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RELATIVISTIC CHARGED BOSE GAS

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Abstract

The excitation spectrum of a relativistic spin-zero charged Bose gas is obtained in a dielectric response formulation. Relativity introduces a dip in the spectrum and the consequences of this dip for the thermodynamic functions are discussed.

The charged Bose gas has been previously studied [1] as a many body problem of great intrinsic interest which can also serve as a model of some real physical systems, for example, superconductors [2], white dwarf stars [3] and neutron stars.

There is now speculation that neutron stars may contain a superconducting proton component [4]. By analogy with a metallic superconductor, the proton "pairs" would be regarded as charged bosons. In addition, considerable effort has been made to elucidate the problem of pion condensation [5], since it is generally considered that this phenomenon is of great importance within neutron stars.

Motivated by these fascinating astrophysical speculations, we have investigated the gas of relativistic charged spin-zero bosons at zero temperature. Free spin-zero bosons are, of course, described by the Klein-Gordon equation. As is well-known, the presence of a second time derivative leads to difficulties in interpreting the wave-functions. However, as was first pointed out by Pauli and Weisskopf [6], there is no problem if the usual probability current is taken to be the charge current. A complete and consistent description of relativistic charged spin-zero bosons is given by the Hamiltonian formalism of Feshbach and Villars [7].

As in the case of the relativistic charged Fermi gas [8], we use a linearized self-consistent field treatment to obtain the dielectric response function. The Hamiltonian for charged bosons in an external field is given by [7,8]

$$H = \int d\underset{\sim}{x}\; \tilde{\Psi}^{+}(x)\tau_3\left[\frac{1}{2m}\left(\underset{\sim}{p} - \frac{e}{c}\underset{\sim}{A}^{(e)}(\underset{\sim}{x})\right)^2(\tau_3 + i\tau_2) + mc^2\tau_3 + e\phi\right]\tilde{\Psi}(x)$$
$$+ \frac{1}{2}\iint \frac{\rho(\underset{\sim}{x})\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d\underset{\sim}{x}\, d\underset{\sim}{x}' \quad , \tag{1}$$

where $\underset{\sim}{A}^{(e)}(\underset{\sim}{x})$ is the external potential excluding electric fields and $\rho(\underset{\sim}{x}) = e\tilde{\Psi}^{+}(x)\tau_3\tilde{\Psi}(x)$ is the charge density. $\tilde{\Psi}(x)$ is the appropriate field operator which is expanded in eigenfunctions that satisfy the Klein-Gordon equation [7]

$$\left[(\tau_3 + i\tau_2)\left(\frac{1}{2m}\right)\left(\underset{\sim}{p} - \frac{e}{c}\underset{\sim}{A}^{(e)}(\underset{\sim}{x})\right)^2 + mc^2\tau_3\right]\Psi(x) = i\hbar\frac{\partial\Psi(x)}{\partial t} \quad . \tag{2}$$

The self-consistent potential ϕ satisfies Poisson's equation

$$\nabla^2\phi(\underset{\sim}{x}) = -4\pi\rho(\underset{\sim}{x}) \quad . \tag{3}$$

The τ_i , i = 1,2,3, are the Pauli spin matrices and $x = (\underset{\sim}{x}, t)$. We assume, as always, the presence of a uniform neutralizing background of charge.

Through a linearized Hartree approximation, these equations yield the following expression for the longitudinal dielectric function:

$$\epsilon(\underset{\sim}{q},\omega) = 1 + \frac{4\pi e^2}{\Omega q^2} \sum_{a,a'} |\langle a| e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} |a'\rangle|^2 \frac{[F(a') - F(a)]}{[\hbar\omega - (E_{a'} - E_a) + i\delta]} \quad , \qquad (4)$$

where $\{a\}$ is the set of quantum numbers specifying the single-particle state, Ω is the volume of the system, E_a is the single-particle energy and

$$\langle a| e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} |a'\rangle = \int \Psi_a^+(x) \tau_3 \, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} \Psi_a(x) \, d\underset{\sim}{x} \quad ,$$

where the $\Psi_a(x)$ are the "positive energy" solutions of eq. (2). F(a) is the ideal Bose-Einstein distribution function. We here consider the case when there is no external field; $\underset{\sim}{A}^{(e)}(\underset{\sim}{x}) = 0$ and the eigenfunctions are those for a free particle [7] :

$$\Psi_0(\underset{\sim}{p}) = \begin{bmatrix} \phi_0(p) \\ \chi_0(p) \end{bmatrix} e^{(i/\hbar)(\underset{\sim}{p}\cdot\underset{\sim}{x} - E_p t)} \quad , \qquad (5)$$

where
$$\left.\begin{aligned} \phi_0 &= (E_p + mc^2)/2(mc^2 E_p)^{1/2} \quad , \\ \chi_0 &= (mc^2 - E_p)/2(mc^2 E_p)^{1/2} \end{aligned}\right\} \qquad (6)$$

and
$$E_p^2 = (cp)^2 + (mc^2)^2 \quad . \qquad (7)$$

At the absolute zero of temperature the ideal Bose-Einstein gas has all particles in the single-particle ground-state. Only two terms, therefore, survive the double sum in eq. (4). Satisfying the requirement $\epsilon(\underset{\sim}{q},\omega) = 0$, we obtain the following exact dispersion relation:

$$(\hbar\omega/E_0)^2 = \tfrac{1}{2}\epsilon^2[1 + (1+Q^2)^{-1/2}] + [1 - (1+Q^2)^{1/2}]^2 \quad , \qquad (8)$$

where $Q = q\lambda_c$, $\epsilon = \hbar\omega_p/E_0$, $E_0 = mc^2$, $\lambda_c = \hbar/mc$ is the Compton wavelength and $\omega_p^2 = 4\pi e^2\rho/m$ is the plasma frequency.

A small-Q expansion yields

$$\hbar\omega = \hbar\omega_p[1 - Q^2/8 + Q^4/8\epsilon^2 + O(Q^4, Q^6/\epsilon^2)] \quad , \qquad (9)$$

when $\epsilon \ll 1$, the low-density regime. This is to be compared with the expression obtained by Foldy [9] for the zero-temperature excitation spectrum of the non-relativistic charged Bose gas,

$$\hbar\omega_1 = \hbar\omega_p[1 + Q^4/8\epsilon^2 + O(Q^8/\epsilon^4)] \quad . \qquad (10)$$

Equation (9) has a dip or local minimum, which phenomenon we refer to as "negative dispersion". This dip is a retardation effect not present in the ultra-non-relativistic ($c = \infty$) result, eq. (10). In fact eq. (9) has its minimum at $Q_M = (\epsilon/\sqrt{2})[1 + O(\epsilon^2)]$. The dip is, then, very close to Q = 0 , so we expect that for temperatures only slightly above zero temperature, the use of eq. (9) will give different thermodynamic behaviour from that obtained when the spectrum eq. (10) is used [10].

Q_M can also be found in the limit of ultra-high density, when $\epsilon \gg 1$, by using the full spectrum in eq. (8). When $\epsilon \gg 1$, $Q_M = (\epsilon/2)^{2/3}[1+O(\epsilon^{-2/3})]$ and the dip occurs over a wide range of Q. This particular density region is, however, somewhat pathological, since physical processes such as pair production, etc., are extremely important at these high densities and these are not taken into account here.

At very low temperatures ($\hbar\omega_p/kT \gg 1$), we can approximate the Bose distribution by the Boltzmann distribution. Then using the spectrum eq. (9), we obtain for the internal energy, E ,

$$E/\Omega \simeq \frac{\hbar\omega_p}{\sqrt{2}\,\pi^2}\left(\frac{m^2\omega_p kT}{\hbar^3}\right)^{3/4} e^{x^2/4}\Big\{\Gamma\left(\tfrac{3}{2}\right)D_{-3/2}(-x) \qquad (11)$$

$$- \frac{\Gamma(5/2)}{4}\left(\frac{m^2\omega_p kT}{\hbar^3}\right)^{1/2}\lambda_c^2\, D_{-5/2}(-x) + \frac{\Gamma(7/2)}{2}\left(\frac{kT}{\hbar\omega_p}\right)D_{-7/2}(-x) + \cdots\Big\}$$

where $D_\nu(z)$ is the parabolic cylinder function [11] and $x^2 = (\epsilon^2/16)(\hbar\omega_p/kT)$.

Even though we are at low temperatures, the result eq. (11) allows for two different asymptotic regions, namely $x \ll 1$ and $x \gg 1$. Expanding the parabolic cylinder functions for the case $x \ll 1$, we obtain

$$E/\Omega \simeq \left[2^{9/4}\pi\,\Gamma(5/4)\right]^{-1}\left(m^3\omega_p^5/\hbar\right)^{1/2}\left(kT/\hbar\omega_p\right)^{3/4} e^{-\hbar\omega_p/kT} , \qquad (12)$$

to leading order. This is exactly the first term given by Fetter [10] in his calculation of the low-temperature behaviour of the thermodynamic functions of the non-relativistic charged Bose gas. The expansion is good in an intermediate region of low temperature and density: the parameters ($kT/\hbar\omega_p$) and ϵ must be small.

The large-argument, $x \gg 1$, expansion of the parabolic cylinder functions yields

$$E/\Omega \simeq \frac{2}{\sqrt{\pi}}\,\alpha\cdot\hbar\omega_p\cdot\rho\cdot e^{-\frac{\hbar\omega_p}{kT}(1-\epsilon^2/32)}\left(kT/\hbar\omega_p\right)^{1/2} , \qquad (13)$$

to leading order, where α is the fine-structure constant. This is a very-low-temperature result ($x \gg 1$, so $\hbar\omega_p/kT \ggg 1$). The "energy gap" in the Boltzmann factor has been shifted slightly as is, of course, to be expected for a spectrum with the "negative dispersion" characterised by eq. (9). Furthermore, the temperature dependence is quite different from that of eq. (12).

For neutron stars [12], $\rho \sim 5\times10^{37}$ cm.$^{-3}$ and $T \sim 10^7$ °K near the centre. The paired protons can indeed be superconducting according to the Schafroth model [2], since at this density the critical temperature is $T_c \sim 10^{11}$ °K. We have $\hbar\omega_p \sim 5\times10^{-6}$ erg. so that $\epsilon \sim 2\times10^{-3}$,$x^2 \sim 10^4/T$ and $(kT/\hbar\omega_p) \sim 3\times10^{-9}\,T$. The small-argument expansion, eq. (12), is applicable.

Correspondingly, for metallic superconductors (paired electrons) $\rho \sim 10^{23}\,\mathrm{cm.}^{-3}$, so $\epsilon \sim 5\times10^{-6}$, $(kT/\hbar\omega_p) \sim 10^{-5}\,T$ and $x^2 \sim 10^{-7}/T$. Thus eq. (12) gives the appropriate result except at extraordinarily low temperatures!

It would be interesting to study eq. (7) at finite temperature to ascertain how the temperature-corrected quasiparticle spectrum modifies the thermodynamic functions [13]. We observe also that there are radiative corrections to the Coulomb potential which should be included in a full relativistic treatment. Although for fermions these corrections come in at $O[(v/c)^2]$, the boson interaction is expected to be corrected first at $O[(v/c)^4]$ [7]. We therefore expect that our main results, eqs. (9) and (14) will not be modified by these relativistic corrections.

In a relativistic charged Fermi gas not subject to external fields [8] there is no dip in the dispersion relation, but in an intense magnetic field [14] "negative dispersion" occurs. We have seen a dip in the Bose gas dispersion relation here in zero field. It would be, therefore, interesting to see what effects an intense magnetic field might introduce in the relativistic charged Bose gas.

All of the above effects are currently being studied and we hope to report on them in the near future.

References

1. S.R. Hore and N.E. Frankel, Phys. Rev. B12 (1975) 2619.
2. M.R. Schafroth, Phys. Rev. 100 (1955) 463.
3. S. Chandrasekhar, Stellar Structure (Dover, New York, 1957).
4. V.L. Ginzburg, J. Stat. Phys. 1 (1969) 3.
5. G.E. Brown and W. Weise, Phys. Repts. 27C (1976) 1 ; A.B. Migdal, Rev. Mod. Phys. 50 (1978) 107.
6. W. Pauli and V.F. Weisskopf, Helv. Phys. Acta 1 (1934) 709.
7. H. Feshbach and F. Villars, Rev. Mod. Phys. 30 (1958) 24.
8. A.E. Delsante and N.E. Frankel, Phys. Lett. 66A (1978) 481.
9. L.L. Foldy, Phys. Rev. 124 (1961) 649.
10. A.L. Fetter, Ann. Phys. (N.Y.) 60 (1970) 464.
11. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products (Academic, New York, 1965).
12. G. Baym and C. Pethick, Ann. Rev. Astron. Astrophys. 14 (1975) 28.
13. S.R. Hore and N.E. Frankel, Phys. Rev. B13 (1976) 2242.
14. A.E. Delsante and N.E. Frankel, Phys. Lett. 67A (1978) 279.