KFKI-1981-16

I. BIAŁYNICKI-BIRULA
E.T. NEWMAN
J. PORTER
J. WINICOUR
B. LUKÁCS
Z. PERJÉS
Á. SEBESTYÉN

A NOTE ON HELICITY

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

KFKI-1981-16

A NOTE ON HELICITY*

I. Białynicki-Birula

University of Warsaw, PL-00-681, Hoża 69, and
University of Pittsburgh, Pittsburgh, Pa. 15260, U.S.A.

E.T. Newman, J. Porter and J. Winicour

University of Pittsburgh, Pittsburgh, Pa. 15260, U.S.A.

B. Lukács, Z. Perjés and Á. Sebestyén

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

HU ISSN 0368 5330
ISBN 963 371 794 9

*Work supported in part by a National Science Foundation cooperation grant.

ABSTRACT

We give a formal definition of the helicity operator for integral spin fields, which does not involve their momentum-space decomposition. We base our discussion upon a representation of the Pauli-Lubanski operator in terms of the action on tensor fields by the Killing vectors associated with the generators of the Poincare group. This leads to an identification of the helicity operator with the duality operator defined by the space-time alternating tensor. Helicity eigenstates then correspond to self-dual or anti-self-dual fields, in agreement with usage implicit in the literature. In addition, we discuss the relationship between helicity eigenstates, which are intrinsically non-classical, and states of right or left circular polarization in classical electrodynamics.

АННОТАЦИЯ

Дается определение оператора спиральности для полей с целыми спинами без употребления конкретной формы генераторов группы Лоренца. Образуется оператор Паули-Лубаньски с помощью генераторов группы Пуанкаре. В результате этого оператор спиральности идентифицируется с помощью оператора дуальности в пространстве-времени, таким образом, собственные состояния являются само-дуальными и антисамодуальными полями.В классической электродинамике трактуется и связь собственных состояний с циркулярной поляризацией.

KIVONAT

Egész spinű terekre olyan definiciót adunk a helicitás-operátorra, amely nem használja az impulzustérbeli felbontásukat. A Pauli-Lubanski operátort a Poincaré-csoport generátor Killing vektoraival ábrázoljuk. Ennek eredményeként a helicitásoperátort a téridő-dualitás operátorával azonositjuk. A helicitás-sajátállapotok ekkor önduált és anti-önduált tereknek felelnek meg. Tárgyaljuk továbbá a helicitás-sajátállapotok és a jobb- illetőleg bal körkörös polarizáció kapcsolatát a klasszikus elektrodinamikában.

I. INTRODUCTION

Helicity, which is defined as the projection of the total angular momentum on the direction of motion of the particle, is usually discussed in the context of particle theory and almost always with the use of momentum eigenstates.[1,2,3,4] In this note we shall extend the notion of helicity to classical fields, such as the Maxwell field or the linearized gravitational field, and shall furthermore show that the helicity operator can be represented as a single algebraic (local) operator on these fields, which is completely independent on the decomposition into momentum states. In the case of classical non-Abelian gauge theory or general relativity, where the generators of the Poincare group seem to be poorly defined, one can still define the helicity operator[5] as a natural generalization from the Maxwell case.

In Section II of this note we discuss, from the point of view of Lie derivatives. the action of the Pauli-Lubanski[3,6] vector on Maxwell and linearized gravitational fields and we discover that the helicity operator is essentially the (Hodge) duality operator. In Section III we discuss the relationship of helicity states with polarization states.

II. THE PAULI-LUBANSKI VECTOR, HELICITY AND DUALITY

We begin with listing the ten (Hermitian) generators of the Poincare group $P_{\underline{a}}$ and $M_{\underline{ab}} = - M_{\underline{ba}}$ and their commutat[illegible] relations. These generators can be represented as ten Killing vector [illegible]s on Minkowski space-time

$$P_{\underline{a}} = i p^a_{\underline{a}} \frac{\partial}{\partial x^a} ,$$
$$M_{\underline{ab}} = i m^{ab}_{\underline{ab}} \eta_{ac} x^c \frac{\partial}{\partial x^b} , \tag{2.1}$$

with

$$p^a_{\underline{a}} = \delta^a_{\underline{a}}$$
$$m^{ab}_{\underline{ab}} = \delta^a_{\underline{a}} \delta^b_{\underline{b}} \quad \delta^b_{\underline{a}} \delta^a_{\underline{b}} \tag{2.2}$$

and $\eta_{ab} = \text{diag}\,(1, -1, -1, -1) = \eta_{\underline{ab}}$. Note that underlined letters label vectors while the non-underlined letters label components.

The commutators are defined as the Lie brackets between the Killing vectors

$$\frac{1}{i}\,[P_{\underline{a}}, P_{\underline{b}}] = 0,$$

$$\frac{1}{i}\,[M_{\underline{ab}}, P_{\underline{c}}] = P_{\underline{a}}\,\eta_{\underline{bc}} - P_{\underline{b}}\,\eta_{\underline{ac}}, \tag{2.3}$$

$$\frac{1}{i}\,[M_{\underline{ab}}, M_{\underline{cd}}] = M_{\underline{ad}}\,\eta_{\underline{bc}} - M_{\underline{bd}}\,\eta_{\underline{ac}} + M_{\underline{bc}}\,\eta_{\underline{ad}} - M_{\underline{ac}}\,\eta_{\underline{bd}}.$$

The action of a Poincaré generator on a tensor field is defined as the Lie derivative of the tensor field by the associated Killing vector. As an example, the Lie derivative of the Maxwell field by ξ^a is

$$\mathcal{L}_\xi\, F_{ab} \equiv \xi^c\,\nabla_c\, F_{ab} + F_{ac}\,\nabla_b\,\xi^c + F_{cb}\,\nabla_a\,\xi^c. \tag{2.4}$$

For classical zero rest-mass particles the Pauli-Lubanski vector, defined as

$$S^a = 1/2\ \varepsilon^{abcd}\,P_b\,M_{cd}, \tag{2.5}$$

where ε^{abcd} is the alternating tensor ($\varepsilon^{0123} = -1$) and P_b and M_{cd} are respectively the particle four-momentum and angular momentum, can be shown[6] to be proportional to the particle momentum P^a. The particle helicity S is defined as the proportionality coefficient:

$$S^a = SP^a. \tag{2.6}$$

With the motivation to find the analog of S for fields, we define the Pauli-Lubanski operator for tensor fields:

$$S^{\underline{a}} = 1/2\ \varepsilon^{\underline{abcd}}\,P_{\underline{b}}\,M_{\underline{cd}} \tag{2.7}$$

$$= 1/2\ \varepsilon^{\underline{abcd}}\,\mathcal{L}_{P_{\underline{b}}}\,\mathcal{L}_{M_{\underline{cd}}} \quad . \tag{2.8}$$

Note that $S^{\underline{a}}$ is independent of the order in which the two Lie derivatives appear as well as the choice of origin for $M_{\underline{cd}}$. We shall now apply this general definition to the Maxwell field. We have

$$\mathcal{L}_{P_{\underline{b}}}\,F_{ab} = i\nabla_{\underline{b}}\,F_{ab} \tag{2.9}$$

$$\mathcal{L}_{M_{\underline{cd}}} \mathcal{L}_{P_{\underline{b}}} F_{ab} = - m^{cd}_{\underline{cd}} \eta_{cf} x^f \nabla_d \nabla_{\underline{b}} F_{ab}$$

$$- m^{cd}_{\underline{cd}} \eta_{db} \nabla_{\underline{b}} F_{ac} - m^{cd}_{\underline{cd}} \eta_{da} \nabla_{\underline{b}} F_{cb}. \quad (2.10)$$

It follows from the definition of $m^{ab}_{\underline{ab}}$ that the first term in (2.10) is symmetric in $\underline{bc}$ and $\underline{bd}$ and thus vanishes when contracted with $\varepsilon^{\underline{abcd}}$. Hence

$$S^{\underline{a}} F_{ab} = - 1/2 \, \varepsilon^{\underline{abcd}} m^{cd}_{\underline{cd}} (\eta_{db} \nabla_{\underline{b}} F_{ac} + \eta_{da} \nabla_{\underline{b}} F_{cb})$$

$$= -\varepsilon^{\underline{abcd}} (\eta_{db} \nabla_{\underline{b}} F_{ac} + \eta_{da} \nabla_{\underline{b}} F_{cb}). \quad (2.11)$$

Using $F^{*ab} = 1/2 \, \varepsilon^{abcd} F_{cd}$, $j_b = \nabla_a F^a_b$ and $j^*_b = \nabla_a F^{*a}{}_b$, we obtain

$$S_{\underline{a}} F_{ab} = \eta_{\underline{ab}} j^*_a - \eta_{aa} j^*_b - \varepsilon_{\underline{a}abc} j^c - \nabla_{\underline{a}} F^*_{ab} \quad . \quad (2.12)$$

If the free Maxwell equations are satisfied, i.e.

$$j_a = 0 = j^*_a,$$

then

$$S_{\underline{a}} F_{ab} = i \, P_{\underline{a}} F^*_{ab} = i \, D \, P_{\underline{a}} F_{ab} \quad (2.13)$$

where D is the duality operator

$$(DF_{ab}) \equiv 1/2 \, \varepsilon_{abcd} F^{cd}. \quad (2.14)$$

Comparing (2.14) with (2.6) we see that we can take the helicity operator $\hat{S}$ as

$$\hat{S} = i \, D. \quad (2.15)$$

If we had done this calculation with the Weyl tensor satisfying the linearized Einstein equations, the results would have been

$$S_{\underline{a}} C_{abcd} = i \, D \, P_{\underline{a}} C_{abcd} \quad (2.16)$$

with $DC_{abcd} \equiv 1/2 \, \varepsilon_{abef} C^{ef}{}_{cd}$.

In general, one would have

$$\hat{S} = |S| \, i \, D, \quad (2.17)$$

with $|S|$ an integer, the spin of the field.

Note that in order to obtain the operator equation

$$S_{\underline{a}} = \hat{S}\, P_{\underline{a}}$$

we needed to impose the field equations on the vector space of spin $|S|$ fields. However, the helicity operator $\hat{S}$ is a purely algebraic operation and, therefore, we can define it on the Weyl tensor for the full Einstein theory or for non-Abelian gauge fields.

III. MEANING OF HELICITY EIGENSTATES

Returning to Maxwell fields we now examine helicity eigenstates. They are respectively the self-dual and anti-self dual fields

$$F^{*}_{+ab} = - i\, F_{+ab}$$
$$F^{*}_{-ab} = i\, F_{-ab}. \tag{3.1}$$

Unfortunately, aside from the zero field, they are never real and hence do not correspond to a classical Maxwell field. Nevertheless a classical "meaning" can still be assigned to (3.1). If we are given a real Maxwell field, it can be uniquely decomposed into the self and anti-self dual parts

$$F_{ab} = F_{+ab} + F_{-ab}$$
$$F_{+ab} = 1/2(F_{ab} + i\, F^{*}_{ab}),\ F_{-ab} = 1/2(F_{ab} - i\, F^{*}_{ab})\ . \tag{3.2}$$

Each part, it being the complex conjugate of the other, carries the full information about the original field. If however we ask for the positive frequency parts of the self and anti-self dual fields, they are independent and carry the full information of respectively the left and right circularly polarized parts of the original field.

To see this note that dualing and extracting the positive frequency parts are commuting linear operations and we can hence look at a single frequency plane wave. For a plane elliptically polarized wave traveling along the x-axis we have[7,8]

$$\vec{E} = \mathrm{Re}\ \vec{b}\ e^{i\theta}$$
$$\vec{B} = \vec{i} \times \vec{E} \tag{3.3}$$

with $\vec{i}$, $\vec{j}$, $\vec{k}$ unit vectors in the x, y, z directions and

$$\vec{b} = b_1\vec{j} + b_2\vec{k};\ \theta = \vec{k} - wt \cdot \alpha \quad .$$

This can be decomposed into right and left circularly polarized parts by

$$\vec{E} = \mathrm{Re}\{b_L(\vec{j} + i\vec{k}) + b_R(\vec{j} - i\vec{k})\}e^{i\theta}$$

with

$$b_1 = b_L + b_R,\ b_2 = b_L - b_R.$$

It is now straightforward to show that

$$\begin{aligned} \vec{E} + i\vec{B} &= b_L[\vec{j} - i\vec{k}]e^{-i\theta} + b_R[\vec{j} - i\vec{k}]e^{i\theta} \\ \vec{E} - i\vec{B} &= b_L[\vec{j} + i\vec{k}]e^{i\theta} + b_R[\vec{j} + i\vec{k}]e^{-i\theta} \quad . \end{aligned} \tag{3.4}$$

We thus have that the positive frequency part of $\vec{E} + i\vec{B}$ and $\vec{E} - i\vec{B}$ are

$$\begin{aligned} (\vec{E} + i\vec{B})^{(+)} &= b_R[\vec{j} - i\vec{k}]e^{i\theta} \\ (\vec{E} - i\vec{B})^{(+)} &= b_L[\vec{j} + i\vec{k}]e^{-i\theta} \end{aligned} \tag{3.5}$$

representing the left and right circularly polarized fields. Since $\vec{E} - i\vec{B}$ and $\vec{E} + i\vec{B}$ are equivalent[9] respectively to the self and anti-self dual parts of F_{ab} we have proved our contention.

In addition to this classical "meaning", the positive-frequency self and anti-self dual fields are, from a quantum mechanical viewpoint, the one-particle wave functions for positive and negative helicity eigenstates. In contrast to the real-valued classical fields, these wave functions are necessarily complex, with $F_{\pm ab}$ having the form (3.2), where F_{ab} is real. This complex nature of $F_{\pm ab}$ combines with the axial-tensor property of ϵ_{abcd} to yield the well-known mapping of positive helicity wave-functions into negative ones by a parity transformation. For instance,

$$PF_{+ab} = 1/2\{PF_{ab} - i(PF_{ab})^{*}\}$$

where PF_{ab} is the image of F_{ab} under a parity transformation. Thus PF_{+ab} has negative helicity, in accordance with (3.2).

VI. DISCUSSION

In this note we have formalized the generally accepted (but no place to our knowledge explicitly stated) idea that helicity states of the Maxwell field are eigenstates of the duality operator. In the process of doing this we saw that the helicity operator could easily be extended to non-linear gauge fields and general relativity. A problem arises here in the interpretation of the classical helicity eigenstates - they cannot be combined with their complex conjugates (as in 3.5) to produce real solutions of the field equations. Only in the asymptotic domain (where the fields become weak) can this be done. It seems reasonable to try to interpret[9,10] these solutions as the one particle quantum mechanical wave functions or, asymptotically, as classical circular-polarization states.

We conclude with a brief summary of the conventions used here. They have been adopted to coincide with those used by many workers in general relativity.

1) The Lorentz metric is diag (+---).

2) The alternating tensor ϵ_{abcd} has $\epsilon_{0123}=1$, $\epsilon^{0123} = -1$

3) The duality operation is $F^{*}_{ab} = 1/2\ \epsilon_{abcd}\ F^{cd}$ and the decomposition of F into respectively self and anti-self dual parts is

$$F_{+ab} = 1/2(F_{ab} + iF^{*}_{ab})$$

$$F_{-ab} = 1/2(F_{ab} - iF^{*}_{ab})$$

from which

$$F^{*}_{+ab} = -\ iF_{+ab}$$

$$F^{*}_{-ab} = iF_{-ab}.$$

4) For spin $|S|$ fields the eigenvalues of the helicity operator $\hat{S} = iD|S|$ are $|S|$ for self-dual fields and $-|S|$ for the anti-self dual fields.

5) The electric vector of a right circularly polarized wave approaching an observer will be seen to rotate clock-wise.

6) Note that the traditional[8] definition of right (left) circular polarization coincides with negative (positive) helicity.

7) Anti-self dual fields correspond to unprimed spinor fields and self dual fields correspond to primed spinor fields. Furthermore one frequently encounters the terminology of self-dual fields being called left-flat and anti-self dual fields being called right-flat.

REFERENCES

[1] William W. Frazer, Elementary Particles, Prentice Hall, 1966

[2] Stephen Gasiorowicz, Elementary Particle Physics, John Wiley & Sons, 1966

[3] Paul Roman, Introduction to Quantum Field Theory, John Wiley & Sons, 1969

[4] L. Landau and E. Lifshitz, Course of Theoretical Physics Vol. 4; V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii: Relativistic Quantum Theory, Pergamon, 1971

[5] For another approach to this question see D. J. Duff and C. J. Isham, Nuclear Physics, B162 (1980) 271.

[6] R. Penrose and M.A.H. MacCallum, Physics Letters (section C) 6C # 4 (1973) 242.

[7] L. Landau and E. Lifshitz, Classical Theory of Fields, Addison-Wesley (1951)

[8] J.D. Jackson, Classical Electrodynamics, John Wiley & Sons (1962)

[9] If $F^{ab} \Longleftrightarrow F^{oi} = E_i$, $F^{12} = B_3$, $F^{13} = -B_2$, $F^{23} = B_1$ then $F_{oi} = -E_i$ and $F^*_{oi} = 1/2\epsilon_{oijk} F^{jk} = B_i$ and hence $F_{oi} + i F^*_{oi} = -(E_i - i B_i)$, etc.

[10] R. Penrose, Gen. Rel. Grav. 7, 31 (1976)

[11] M. Ko, M. Ludvigsen, E.T. Newman and K.P. Tod, The Theory of H-Space, to be published in Phys. Rep.

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Borbély Imre
Nyelvi lektor: Doleschall Pál
Gépelte: Balczer Györgyné
Példányszám: 450 Törzsszám: 81-162
Készült a KFKI sokszorosító üzemében
Felelős vezető: Nagy Károly
Budapest, 1981. március hó