BIANCHI-BÄCKLUND TRANSFORMATIONS, CONSERVATION LAWS, AND LINEARIZATION OF VARIOUS FIELD THEORIES

L.L. Chau Wang

CONTENTS

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Ling-Lie Chau Wang

Physics Department
Brookhaven National Laboratory
Upton, NY 11973

I. INTRODUCTION

My friend and colleague Maurice Goldhaber likes to classify physics activities into two kinds: the Olympian kind and the bloodhound kind. The latter refers to physics research activities that deal more directly with the comparison between theory and experimental results. The former refers to physics activities more of esertoric kind which is intelectually satisfying and mathematically oriented, but its immediate relevance respect to experimental results is not clear. My talk today falls into this category. I hope that I can convey to you some flavor f such research in a quite specialized field.

It has become increasingly clear that, besides its mathematical beauty, the Yang-Mills[1] theory may provide the key to our understanding of strong interactions. With the recent experimental observation of gluon jets[2], the idea of non-Abelian gauge theory for strong interactions is brought one step futhter to reality. Despite many interesting theoretical and phenomenological observations, like confinement[3], asymptotic freedom[4] and QCD perturbative studies[5], the non-Abelian gauge theory is far from fully solved.

In the past ten years or so, powerful mathematical tools have been developed in completely solving many two-dimensional non-linear differential equations[6], e.g., obtaining infinite number of conservation laws, soliton solutions and even the construction of S-matrix.[7] Interestingly it has been observed that many of these two-dimensional theories bear resemblances[8] to that of Yang-Mills theory, e.g., topological solutions, asymptotic freedom and confinement. In addition, it has been shown that if properly formulated, the equations for the self-dual Yang-Mills[9,10] as well as the full Yang-Mills[11] resemble those of the two-dimensional chiral fields. Therefore it is hopeful that the mathematical tools used for the two-dimensional theories can also be developed for the four-dimensional non-Abelian gauge theory. Here I shall report some of the progress we made in this direction.[10,12,13,14,15]

II. THE SINE-GORDON EQUATION: PARAMETRIC BIANCHI-BACKLUND TRANSFORMATIONS (BT) AND THE DERIVATION OF LOCAL CONSERVATION LAWS

To understand what BT is, we give the concrete and classical example of the Sine-Gordon equation (S-G), which is a second-order differential equation,

$$\alpha_{,\zeta\eta} = \sin\alpha . \tag{2.1}$$

Its BT is a pair of coupled 1st-order differential equations:

$$\tfrac{1}{2}(\alpha' - \alpha)_{,\zeta} = \gamma\sin\tfrac{1}{2}(\alpha' + \alpha), \tag{2.2a}$$

$$\tfrac{1}{2}(\alpha' + \alpha)_{,\eta} = \gamma^{-1}\sin\tfrac{1}{2}(\alpha' - \alpha), \tag{2.2b}$$

where γ is a constant; α and α' are functions of ζ,η; the subscript with a comma means differentiation. Given a function α, which is a solution to Eq. (2.1), α' satisfying Eqs. (2.2a) and (2.2b) will also satisfy Eq. (2.1). Thus BT can be loosely defined as coupled set of 1st-order differential equations which relate two functions such that each of the two functions satisfies a designated second-order equation. It is elementary to show that Eqs. (2.2a) and (2.2b) have such properties:

$$(2.2a),_{\eta} \rightarrow \tfrac{1}{2}(\alpha'-\alpha),_{\zeta\eta} = \gamma\tfrac{1}{2}(\alpha'-\alpha),_{\eta}\cos\tfrac{1}{2}(\alpha'+\alpha)$$

$$= \sin\tfrac{1}{2}(\alpha'-\alpha)\cos\tfrac{1}{2}(\alpha'+\alpha), \qquad (2.3a)$$

$$(2.2b),_{\zeta} \rightarrow \tfrac{1}{2}(\alpha'+\alpha),_{\eta\zeta} = \frac{1}{\gamma}\tfrac{1}{2}(\alpha'-\alpha),_{\zeta}\cos\tfrac{1}{2}(\alpha'-\alpha)$$

$$= \sin\tfrac{1}{2}(\alpha'+\alpha)\cos\tfrac{1}{2}(\alpha'-\alpha), \qquad (2.3b)$$

$$(2.3a)+(2.3b) \rightarrow \alpha',_{\zeta\eta} = \sin\alpha', \qquad (2.4a)$$

$$(2.3a)-(2.3b) \rightarrow \alpha,_{\zeta\eta} = \sin\alpha. \qquad (2.4b)$$

So α and α' <u>individually</u> ("strong" BT) satisfy the <u>same</u> ("auto" BT) second-order differential equation, the S-G equation; these two properties are characterized as "strong" and "auto" respectively as written in the parentheses. For examples of non-auto BT, see Ref. 16.

The first use of the BT is to generate new solutions from a given solution (which can be a very trivial kind) by solving first-order differential equations rather than the original second-order differential equations, e.g., given $\alpha = 0$, a trivial solution to Eq. (2.1), we can solve Eqs. (2.2a), (2.2b) to obtain the non-trivial one soliton solution

$$\alpha' = 4\tan^{-1}[\pm\mathrm{Exp}(\gamma\zeta + \gamma^{-1}\eta + c)], \qquad (2.5)$$

where c is an integration constant. Multisoliton solutions can also be generated.[6,17]

The second use of the BT is to obtain infinite number of local conservation laws from the parameter γ. First we need to obtain a continuity-like equation involving both α and α': multiplying Eq. (2.2a) by $\tfrac{1}{2}(\alpha' + \alpha),_{\eta}$ and Eq. (2.2b) by $\tfrac{1}{2}(\alpha' - \alpha),_{\zeta}$, and then subtracting, we obtain

$$\gamma[\cos\tfrac{1}{2}(\alpha'+\alpha)],_{\eta} - \frac{1}{\gamma}[\cos\tfrac{1}{2}(\alpha'-\alpha)],_{\zeta} = 0. \qquad (2.6)$$

Note that as $\gamma = 0$, $\alpha' = \alpha$ in Eq. (2.2a). So we expand α' near $\gamma = 0$, i.e.,

$$\alpha' = \alpha_0 + \gamma\alpha_1 + \gamma^2\alpha_2 + \cdots . \qquad (2.7)$$

Substituting Eq. (2.7) into the BT Eqs. (2.2a), (2.2b), we can solve for α_i,

$$\alpha_0 = \alpha, \quad \alpha_1 = 2\alpha_{,\zeta}, \quad \alpha_2 = \alpha_{,\zeta\zeta}, \quad \cdots . \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.7) and then into Eq. (2.6) and collecting terms of the same power in γ, one can then obtain an infinite number of constraints, which relate α with higher and higher order of differentiations.[18] These infinite number of conservation equations, in the quantum mechanical version, gives severe constraints and implies no particle production in a scattering[7]. This forms the basis of construction the S-matrix[7] for the S-G equation. Though the existence of both solitons and the infinite set of conservation equations must be related, it appears that the latter is more relevant for the quantum mechanical system.

Though physicists recently christened Eq. (2.1) as the Sine-Gordon equation, the equation was well known to the mathematicians in the 19th century. Eq. (2.1) defines a surface of constant Gaussian curvature -1, i.e., a pseudosphere, if α is taken to be the angle between two asymptotes.[19] Bianchi first introduced the transformation Eq. (2.2) of $\gamma = 1$ and then Bäcklund generalized it to arbitrary γ in 1875. The BT generates a new pseudosphere from the old one with the following characteristics[20]: they share common tangents, the distance between the two points sharing the common tangent and the angle θ between the normals at the two points
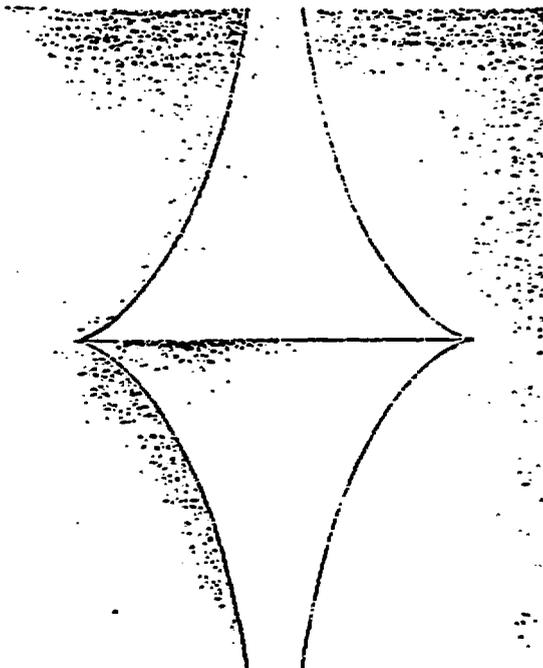
Fig.21. The surface of the revolution of a tractrix.

are kept constant. The free parameter γ in Eqs. (2.2a) and (2.2b) is related to θ by $\gamma = \cot\theta + \csc\theta$. $\theta = 90^O$ is a special case first given by Bianchi. The geometric meaning of the first soliton generation by the BT is shown in Fig. (1). Starting with $\alpha = 0$, a limiting case of a pseudosphere, the first soliton α' as given by Eq. (2.5) with $\gamma = 1$ corresponds to the surface of the revolution of a tractrix.

Therefore, whenever a BT is constructed, mathematicians will ask its geometric interpretations and physicists will look for new soliton solutions and possible conservation laws if there are parameters in the BT.

III. CHIRAL FIELDS: PARAMETRIC BT, LOCAL AND NON-LOCAL CONSERVATION LAWS AND LINEARIZATION

Chiral theories are theories with geometrical constraints. They have rich geometrical structure and are relevant to certain physical systems. In particle physics, these models are studied because of their similarity (asymptotic freedom, instantons, etc.) with four dimensional Yang-Mills fields. Further, as we shall briefly mention in Section VI, it has recently been shown that the classical Yang-Mills theory can be formulated as chiral fields in the loopspace.

As shown in Section II BT is so useful. One may ask if there is a systematic way of finding BT for a given second-order differential equation. As far as I know, the answer is negative. BT is found by guesswork. Here I shall discuss the BT with two parameters for the principal chiral fields (generalization to chiral field is trivial). The BT here is guessed and motivated from our work for the self-dual Yang-Mills field in four dimensions.[10] However, logically it is more natural to discuss the two dimensional chiral first.[12]

The principal chiral fields $g(\xi,\eta)$ of group SU(N) are $n \times n$ matrix fields, which have the following Lagrangian density and constraints:

$$\mathscr{L} = \mathrm{Tr}(\partial_\zeta g)(\partial_\eta g^+) \text{ , with } g^+ g = g g^+ = I. \qquad (3.1)$$

Defining

$$A_\zeta \equiv g^+ \partial_\zeta g, \quad A_\eta \equiv g^+ \partial_\eta g, \qquad (3.2)$$

equation of motion obtained from Eq. (3.1) is

$$\partial_\zeta A_\eta + \partial_\eta A_\zeta = 0. \qquad (3.3)$$

Here we use the light cone variable $\zeta \equiv x_0 + x_1$, $\eta \equiv x_0 - x_1$. Notice here that A_ζ, A_η from the algebra of SU(N) and they can be considered as pure gauge potential due to $\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta] = 0$. Eq. (3.3) has the appearances of a continuity equation. Eq. (3.2) and Eq. (3.3) characterize the most important properties of the system, i.e., curvatureless gauge potential and continuity like equation satisfied by the potential. These properties are shared by many non-linear differential equations including the properly formulated self-dual Yang-Mills equations, as we shall discuss later in Section V.

Parametric BT and local conservation laws: The BT we have constructed for the principal chiral fields is:

$$g'^+ \partial_\zeta g' - g^+ \partial_\zeta g = \partial_\zeta (g^+ g') , \qquad (3.4)$$

$$g'^+ \partial_\eta g' - g^+ \partial_\eta g = - \partial_\eta (g^+ g'), \qquad (3.5)$$

with the constraint

$$g^+ g' + g'^+ g = 2\beta I, \text{ where } \beta \leq 1 \qquad (3.6)$$

and

$$g'^+ g' = g^+ g = 1 . \qquad (3.7)$$

β is a constant parameter. It is easy to show that Eq. (3.4) and (3.5) are the BT. $(3.4)_{,\eta} + (3.5)_{,\zeta} \rightarrow \partial_\eta A'_\zeta + \partial_\zeta A'_\eta - \partial_\eta A_\zeta - \partial_\zeta A_\eta = 0$. So A'_ζ and A'_η satisfy the equation of motion Eq. (3.3), if A_ζ, A_η do. Unlike the BT for the S-G equation, this is a "weak" BT. The condition Eq. (3.6) can be shown to follow from Eqs. (3.4) and (3.5).

Since there is a parameter in the BT, we can now try to derive local currents just like in the S-G case. For this purpose our experience taught us that it's better to re-write Eq. (3.4), (3.5) in the following form:

$$2(1-\beta)\partial_\zeta(g'+g) = (g'-g)[(\partial_\zeta g^+)g'+g'^+(\partial_\zeta g)] ,$$ (3.8)

$$2(1+\beta)\partial_\eta(g'-g) = -(g'+g)[(\partial_\eta g^+)g'+g'^+(\partial_\eta g)] .$$ (3.9)

Then incorporating equation of motion Eq. (3.3), a continuity-like equation could be obtained,

$$(1-\beta)\partial_\zeta\{Tr[(\partial_\eta g^+)g'+g'^+(\partial_\eta g)]\}+(1+\beta)\partial_\eta\{Tr[(\partial_\zeta g^+)g'+g'^+(\partial_\zeta g)]\} = 0 .$$ (3.10)

This is analogous to Eq. (2.6) for the S-G equation. Using the procedure demonstrated in Section II, local conservation laws can be derived by expanding around $\beta = 1$.

We first discuss a special case of the $O(3)$ σ-model. In this case,

$$g^+ = g = \vec{q} \cdot \vec{\sigma} \text{ and } \vec{q} \cdot \vec{q} = 1,$$ (3.11)

where $\vec{\sigma}$ are the Pauli matrices. The BT of Eqs. (3.8) and (3.9) and the constraint equation (3.6) become

$$(1-\beta)\partial_\zeta(\vec{q}' + \vec{q}) = (\vec{q}' - \vec{q})[(\partial_\zeta\vec{q})\cdot\vec{q}'],$$ (3.12)

$$(1+\beta)\partial_\eta(\vec{q}' - \vec{q}) = -(\vec{q}' + \vec{q})[(\partial_\eta\vec{q})\cdot\vec{q}'],$$ (3.13)

and

$$\vec{q}' \cdot \vec{q} = \beta .$$ (3.14)

The continuity equation Eq. (3.10) becomes[21]

$$(1-\beta)\partial_\zeta[(\partial_\eta\vec{q})\cdot\vec{q}'] + (1+\beta)\partial_\eta[(\partial_\zeta\vec{q})\cdot\vec{q}'] = 0.$$ (3.15)

Define θ by

$$\tfrac{1}{2}(1-\beta) \equiv \sin^2\theta = \theta^2 - \theta^4/3 + 2\theta^6/45 + \ldots ,$$
$$\tfrac{1}{2}(1+\beta) \equiv \cos^2\theta = 1 - \theta^2 + \theta^4/3 - 2\theta^6/45 + \ldots ,$$ (3.16)

and expand $\vec{q}'$ around $\theta = 0$,

$$\vec{q}' = \vec{a}_0 + \theta\vec{a}_1 + \theta^2\vec{a}_2 + \dots \quad . \tag{3.17}$$

Substituting Eqs. (3.17) into Eq. (3.12) and (3.13) and collecting coefficients of different powers in θ, one can obtain

$$\vec{a}_0 = \vec{q}, \ \vec{a}_1 = \pm\partial_\zeta\vec{q}/|\partial_\zeta\vec{q}|, \ \vec{a}_2 = \partial_\zeta(\partial_\zeta\vec{q}/|\partial_\zeta\vec{q}|)/|\partial_\zeta\vec{q}| \ , \ \dots \ . \tag{3.18}$$

Substituting Eq. (3.18) into Eq. (3.17) and then into Eq. (3.15), an infinite number of local constraints on q and its derivatives are obtained:

$$\partial_\eta[(\partial_\zeta q)^2] = 0, \ \partial_\eta\{[\partial_\zeta(\partial_\zeta\vec{q}/|\partial_\zeta q|)]^2\} = \partial_\zeta[(\partial_\zeta\vec{q})\cdot(\partial_\eta\vec{q})/|\partial_\zeta\vec{q}|], \dots \ . \tag{3.19}$$

These are the well-known local conserved local currents first obtained by Pohlmeyer[21] by a very different method. In the quantum version, these local currents were shown by Polyakov[22] to survive, actually in much simpler forms, and put severe constraint of no-particle production in an interaction, thus providing a basis for the construction of S-matrices[23] for the σ-model.

Now come back the general chiral fields. Following the same procedure as in Eq. (3.16), (3.17) and (3.18), except now we are working with matrix:

$$g' = a_0 + \theta a_1 + \theta^2 a_2 + \dots \quad . \tag{3.20}$$

Substituting into Eqs. (3.8) and (3.9) and the constraint equations (3.6) and (3.7), consistent solutions for a_0 and a_1 were found

$$a_0 = g, \ a_1 = 2(\partial_\zeta g)^{-1}\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}. \tag{3.21}$$

Substituting Eqs. (3.21), (3.20) into Eq. (3.10) we obtained the first non-trivial conserved quantity

$$\partial_\eta\left(\text{Tr}\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}\right) = 0, \tag{3.22}$$

Using equation of motion Eq. (3.3) and Eq. (3.22) one can obtain for arbitrary n

$$\partial_n \left[\mathrm{Tr}\left(\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)} \right)^n \right] = 0, \tag{3.23}$$

i.e., the eigenvalues of $\sqrt{(\partial_\zeta g^+)(\partial_\zeta g)}$ are independent of n. For the next order, a_2 must satisfy the following matrix equations:

$$a_1^+ \partial_\zeta a_1 = (\partial_\zeta g^+)a_2 - a_2^+(\partial_\zeta g),$$
$$g^+ a_2 + a_2^+ g = -4I, \tag{3.24}$$
$$a_1^+ a_2 + a_2^+ a_1 = 0,$$

where a_1 is given by Eq. (3.21). So far we have not been able to solve these equations for a_2. However, we have demonstrated that local conservation laws do exist for the chiral field.

Actually the BT we found has another parameter. The general BT is

$$g'^+ \partial_\zeta g' - g^+ \partial_\zeta g = \lambda \partial_\zeta (g^+ g'), \tag{3.25}$$
$$g'^+ \partial_n g' - g^+ \partial_n g = -\lambda \partial_n (g^+ g') \tag{3.26}$$

and the constraint $\lambda g^+ g' + \lambda^* g'^+ g = 2\beta I$; where β is a real constant, λ and its complex conjugate λ^* are also constants. The reader is referred to in Ref. 12 for a more detailed exposition.

Non-local conservation laws: Besides local conservation laws, the chiral fields also have non-local conservation laws. The existence of such non-local currents for the σ-model was first obtained by Lüscher and Pohlmeyer.[24] Here I shall demonstrate using the method of Brezin, et al.[25] As I mentioned before, Eq. (3.3) is like a continuity equation. So let's consider A_ζ, A_n to be the first currents, i.e.,

$$V_\zeta^{(1)} \equiv A_\zeta = \partial_\zeta \chi^{(1)}, \quad V_n^{(1)} \equiv A_n = -\partial_n \chi^{(1)}. \tag{3.27}$$

Such $\chi^{(1)}$ exists and can be obtained from the A's by integration because of the equation of motion Eq. (3.3). The higher currents are then obtained by an iteration procedure. Suppose the nth currents $V_\zeta^{(n)}$, $V_n^{(n)}$ exist, i.e.,

$$\partial_\eta V_\zeta^{(n)} + \partial_\zeta V_\eta^{(n)} = 0, \text{ and } V_\zeta^{(n)} = \partial_\zeta \chi^{(n)}, \ V_\eta^{(n)} = -\partial_\eta \chi^{(n)}. \tag{3.28}$$

Then the $(n+1)^{th}$ currents can be constructed from $\chi^{(n)}$ by

$$V_\zeta^{(n+1)} = \mathcal{D}_\zeta \chi^{(n)}, \ V_\eta^{(n+1)} = \mathcal{D}_\eta \chi^{(n)}, \tag{3.29}$$

where

$$\mathcal{D}_\zeta \equiv \partial_\zeta + A_\zeta, \ \mathcal{D}_\eta = \partial_\eta + A_\eta. \tag{3.30}$$

Now we need to prove $\partial_\eta V_\zeta^{(n+1)} + \partial_\zeta V_\eta^{(n+1)} = 0$.

$$\partial_\eta V_\zeta^{(n+1)} + \partial_\zeta V_\eta^{(n+1)} = (\partial_\eta \mathcal{D}_\zeta + \partial_\zeta \mathcal{D}_\eta)\chi^{(n)}, \quad \text{from Eq. (3.29)}$$

$$= (\mathcal{D}_\zeta \partial_\eta + \mathcal{D}_\eta \partial_\zeta)\chi^{(n)}, \quad \text{from Eq. (3.3)}$$

$$= -(\mathcal{D}_\zeta V_\eta^{(n)} - \mathcal{D}_\eta V_\zeta^{(n)}) \quad \text{from Eq. (3.28)}.$$

$$= -(\mathcal{D}_\zeta \mathcal{D}_\eta - \mathcal{D}_\eta \mathcal{D}_\zeta)\chi^{(n-1)},$$

$$= 0, \quad \text{because of Eq. (3.2)}.$$

Therefore the $(n-1)^{th}$ currents constructed from Eq. (3.29) are conserved. To obtain the $(n+1)^{th}$ charge we need integration and differentiation of lower charges at different points in ζ and η:

$$\chi^{(n+1)} = \int^\zeta \mathcal{D}_{\zeta'} \chi^{(n)} d\zeta' = \int^\zeta \mathcal{D}_{\zeta'} [\int^{\zeta'} \mathcal{D}_{\zeta''} \chi^{(n-1)} d\zeta''] = \cdots. \tag{3.31}$$

Thus the term "non-local" is used for these conservation laws.

In the quantum mechanical version, Lüscher[26] showed that in the σ-model these non-conservation laws also imply no particle production, which is the basis for constructing the S-matrix.[23]. However, so far the physical origin and meaning of these non-local currents for the chiral field have not been exploited.

Linearization: Now we want to show how to obtain the "linearized" equations, or sometimes named the inverse-scattering equations, for the chiral equations. These equations were known.[27] Here we

like to demonstrate the method, which is new and will be used for the self-dual Yang-Mills fields in Section V.

From Eqs. (3.28) and (3.29) we obtain

$$\partial_\zeta \chi^{(n)} = \mathcal{D}_\zeta \chi^{(n-1)}, \tag{3.31}$$

$$\partial_\eta \chi^{(n)} = -\mathcal{D}_\eta \chi^{(n-1)}. \tag{3.32}$$

Multiply Eq. (3.31) by L^n and sum,

$$\sum_{n=1}^{\infty} L^n \partial_\zeta \chi^{(n)} = \sum_{n=1}^{\infty} L^n \mathcal{D}_\zeta \chi^{(n-1)}, \tag{3.33}$$

where L is an arbitrary constant. Eq. (3.33) can be rewritten as

$$\partial_\zeta \left(\sum_{n=0}^{\infty} L^n \chi^{(n)} \right) = L \mathcal{D}_\zeta \left(\sum_{n=0}^{\infty} L^n \chi^{(n)} \right), \tag{3.34}$$

where the sum on the left-hand side of Eq. (3.33) can be extended to $n = 0$ because $\chi^{(0)} = 1$. Now define

$$\Psi \equiv \sum_{n=0}^{\infty} L^n \chi^{(n)}, \tag{3.35}$$

which is a function of ζ, η, and L. Eq. (3.34) becomes

$$\partial_\zeta \Psi = L \mathcal{D}_\zeta \Psi. \tag{3.36}$$

By similar procedure we obtain

$$\partial_\eta \Psi = -L \mathcal{D}_\eta \Psi. \tag{3.37}$$

To claim that Eq. (3.36) and (3.37) are the linearized equations for the chiral fields, we need to show that the integrability condition of Ψ from Eqs. (3.36) and (3.37) implies the chiral field. Eqs. (3.36) and (3.37) can be rewritten as

$$\partial_\zeta \Psi = L(1-L)^{-1} A_\zeta \Psi, \tag{3.36}'$$

$$-\partial_\eta \Psi = L(1+L)^{-1} A_\eta \Psi. \tag{3.37}'$$

$$\partial_\eta (3.36)' + \partial_\zeta (3.37)' \rightarrow$$

$$L(1-L)^{-1}[(\partial_\eta A_\zeta)\Psi + A_\zeta \partial_\eta \Psi] + L(1+L)^{-1}[(\partial_\zeta A_\eta)\Psi + A_\eta \partial_\zeta \Psi] = 0. \tag{3.38}$$

Using Eqs. (3.36)' and (3.37)' in Eq. (3.38) and after simple manipulations, one obtains

$$\{\partial_\eta A_\zeta + \partial_\zeta A_\eta + \ell(\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta])\}\psi = 0. \qquad (3.39)$$

We see that the integrability of ψ for arbitrary L implies $\partial_\eta A_\zeta - \partial_\zeta A_\eta - [A_\zeta, A_\eta] = 0$, and $\partial_\eta A_\zeta + \partial_\zeta A_\eta = 0$. These are just the conditions of curvaturelessness of the gauge potential A and the continuity-like equation of Eq. (3.2) and (3.3). Notice that if we define $\lambda = L^{-1}$, Eqs. (3.36)' and (3.37)' are just the inverse-scattering equations of Zakharov and Mikhailov[27] for the chiral fields. Thus we see that the existence of conserved non-local currents is closely related to the fact that there is an arbitrary parameter in the linearized equations. This aspect should be further analyzed in order to shed light on the meaning of those non-local currents.

Lastly we note that writing in the x_1, x_2 coordinate, Eqs. (3.36)' and (3.37)' become

$$[\partial_\mu - A_\mu - L^{-1}\epsilon_{\mu\nu}\partial_\nu]\,\phi(x,L) = 0, \quad \mu = 1,2. \qquad (3.40)$$

IV. SUPERCHIRAL FIELDS: A PARALLEL DEVELOPMENT SIMILAR TO THE CHIRAL FIELDS.

It is known that a natural and elegant way to incorporate fermions is via super symmetric space extensions.[28] As in chiral theory, we hope that σ-model with fermion may shed light on four-dimensional gauge theory with fermions. Since the construction[29] of super-chiral field some progress has been made, e.g., the inverse-scattering equation·formulation,[30] the derivation of conserved currents as non-local functionals of the fields in the special case of the non-linear super σ-model,[31] and the conserved charges[32] as non-local functional of the fields in general.

In a recent paper with Z. Popowicz,[14] we show that exact parallel development like those for the ordinary chiral equation, i.e., construction of parametric BT, derivation of local conservation laws, non-local conservation laws and the inverse scattering formu-

lation. From this exercise we can appreciate the elegance of introducing fermion fields via super-space symmetry and view these theories in a unified way.

Following the formulation of Refs. 29 and 30 we consider a super-chiral field $\hat{g}(\xi, \eta, \theta_1 \theta_2)$, which is an element of the group SU(n) in the space of the usual coordinates x_0, x_1 (with $\xi \equiv x_0 + x_1$, $\eta \equiv x_0 - x_1$) and the two anti-commuting coordinates θ_1 and θ_2, which are Majorana spinors. The $\hat{g}$ satisfies the unitary condition

$$\hat{g} \, \hat{g}^+ = \hat{g}^+ \hat{g} = I. \qquad (4.1)$$

Due to the anti-commuting properties of θ_1 and θ_2, $\hat{g}$ can in general be expanded in the following form:

$$\hat{g} = (1 + i\theta_2 \Lambda_1 - i\theta_1 \Lambda_2 + i\theta_1 \theta_2 F) \, g \,. \qquad (4.2)$$

Since $[g, \theta_i] = 0$, we have $\{\Lambda_i, \theta_j\} = 0$, $[F, \theta_i] = 0$, where $i, j = 1, 2$. The unitary condition Eq. (4.1) imposes the following constraints

$$gg^+ = g^+ g = 1, \quad \Lambda_i + \Lambda_i^+ = 0, \text{ where } i = 1, 2$$
$$F + F^+ - i\Lambda_1 \Lambda_2^+ + i\Lambda_2 \Lambda_1^+ = 0 \,. \qquad (4.3)$$

The action of the super chiral field is

$$S = \int d\theta_2 d\theta_1 d\xi d\eta \ \text{Tr} \ (\hat{d}_1 \hat{g} \hat{d}_2 \hat{g}^+), \qquad (4.4)$$

where

$$\hat{d}_1 \equiv \frac{\partial}{\partial \theta_2} - i \, \theta_2 \frac{\partial}{\partial \xi}, \quad \hat{d}_2 \equiv -\frac{\partial}{\partial \theta_1} + i\theta_1 \frac{\partial}{\partial \eta} \,. \qquad (4.5)$$

The $\hat{d}$'s have the anticommuting property

$$\hat{d}_1 \hat{d}_2 + \hat{d}_2 \hat{d}_1 = 0. \qquad (4.6)$$

The equation of motion following from such an action is

$$\hat{A}_i \equiv (\hat{d}_i \hat{g}) \, \hat{g}^+, \quad i = 1, 2 \qquad (4.7)$$

$$\hat{d}_1 \hat{A}_2 - \hat{d}_2 \hat{A}_1 = 0. \qquad (4.8)$$

This is the similar equation to Eq. (3.3) of the chiral equation, except that here the light cone variables are not used. Now we can derive almost one-one correspondingly all those equations in Section

III.   In the manipulations we must remember the differences in the differential operators: $\hat{d}_1, \hat{d}_2$ anti-commutes but $\partial_\zeta, \partial_\eta$ commute; and the super-covariant-differentiation $\hat{\mathscr{D}}_1 \equiv \hat{d}_1 - \hat{A}_1$, $\hat{\mathscr{D}}_2 \equiv \hat{d}_2 - \hat{A}_2$ anti-commutes $\hat{\mathscr{D}}_1 \hat{\mathscr{D}}_2 + \hat{\mathscr{D}}_2 \hat{\mathscr{D}}_1 = 0$, while the ordinary-covariant-differentiations commutes. So we can construct <u>the two-parametric BT</u>, <u>local conservation laws, non-local conservation laws</u> and <u>the linearization</u> for the super chiral equations just the same. We refer the reader to Ref. 14 for details.

The natural next step is the construction of S-matrix, at least for the special case of super-σ model. However so far this has not been done.

V.   SELF-DUAL YANG-MILLS FIELDS IN 4-DIMENSIONAL EUCLIDEAN SPACE: VERY SIMILAR DEVELOPMENT

Here we shall briefly describe the similar development in the case of self-dual Yang-Mills equations.

After the historical discovery of the one instanton solution[33] to the self-dual Yang-Mills field in 4-dimensional Euclidean space, general multi-instanton solutions were constructed.[34] They are actually proven to be all the solutions giving finite, minimum actions of the Yang-Mills field.[35] However, since the construction of the Polyakov-'t Hooft[36] monopole, which in the limit of vanishing Higgs potential[37] is a self-dual finite energy solution depending[38] on three dimensions in the 4-dimensional Euclidean space, no finite-energy multi-monopole solutions have been found, nor the proof of their non-existence.   It is therefore important to pursue other methods which may help to find such solutions. Further, it is interesting in its own right to exploit the possibility of making similar, successful mathematical developments for 4-dimensional fields analogous to those for the two-dimensional theories. Indeed we find that the similarities have been striking, e.g., the BT, the conservation laws, and now even the inverse-scattering formulation. Conversely, our studies of 4-dimensional theories also helped to make progress for the 2-dimensional theories, like the BT for chiral fields discussed in Section III.

After a lengthy reformulation,[39] the self-dual equations

$$F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{5.1}$$

can be written as

$$B_y \equiv J^{-1}\partial_y J, \quad B_z \equiv J^{-1}\partial_z J, \tag{5.2}$$

$$\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z = 0 \tag{5.3}$$

where $y \equiv (x_1+ix_2)/\sqrt{2}$, $\bar{y} = (x_1-ix_2)/\sqrt{2}$, $z = (x_3-ix_4)/\sqrt{2}$, $\bar{z} = (x_3+ix_4)/\sqrt{2}$; and J is an n x n Hermitean matrix with det $J = 1$. The problem of finding solutions to Eq. (5.1) becomes to find a n x n Hermitian matrix J satisfying Eq. (5.3) with det $J = 1$. The field curvature can be constructed by $F_{u\bar{v}} = -D^{-1}\partial_{\bar{v}}(J^{-1}\partial_u J)D$, where $u,v = y,z$ and the D is any n x n matrix with det $D = 1$, and picking a different D corresponds to choosing a different gauge.

<u>Bäcklund transformation</u>: For Eq. (5.3) a BT can be constructed[10]

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = e^{i\alpha}\partial_{\bar{z}}(J^{-1}J'), \tag{5.4}$$

with the algebraic constraint

$$J'J^{-1} - J'^{-1}J = \beta I, \tag{5.5}$$

where α, β are real constants. To show Eq. (5.4) with Eq. (5.5) is a BT, we first obtain the following equation

$$J^{-1}\partial_z J - J'^{-1}\partial_z J' = -e^{i\alpha}\partial_{\bar{y}}(J^{-1}J'), \tag{5.6}$$

from Eq. (5.4), by taking Hermitian conjugate of Eq. (5.4), and then multiplying J'^{-1} to the left and J to the right, and finally using Eq. (5.5). Now differentiate Eq. (5.4) by $\bar{y}$ and Eq. (5.6) by $\bar{z}$, and then add

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) - \partial_{\bar{y}}(J'^{-1}\partial_y J') - \partial_{\bar{z}}(J'^{-1}\partial_z J') = 0.$$

So J' satisfies the self-dual equation Eq. (5.3) if J does. The BT Eq. (5.4) and (5.5) is thus "weak". It has two parameters[40] α, β. We could also show from Eq. (5.5) alone that the BT is non-

auto. For example, starting with a self-dual J giving field of SU(2), J' gives a field of SU(1,1) and vice-versa. In general for a J giving a field of SU(n) with n > 2, using the algebraic equation Eq. (5.5) alone, we can show that J' must not give a field of SU(n), but we do not know the final group explicitly without solving Eq. (5.4). So far no local conservation laws have been found from this BT, though it has two parameters. This is probably related to the fact that the BT is non-auto. The topological-solution-generating properties of the BT are yet to be studied.

<u>Non-local conservation laws</u>: Notice the striking similarity between the self-dual equations (5.2) and (5.3) and the chiral equations (3.2) and (3.3). Therefore non-local conservation laws can similarly be constructed. Consider B_y and B_z of Eq. (5.3) being the first conserved currents,

$$\dot{v}_y^{(1)} \equiv B_y = \partial_{\bar{z}}\chi^{(1)}, \quad v_z^{(1)} \equiv B_z = -\partial_{\bar{y}}\chi^{(1)}, \tag{5.7}$$

$\chi^{(1)}$ exists because Eq. (5.3). Now suppose that the nth currents exist, i.e.,

$$\partial_{\bar{y}}v_y^{(n)} + \partial_{\bar{z}}v_z^{(n)} = 0, \tag{5.8}$$

$$v_y^{(n)} = \partial_z\chi^{(n)}, \quad v_z^{(n)} = -\partial_y\chi^{(n)}. \tag{5.9}$$

Then the $(n+1)^{th}$ currents are

$$v_y^{(n+1)} = \mathscr{D}_y\chi^{(n)}, \quad v_z^{(n+1)} = \mathscr{D}_z\chi^{(n)}, \tag{5.10}$$

where $\mathscr{D}_u \equiv \partial_u + B_u$, $u=y,z$.

Now we show that the $v^{(n+1)}$'s are conserved:

$$\partial_{\bar{y}}v_y^{(n+1)} + \partial_z v_z^{(n+1)} = (\partial_{\bar{y}}\mathscr{D}_y + \partial_{\bar{z}}\mathscr{D}_z)\chi^{(n)}, \text{ from Eq. (5.10)};$$

$$= (\mathscr{D}_y\partial_{\bar{y}} + \mathscr{D}_z\partial_{\bar{z}})\chi^{(n)}, \text{ due to Eq. (5.3)};$$

$$= -\mathscr{D}_y v_z^{(n)} + \mathscr{D}_z v_y^{(n)}, \text{ using Eq. (5.8)};$$

$$= (-\mathscr{D}_y\mathscr{D}_z + \mathscr{D}_z\mathscr{D}_y)\chi^{(n-1)}, \text{ using Eq. (510)}$$
$$\text{again;}$$

$$= 0, \text{ due to Eq. (5.2).}$$

Therefore, the self-dual Yang-Mills fields have these non-local conservation laws. However their physical meaning is not yet clear.

<u>Linearization</u>: Using the method demonstrated in Sec. III, from Eqs. (5.9) and (5.10):

$$\partial_{\bar{z}} \chi^{(n)} = \mathcal{D}_y \chi^{(n-1)}, \tag{5.11}$$

$$-\partial_{\bar{y}} \chi^{(n)} = \mathcal{D}_z \chi^{(n-1)}. \tag{5.12}$$

Multiplying Eqs. (5.11), (5.12) by L^n (L being an arbitrary constant), summing over n and defining

$$\Psi \equiv \sum_{L=0} L^n \chi^{(n)}, \text{ we obtain}$$

$$\partial_{\bar{z}} \Psi = L \mathcal{D}_y \Psi, \tag{5.13}$$

$$-\partial_{\bar{y}} \Psi = L \mathcal{D}_z \Psi, \tag{5.14}$$

To show that these equations are indeed linearized equations for the self-dual Yang-Mills equation, we need to show that the integrability of Ψ from Eqs. (5.13), (5.14) gives Eqs. (5.2), (5.3). Eqs. (5.13), (5.14) can be rewritten as

$$(\partial_{\bar{z}} - L\partial_y)\Psi = LB_y \Psi, \tag{5.15}$$

$$-(\partial_{\bar{y}} + L\partial_z)\Psi = LB_z \Psi. \tag{5.16}$$

Differentiating Eq. (5.15) by $\bar{y}$ and Eq. (5.16) by $\bar{z}$, and after some simple manipulations, we obtain

$$L(\mathcal{D}_y \mathcal{D}_z - \mathcal{D}_z \mathcal{D}_y)\Psi + (\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z)\Psi = 0. \tag{5.17}$$

For Eq. (5.17) to be true for all L, we need $\mathcal{D}_y \mathcal{D}_{\bar{z}} - \mathcal{D}_z \mathcal{D}_y = 0$ $\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z = 0$, which are precisely Eqs. (5.2) and (5.3). Therefore, Eqs. (5.15), (5.16) are the linearized equations[41] for Eqs. (5.2) and (5.3).

For multi-instanton solutions in the ansatz of 't Hooft et al.[42], the matrix J is

$$J_n = \frac{1}{\phi_n} \begin{pmatrix} 1 & \bar{\rho}_n \\ \rho_n & \phi_n^2 + \rho_n \bar{\rho}_n \end{pmatrix}, \tag{5.18}$$

where

$$\phi_n = 1 + \sum_{i=1}^{n} \frac{\lambda_i^2}{R_i^2},$$

$$\rho_n = -\sum_{i=1}^{n} \frac{\lambda_i^2}{R_i^2} \frac{y_i}{\bar{z}_i},$$

$$\bar{\rho}_n = -\sum_{i=1}^{n} \frac{\lambda_i^2}{R_i^2} \frac{\bar{y}_i}{z_i},$$

where $u_i \equiv u - u_{0,i}$ with $u_{0,i}$ being constants, and $u = y$, $\bar{y}$, z or or $\bar{z}$, and $R_i^2 \equiv y_i \bar{y}_i + z_i \bar{z}_i$. It turns out that we can solve Eqs. (5.15), (5.16),

$$\psi_n = 1 + \frac{1}{\phi_n} \begin{pmatrix} L\bar{\rho}_n & L\bar{\rho}_n \theta_n + (\theta_n - \bar{\rho}_n)\phi_n \\ -L & -L\theta_n \end{pmatrix},$$

where

$$\theta_n \equiv \sum_{i=1}^{n} \frac{\lambda_i^2}{R_i^2} \frac{\bar{y}_i}{L\bar{y}_i - z_i}.$$

It is interesting to note that χ_n has pole in L depending on co-ordinates. Details on the subject will be discussed in Ref. 43.

Though much work is still needed, it is already quite sur-prising that many of the mathematical formulations for 2-dimen-sional theories can be carried over to 4-dimension via the self-dual Yang-Mills fields. Still our horizon should not end here. Our real goal is to solve the 4-dimensional non-Abelian gauge theory. Such a hope has been raised by the loop formulation,[11] i.e., the classical Yang-Mills theory can be formulated as chiral field in the loop space, which we shall briefly describe in the next section. It is foreseeable that the methods discussed here will be useful even for the full Yang-Mills theory.

VI. **LOOP SPACE CHIRAL EQUATIONS: PARALLEL DEVELOPMENT BUT WITH SUBTLETY.**

The path ordered phase factor $\phi_{ab} \equiv P \exp\int_a^b dx_\mu A_\mu(x)$ along a loop (or part of a loop) has many desirable features. It is the

minimal necessary object[14] describing a gauge theory as demonstrated by the Aharnov-Bohm experiment[45] for the quantum mechanical electromagnetism. Also it has simple gauge transformation properties[46] and is the spin-like quantity for lattice gauge theory.[47] Thus it is natural, may even be essential, that we formulate gauge theory in the loop space. Last year (1979), it has been realized that Yang-Mills equations give chiral-like equations in the loop space.[11] This beautiful realization immediately raised the hope that the loop-space chiral equations may also be a total integrable system, just like the ordinary-space chiral equations as discussed in Section III, and thus might lead to the full solutions of the Yang-Mills equations.

Here we shall briefly introduce the loop-space chiral equations and then discuss the integrability conditions of the non-local currents in two possible different situations. We shall demonstrate some intricate properties of the integrability conditions of the loop space chiral equations, which do not have their correspondence in the ordinary chiral equations. Therefore, the implication of this formulation to the solution of the full Yang-Mills equation needs much further study.

Let us consider the phase factor along the loop $\ell = x^\mu(s)$ as shown in Fig. (6.1a),

$$\Phi_{02S10} = \psi(\ell) = P \exp(i\oint A_\mu dx_\mu). \qquad (6.1)$$

The functional differentiation of the loop phase factor is defined as the change in $\psi(\ell)$, as ℓ change to ℓ', which is infinitesimally deformed from ℓ at s (Fig. 6.1b)
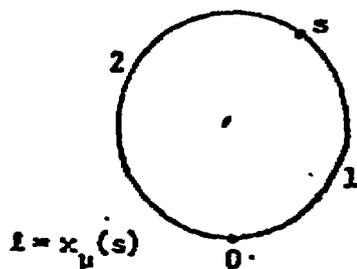

Fig. (6.1a)
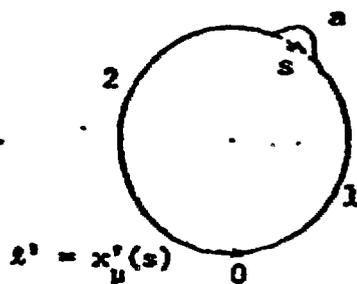
Fig. (6.1b)

$$\frac{\delta\psi(\ell)}{\delta x^\mu(s)} = \frac{\psi(\ell')-\psi(\ell)}{ds\,dx^\mu(x)} = \phi_{02s}f_{\lambda\mu}[x(s)]\frac{dx_\lambda(x)}{ds}\phi_{s10}. \quad (6.2)$$

It is just the parallel transported "normal flux" per unit area that went through the little area as shown in Fig. (6.1b). Define the loop space gauge potential

$$\mathcal{F}_\mu(\ell,s) \equiv \psi(\ell)^{-1}\frac{\delta\psi(\ell)}{\delta x^\mu(s)} = \phi_{01s}f_{\lambda\mu}[x(s)]\dot{x}_\lambda(s)\phi_{s10}. \quad (6.3)$$

Functionally differentiating it again, and after some work, one can show[11]

$$\frac{\delta\mathcal{F}_\mu(\ell,s)}{\delta x^\nu(s')} - \frac{\delta\mathcal{F}_\nu(\ell,s')}{\delta x^\mu(s)} + [\mathcal{F}_\mu(\ell,s),\mathcal{F}_\nu(\ell,s')] = 0, \quad (6.4)$$

which just gives the projected Bianchi identity:

$$(\mathcal{D}_\mu f_{\nu\lambda}[x(s)])\dot{x}_\lambda + (\mathcal{D}_\nu f_{\lambda\mu}[x(s)])\dot{x}_\lambda + (\mathcal{D}_\lambda f_{\mu\nu}[x(s)])\dot{x}_\lambda = 0,$$

$$(6.4a)$$

and

$$\frac{\delta\mathcal{F}_\mu(\ell,s)}{\delta x_\mu(s)} = 0, \quad (6.5)$$

which gives the projected Yang-Mills equation

$$\mathcal{D}_\mu f_{\mu\nu}[x(s)]\dot{x}_\nu(s) = 0. \quad (6.5a)$$

The geometric meaning of the loop space equation (6.4) is that the loop phase factor arrived from an initial loop to a given final loop is independent of the different volumes swapped out by the intermediate loops.

Again Eq. (6.5) is like a continuity equation so we try to follow the same procedure as for the chiral fields and identify the first current, here we specify in two dimension though the conclusion is general,

$$v_\mu^{(1)}(\ell,s) \equiv \mathcal{F}_\mu(\ell,s) = \varepsilon_{\mu\nu}\frac{\delta\chi^{(1)}}{\delta x^\nu(s)}. \quad (6.6)$$

This satisfies Eq. (6.5), but the question is whether Eq. (6.5) provides the sufficient conditions for the integration of χ from Eq. (6.6). We shall discuss separately the following two possible cases.

Case (1): $\chi^{(1)}$ is a functional of the loop above, i.e., Eq. (6.6) reads

$$\mathscr{F}_\mu(\ell,s) = \varepsilon_{\mu\nu} \frac{\delta \chi^{(1)}(\ell)}{\delta x^\nu(s)} . \tag{6.6}'$$

Just as the finite dimensional case, the integrability conditions of $\chi^{(1)}(\ell)$ is

$$\frac{\delta \chi^{(1)}(\ell)}{\delta x^\nu(s')\delta x^\mu(s)} - \frac{\delta \chi^{(1)}(\ell)}{\delta x^\mu(s)\delta x^\nu(s')} = 0. \tag{6.7}$$

From $\dfrac{\delta \chi^{(1)}(\ell)}{\delta x^\nu(s)} = -\varepsilon_{\nu\mu}\mathscr{F}_\mu(\ell,s)$, eq. (6.7) gives

$$\varepsilon_{\mu\alpha} \frac{\delta \mathscr{F}_\alpha(\ell,s)}{\delta x^\nu(s')} = \varepsilon_{\nu\alpha} \frac{\delta \mathscr{F}_\alpha(\ell,s')}{\delta x^\mu(s)} , \tag{6.8}$$

for $\mu = \nu = 1$, $\dfrac{\delta \mathscr{F}_2(\ell,s)}{\delta x^1(s')} = \dfrac{\delta \mathscr{F}_2(\ell,s')}{\delta x^1(s)}$, $\tag{6.8a}$

which is false, unless $s' \to s$; for $\mu = 1$, $\nu = 2$,

$$\frac{\delta \mathscr{F}_2(\ell,s)}{\delta x^2(s')} = -\frac{\delta \mathscr{F}_1(\ell,s')}{\delta x^1(s)} , \tag{6.8b}$$

which become Eq. (6.5) only in the limit $s' \to s$. Therefore, we see that higher conserved currents cannot be constructed by this procedure. In Ref. 15 we demonstrate this point by solving the 2-dimension Yang-Mills equations explicity.

Since the higher conserved currents do not exist, we cannot construct the inverse-scattering equation following the procedure given in Section III; however, by analog to Eq. (3.40), we can construct one,

$$\left[\frac{\delta}{\delta x^\mu(s)} + \mathscr{F}_\mu(\ell,s) - \gamma\varepsilon_{\mu\nu} \frac{\delta}{\delta x^\nu(s)} \right] \Diamond(\ell,\gamma) = 0. \tag{6.9}$$

What are the conditions for the integration of $\Diamond(\ell,\gamma)$? Equation (6.9) can be rewritten as

$$\frac{\delta}{\delta x_\mu(s)} \Diamond(\ell,\gamma) = -\frac{1}{1+\gamma^2}[\mathscr{F}_\mu(\ell,s) + \gamma\varepsilon_{\mu\alpha}\mathscr{F}_\alpha(\ell,s)]\Diamond(\ell,\gamma). \tag{6.9a}$$

Requiring, for arbitrary γ,

$$\frac{\delta^2\phi(\ell,\gamma)}{\delta x^\mu(s)\delta x^\nu(s')} - \frac{\delta^2\phi(\ell,\gamma)}{\delta x^\nu(s')\delta x^\mu(s)} = 0; \qquad (6.10)$$

one obtains the following conditions for arbitrary s, and s'

$$\frac{1}{\gamma^3} \text{ term: } \epsilon_{\mu\alpha}\frac{\delta\mathcal{F}_\alpha(\ell,s)}{\delta x^\nu(s')} - \epsilon_{\nu\beta}\frac{\delta\mathcal{F}_\beta(\ell,s')}{\delta x^\mu(s)} = 0 \qquad (6.10)$$

$$\mu=1, \quad \nu=1, \quad \frac{\delta\mathcal{F}_2(\ell,s)}{\delta x^1(s')} - \frac{\delta\mathcal{F}_2(\ell,s')}{\delta x^1(s)} = 0, \qquad (6.10a)$$

$$\mu=1, \quad \nu=2, \quad \frac{\delta\mathcal{F}_2(\ell,s)}{\delta x^2(s')} + \frac{\delta\mathcal{F}_1(\ell,s')}{\delta x^1(s)} = 0; \qquad (6.10b)$$

$$\frac{1}{\gamma^2} \text{ term: } -\frac{\delta\mathcal{F}_\mu(\ell,s')}{\delta x^\nu(s)} + \frac{\delta\mathcal{F}_\nu(\ell,s)}{\delta x^\mu(s')} + [\epsilon_{\mu\alpha}\mathcal{F}_\alpha(\ell,s'),\epsilon_{\nu\beta}\mathcal{F}_\beta(\ell,s)] = 0, \qquad (6.11)$$

$$\frac{1}{\gamma} \text{ term: } -\epsilon_{\mu\alpha}\frac{\delta\mathcal{F}_\alpha(\ell,s')}{\delta x^\nu(s)} + \epsilon_{\nu\beta}\frac{\delta\mathcal{F}_\beta(\ell,s)}{\delta x^\mu(s')} +$$

$$[\epsilon_{\mu\alpha}\mathcal{F}_\alpha(\ell,s'),\mathcal{F}_\nu(\ell,s)] + [\mathcal{F}_\mu(\ell,s'),\epsilon_{\nu\beta}\mathcal{F}_\beta(\ell,s)] = 0, \qquad (6.12)$$

$$\gamma^0 \text{ term: } \frac{\delta\mathcal{F}_\mu(\ell,s')}{\delta x^\nu(s)} - \frac{\delta\mathcal{F}_\nu(\ell,s)}{\delta x^\mu(s')} + [\mathcal{F}_\mu(\ell,s'), \mathcal{F}_\nu(\ell,s)] = 0. \qquad (6.13)$$

We see that they require much more than the loop space chiral equations (6.4) and (6.5) for integrability.

Case (2): $\chi^{(1)}$ is not only a functional of the loop but also a parameter s. Now Eq. (6.6) becomes

$$\mathcal{F}_\mu(\ell,s) = \lim_{s'\to s} \epsilon_{\mu\nu}\frac{\delta\chi^{(1)}(\ell,s)}{\delta x_\mu(s')} = \epsilon_{\mu\nu}\frac{\delta\chi^{(1)}(\ell,s)}{\partial x_\mu(s)} \qquad (6.6)''$$

Thus the integrability condition of Eq. (6.6) becomes

$$\frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\nu(s)\delta x^\mu(s)} - \frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\mu(s)\delta x^\nu(s)} = 0. \qquad (6.7)'$$

Notice that all parameters coincide at a point at s. Then from $\frac{\delta\chi^{(1)}(\ell,s)}{\delta x^\nu(s)} = \epsilon_{\nu\mu}\mathcal{F}_\mu(\ell,s)$ the integrability condition becomes Eq. (6.8) with $s' \to s$. Thus the equation of motion Eq. (6.5) does provide integrability of $\chi^{(1)}(\ell,s)$ from (6.6)'. However the peculiar situation here is that Eq. (6.6)' constraints $\chi(\ell,s)$ only when the

parameter s' of $\hat{o}_{x_\mu}$(s') coincide with s of $\chi(\ell,s)$, thus does not constraint $\chi(\ell,s)$ enough and there are infinite many $\chi(\ell,s)$'s that can satisfy Eq. (6.6)". This is another manifestation that additional informations are needed in order to integrate the system from one point of the loop to the other uniquely.

Since $\chi^{(1)}(\ell,s)$ can be constructed, now we can follow the same procedure as Eq. (3.29) and (3.30) in the ordinary chiral field to construct the higher currents.

$$V_\mu^{(n)}(\ell,s) = \lim_{s'\to s} \left[\frac{\delta}{\delta x_\mu(s')} + \mathscr{F}_\mu(\ell,s)\right] \chi^{(n-1)}(\ell,s) \quad (6.8)$$

Notice here the arbitrariness in $\chi^{(n-1)}(\ell,s)$ reflects directly in the next current $V_\mu^{(n)}(\ell,s)$. Similarly following the same procedure as in Eqs. (3.31) to (3.37), one obtains the linearized equation for $\phi(\ell,s,L)$, which is in the form of Eq. (6.9) with $\gamma = L^{-1}$,

$$\lim_{s'\to s} \left[\frac{\delta}{\delta x^\mu(s')} + \mathscr{F}_\mu(\ell,s) - L^{-1}\,\epsilon_{\mu\nu}\frac{\delta}{\delta x_\nu(s)}\right]\phi(\ell,s,L) = 0. $$

$$(6.9)'$$

The integrability condition in this limit of $s' \to s$ are just Eqs. (6.10) to (6.13) with $s' \to s$, which imply Eqs. (6.4) and (6.5), the equations of motion.

In conclusion, the above discussions indicate that the loop-space chiral equations are not a totally integrable system in the ordinary sense. The loop-space chiral equations do not provide enough information for the integration of loop space currents from one point of the loop to another in a unique way. However, in spite of such difficulties, the observation that the Yang-Mills equations give the loop-space chiral equations is such a beautiful one, with further insight it is bound to lead to new understanding of the gauge theories.

VII. SUMMARY

From the above discussions, we see the striking similarities of many non-linear systems. They all have the basic characteristic equations of being curvatureless plus a continuity-like equation.

(1) Chiral fields

$$A_\mu(x) \equiv g^{-1}(x)\, \partial_\mu g(x), \quad \mu = 1,2$$
$$\partial_\mu A_\mu(x) = 0.$$

(2) Super-chiral fields

$$\hat{A}_\mu(x) \equiv (\hat{d}_\mu \hat{g})\, \hat{g}^+, \quad \mu = 1,2$$
$$\hat{d}_1 \hat{A}_2 - \hat{d}_2 \hat{A}_1 = 0.$$

(3) Self-dual Yang–Mills equations in 4-dimension

$$B_y \equiv J^{-1} J,_y, \quad B_z = J^{-1} J,_z,$$
$$B_{y,\bar{y}} + B_{z,\bar{z}} = 0.$$

(4) Loop-space chiral equations (Yang–Mills equations)

$$\mathscr{F}_\mu(\ell,s) \equiv \psi(c)^{-1} \frac{\delta\psi(\ell)}{\delta x^\mu(s)}, \quad \mu = 1,2,\cdots.$$

$$\frac{\delta\mathscr{F}_\mu(\ell,s)}{\delta x^\mu(s)} = 0 .$$

I shall summarize the status of development in the following table.

STATUS OF VARIOUS FIELD THEORIES

Non-linear differential equations	BT	Number of parameters	Conservation laws		Inverse scattering (linearization)	S-Matrix
			local	non-local		
Sine-Gordon	✓	1	✓	?	✓	✓
Non-linear σ-model	✓	2	✓	✓	✓	✓
Chiral fields	✓	2	✓	✓	✓	?
Super-chiral fields	✓	2	✓	✓	✓	?
Self-dual Yang-Mills	✓	2	?	✓	✓	?
Loop-space Chiral fields (Yang-Mills)	✓	?	?	?	?	?

Hopefully one day all those question marks in the Table will be replaced by check marks.

<u>Note added in proof</u>. After the completion of this manuscript, a
new set of axially symmetrical self-dual monopole solution with
multi-topological charge has been found. See R.S. Ward "A Yang-
Mills Monopole of Charge 2" (to be published in Comm. Math. Phys.).
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