\epsilon_{\nu\beta}\mathcal{F}_\beta(\ell,s)] = 0, \tag{2.11}$$

$$\frac{1}{\gamma}\ \text{term:}\quad -\epsilon_{\mu\alpha}\frac{\delta \mathcal{F}_\alpha(\ell,s')}{\delta x^\nu(s)} + \epsilon_{\nu\beta}\frac{\delta \mathcal{F}_\beta(\ell,s)}{\delta x^\mu(s')} + [\epsilon_{\mu\alpha}\mathcal{F}_\alpha(\ell,s'),\mathcal{F}_\nu(\ell,s)] + [\mathcal{F}_\mu(\ell,s'),\epsilon_{\nu\beta}\mathcal{F}_\beta(\ell,s)] = 0, \tag{2.12}$$

$$\gamma^\circ\ \text{term:}\quad \frac{\delta \mathcal{F}_\mu(\ell,s')}{\delta x^\nu(s)} - \frac{\delta \mathcal{F}_\nu(\ell,s)}{\delta x^\mu(s')} + [\mathcal{F}_\mu(\ell,s'),\ \mathcal{F}_\nu(\ell,s)] = 0. \tag{2.13}$$

We see that they require much more than the loop space chiral equations (2.4) and (2.5) for integrability.

Case (2), $\chi^{(1)}$ is not only a functional of the loop but also a parameter s. Now Eq. (2.6) becomes

$$\mathscr{F}_\mu(\ell,s) = \lim_{s'\to s} \varepsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell,s)}{\delta\chi_\mu(s')} = \varepsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell,s)}{\partial x_\mu(s)} \qquad (2.6)''$$

Thus the integrability condition of Eq. (2.6) becomes

$$\frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\nu(s)\delta x^\mu(s)} - \frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\mu(s)\delta x^\nu(s)} = 0. \qquad (2.7)'$$

Notice that all parameters coincide at a point at s. Then from $\frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\nu(s)} = \varepsilon_{\nu\mu}\mathscr{F}_\mu(\ell,s)$ the integrability condition becomes Eq. (2.8) with $s'\to s$. Thus the equation of motion Eq. (2.5) does provide integrability of $\chi^{(1)}(\ell,s)$ from (2.6)'. However the peculiar situation here is that Eq. (2.6)' constraints $\chi(\ell,s)$ only when the parameter s' of $\delta_{\chi_\mu}(s')$ coincide with s of $\chi(\ell,s)$, thus does not constraint $\chi(\ell,s)$ enough and there are infinite many $\chi(\ell,s)$'s that can satisfy Eq. (2.6)". This is another manifestation that additional informations are needed in order to integrate the system from one point of the loop to the other uniquely.

Since $\chi^{(1)}(\ell,s)$ can be constructed, now we can follow the same procedure as Eq. (1.6) and (1.7) in the ordinary chiral field to construct the higher currents.

$$V_\mu^{(n)}(\ell,s) = \lim_{s'\to s} \left[\frac{\delta}{\delta x_\mu(s')} + \mathscr{F}_\mu(\ell,s)\right] \chi^{(n-1)}(\ell,s).$$

Notice here the arbitrariness in $\chi^{(n-1)}(\ell,s)$ reflects directly in the next current $V_\mu^{(n)}(\ell,s)$. Similarly following the same procedure as in Eqs. (1.8) to (1.10), one obtains the linearized equation for $\phi(\ell,s,L)$, which is in the form of Eq. (2.9) with $\gamma = L^{-1}$,

$$\lim_{s'\to s} \left[\frac{\delta}{\delta x^\mu(s')} + \mathscr{F}_\mu(\ell,s) - L^{-1}\,\varepsilon_{\mu\nu}\frac{\delta}{\delta x_\nu(s)}\right] \phi(\ell,s,L) = 0. \qquad (2.9)'$$

The integrability conditions in this limit of $s'\to s$ are just Eqs. (2.10) to (2.13) with $s'\to s$, which imply eqs. (2.4) and (2.5), the equations of motion.

In conclusion, the above discussions indicate that the loop-space chiral equations are not a totally integrable system in the ordinary sense. The loop-space chiral equations do not provide enough information for the integration of loop space currents from one point of the loop to another in a unique way. However, in spite of such difficulties, the observation that the Yang-Mills equations give the loop-space chiral equations is such a beautiful one, with further insight it is bound to lead to new understanding of the gauge theories.

Acknowledgement: We would like to thank A. M. Polyakov and I. Ya. Arefeva for their responsive correspondences after receiving our manuscript and the warm discussions afterwards. They obviously were aware of the points elaborated here though they did not spell them out in detail in their papers. One of us (LLCW) would like to thank Prof. L. D. Soloviev for inviting her to attend the International Workshop on High Energy Physics at Protvino (Serpukov) Sept. 22-28, which made such direct discussions possible.

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9. In reference 7 we did not discuss this case because we felt that the equations were not constraining enough. We thank A. M. Polyakov and I. Ya. Arefeva for stressing the importance of this alternative.