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ASSOCIATION EURATOM - C.E.A.

92260 Fontenay-aux-Roses (France)

DEPARTEMENT DE PHYSIQUE DU PLASMA
ET DE LA FUSION CONTROLEE

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THREE DIMENSIONAL TRANSPORT MODEL FOR TOROIDAL PLASMAS

C. COPENHAUER

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C. Copenhauer

ASSOCIATION EURATOM-CEA SUR LA FUSION
Département de Recherches sur la Fusion Contrôlée
Centre d'Etudes Nucléaires
Boite Postale n° 6, 92260 FONTENAY-AUX-ROSES (FRANCE)

ABSTRACT

A nonlinear MHD model, developed for three-dimensional toroidal geometries (asymmetric) and for high β ($\beta \sim \varepsilon$), is used as a basis for a three-dimensional transport model. Since inertia terms are needed in describing evolving magnetic islands, the model can calculate transport, both in the transient phase before nonlinear saturation of magnetic islands and afterwards on the resistive time scale. In the $\beta \sim \varepsilon$ ordering, the plasma does not have sufficient energy to compress the parallel magnetic field, which allows the Alfven wave to be eliminated in the reduced nonlinear equations, and the model then follows the slower time scales. The resulting perpendicular and parallel plasma drift velocities can be identified with those of guiding center theory.

I. INTRODUCTION

A principle object of the controlled fusion program is the containment of a thermonuclear plasma in toroidal devices by means of magnetic fields. With tokamak confinement devices, the confinement problems, although still present, have been reduced in importance and interest has turned to plasma transport and the development of fusion reactor prototypes. With compact toruses, the plasma confinement and transport are intimately related, and require a time scale much faster than the resistive time scale. This faster time scale is also required to represent the development and saturation of magnetic islands in tokamaks.

In present and near future experiments, the plasma can be considered to be in an evolving quasi-equilibrium. Presently, experimental pulse lengths do not exceed one second and the magnetic configuration and plasma flow are still evolving.

Experimentally, it is easiest to ramp the plasma density on sawtooth (m=1) oscillations. Thus, it is reasonable to consider these oscillations as part of the quasi-equilibrium. Further, the m=1 oscillatory flow is typically 10^4 times the residual time-averaged neoclassical resistive flow. Thus, the development of m=1 islands, and magnetic islands in general, requires that the inertia of the fluid and the velocity field be included explicitly. Therefore, the transport studies of present and future toroidal experiments requires going to a time scale faster than the resistive time scale. Also, since magnetic islands are asymmetric, the transport theory of toroidal plasma devices needs to be fully three-dimensional, rather than axisymmetric. Further, non-axisymmetric magnetic fields are produced by many external sources including bundle divertors, ripple coils, helical coils and discrete field coils.

This paper is concerned with the proposed development of a three-dimensional transport model. The model here is not a three-dimensional extension of existing two-dimensional models, but uses a different approach and incorporates a faster time scale than previous models.

In the tokamak model described here, the fast toroidal Alfven wave is removed and the model is structured to follow the convection time scale, which in intermediate between the resistive time scale and the toroidal Alfven time scale, but closer to the latter. The three velocities : plasma flow, sound wave and poloidal Alfven wave are all of the same magnitude, and this is what is meant by the convection time scale.

The suggested transport model presented later is essentially a one-fluid model, except for the energy equations, where the separate electron and ion equations result from the reduced two-fluid model. Separate specie energy equations allow for different electron and ion temperatures, and thermal conductivities, in the nonlinear part. Presently, experimental electron thermal conductivities are 100 times neoclassical. In addition, the electron, ion and magnetic field velocities, although directly related, are now separately calculated. The electron, ion and magnetic field velocities are coupled by the ($\beta \sim \epsilon$) ordering, and the requirements of charge conservation and ambipolar diffusion.

The nonlinear toroidal one-fluid model presented is similar to the original toroidal model developed by Potter et al [1,2], and is even closer to the recent model of Edery et al [3,4], with the exception of the inclusion of diamagnetic effects and parallel velocity here. The principal difference between the original model and the recent model is the numerical method of solution.

In the usual transport model [5,6], the equations for the evolution of the mass, the internal energy, and the poloidal flux (or toroidal flux) specify the source terms in the equilibrium equation. In turn, the solution of the equilibrium equation specifies the geometric variables, which define the integrals in the diffusion coefficients.

In the transport model here, in the nonlinear part, the equations for the evolution of the mass, internal energy and poloidal flux are solved together with the equation for the evolution of vorticity, which takes the place of the equilibrium equation. The consistent solution of the set of nonlinear equations specifies the three-dimensional flux surfaces. The particle and energy diffusion are then obtained using flux surface integration of integrands which include the differences in velocities that are locally calculated. The fact that the flow velocities here are consistent with those of guiding center theory allows stress tensor terms, and thus neoclassical diffusion, to be included in a self-consistent way.

The reasons for three-dimensional transport have already been partly stated, but are summarized here :

(i) The incorporation of magnetic islands results in asymmetry and thus requires three-dimensional space. Magnetic islands change transport and result in anomalous transport. For example, the m=1, n=1 islands show a periodic behaviour with expulsion of plasma from the core. The m=2, n=1 island oscillations stabilize nonlinearly and create finite amplitude asymmetric "helical" field structures. Particles can move rapidly across these saturated island regions much more quickly than by standard axisymmetric neoclassical diffusion.

(ii) Transport studies of asymmetric geometries in general are possible, which include Doublet-shaped and D-shaped magnetic field configurations. Bundle divertors, other asymmetric divertors, and magnetic field ripple transport can be studied.

(iii) Three-dimensional transport, in particular magnetic isl. ds, requires the incorporation of the convective time scale, which allows neutral beam heating, adiabatic compression and rapid heating of flux conserving tokamak equilibria to be realistically evaluated. The quantitative study of transport on a realistic time scale in present and future experiments is of particular interest.

(iv) The three-dimensional geometry of the flux surfaces modifies the values of the plasma transport coefficients. Correspondingly, the three components of velocity are coupled and effect plasma transport directly, and indirectly, since the saturated flux surfaces are dependent on the transient velocity structure.

(v) The asymmetric nature of magnetic islands in toroidal devices requires three-dimensional geometry and, for example, toroidal geometry rather than helical. It will develop that in helical geometry the plasma can be considered as incompressible, whereas, in toroidal geometry this is not the case. Since, the geometry is asymmetric, the gradient operators along the field lines play an important role.

(vi) The proposed transport code would be useful for stability calculations. For example, a realistic study of the disruptive instability requires the use of a quasi-equilibrium which should include saturated magnetic islands.

(vii) Three-dimensional geometry is needed for a great variety of reasons. For example, anisotropic pressure distribution drives turbulence, but axisymmetric codes do not permit asymmetric modes, so that these codes underestimate flux annihilation.

In summary, the transport model presented here principally differs from the usual transport model in that :

(i) The nonlinear one-fluid model replaces the usual Grad-Shafranov equilibrium equation model in obtaining the evolving magnetic flux surfaces.

(ii) The convection time scale is used instead of the much slower resistive time scale allowing transport to be calculated in the transient phase.

The analysis proceeds as follows : In Sec.II, we give the nonlinear two-fluid model and discuss the toroidal incompressibility condition, and how the toroidal Alfven wave is eliminated. The equations for the perpendicular and parallel components of the electron, ion and magnetic field velocities are given. Their relationship to guiding center velocities is then shown. In Sec.III, the nonlinear MHD equations are derived from the two-fluid equations in the one-fluid limit. The nonlinear MHD model is discussed and several reasons for retaining diamagnetic effects and parallel velocity are given. Then, it is shown that the nonlinear MHD toroidal model here reduces to the three-dimensional high beta ($\beta \sim \epsilon$) nonlinear MHD helical model [7] in the large aspect ratio limit. Next, the comparison of the nonlinear toroidal MHD model here with the compact torus model [8] is made. This comparison is of great interest, not only because it shows the general applicability of the model here, but also because the differences in the models illustrate some of the fundamental physical, numerical and mathematical points. The three-dimensional transport model is given in Sec.IV. Included is a discussion of the differences between the three-basic transport models : (1) standard transport model ; (2) time-differentiated transport model ; and (3) the reduced nonlinear MHD transport model presented here. In Sec.V, a summary of the principal results is given.

II. NONLINEAR TWO-FLUID MODEL

A. Basic equations

The two-fluid equations treat the ions and electrons each as conducting fluids that are coupled through momentum transfer equations and by Maxwell's equations. The basic equations used here are :

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla . (\rho_\alpha \vec{V}_\alpha) = 0 \qquad (2.01)$$

$$\rho_\alpha \frac{d\vec{V}_\alpha}{dt} + n_\alpha m_\alpha (\vec{V}_\alpha - \vec{V}_\beta) \nu_{\alpha\beta} = n_\alpha q_\alpha (\vec{E} + \vec{V}_\alpha \times \vec{B}) - \nabla . \overleftrightarrow{P}_\alpha \qquad (2.02)$$

$$\frac{\partial p_\alpha}{\partial t} + \gamma \nabla . (p_\alpha \vec{V}_\alpha) = Z_\alpha \qquad (2.03)$$

$$Z_i = (\gamma - 1) \left[\nabla . (\overleftrightarrow{K}_i \nabla T_i) + Q_{ei} + S_i\right] \qquad (2.04)$$

$$Z_e = (\gamma - 1) \left[\nabla . (\overleftrightarrow{K}_e \nabla T_e) + \eta J^2 - Q_{ei} + S_e\right] \qquad (2.05)$$

$$\nabla \times \vec{B} = n_i e \vec{V}_i - n_e e \vec{V}_e = \vec{J} \qquad (2.06)$$

$$\frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E} \qquad (2.07)$$

$$\nabla . \vec{B} = 0 \qquad (2.08)$$

Here $\vec{V}$, $\vec{E}$, $\vec{B}$ and $\vec{J}$ denote, respectively, the drift velocity of the particle, the electric field, the magnetic induction and current density vectors, $\overleftrightarrow{P}$ the pressure tensor, $\overleftrightarrow{K}$ the thermal conductivity tensor.

ρ the density ($\rho_\alpha = n_\alpha m_\alpha$), q the charge, γ the gas constant (taken to be 5/3) and η the resistivity. In addition, Q_{ei} is the electron-ion thermalization, and S_i and S_e are momentum sources.

There are clearly inconsistencies here, and these can be related to differences in Chew-Goldberger-Low (CGL), guiding center theory (GCT) and magnetohydrodynamics (MHD). The relation between MHD and CGL is well known ; the relation between CGL and GCT is less well known. Here, because drift velocities are important, the standard derivation, that the drift currents in CGL and GCT are equivalent, is included. The MHD toroidal incompressibility condition, that the plasma cannot compress the parallel magnetic field giving a condition on the perpendicular drift velocity, is here related to guiding center theory. The equations used in the full transport model are a composite of CGL and MHD and as such are not rigorously valid in either. CGL theory requires negligible collisional effects and dropping the heat flow tensor, both of which are included here. In order to obtain neoclassical transport and trapped particle effects in MHD requires including the pressure tensor as in CGL theory. Strong magnetic fields are assumed throughout. It should also be remarked that here $\frac{\partial \vec{E}}{\partial t} = 0$ and $\rho_q = 0$; the latter implies charge conservation.

Now, the components of the electron and ion velocities in the poloidal plane can be obtained from Eq. (2.02) x $\nabla \phi$. Here ϕ is in the toroidal direction with physical variables depending on ϕ, and $|\nabla \phi| = R^{-1}$. First, the collision terms are replaced by the linear approximation

$$n_\alpha m_\alpha (\vec{V}_\alpha - \vec{V}_\beta) \nu_{\alpha\beta} = \eta \vec{J} \qquad (2.09)$$

Then, the velocity components become

$$\vec{V}_i = V_{i\phi} \frac{\vec{B}}{B_\phi} + \frac{R^2}{f} \left[\vec{E} - \frac{\nabla \cdot \overleftrightarrow{P}_i}{ne} - \eta \vec{J} \right] \times \nabla\phi \qquad (2.10)$$

$$\vec{V}_e = V_{e\phi} \frac{\vec{B}}{B_\phi} + \frac{R^2}{f} \left[\vec{E} + \frac{\nabla \cdot \overleftrightarrow{P}_e}{ne} - \eta \vec{J} \right] \times \nabla\phi \qquad (2.11)$$

where $f = RB_\phi$ is the toroidal flux and R is the major radius of the torus. The inertia terms could have been retained, however, they are an order $\in$ lower than the first three terms on the right hand side of Eqs.(2.10) and (2.11), so that they are neglected. The $\eta \vec{J}$ term is of order $\in^3$ and is retained because it represents resistive diffusion. Now, let us consider the component of the momentum equation in the ϕ direction

$$\rho_\alpha \frac{d\vec{V}_{\alpha\phi}}{dt} = n_\alpha q_\alpha \vec{E}_\phi - (\nabla \cdot \overleftrightarrow{P}_\alpha)_\phi - n_\alpha q_\alpha \eta \vec{J}_\phi \qquad (2.12)$$

The $(\vec{V}_\alpha \times \vec{B})_\phi$ term would be identically zero if $\vec{B}$ was in the ϕ direction. Here it is small, but cannot be neglected. Now because the gradients in the parallel direction are small, it is necessary to retain the inertia terms. Indeed, for $V_{\alpha\phi} \sim V_{\alpha p} \sim \in$, the ϕ inertia terms dominate and $\frac{d\vec{V}_{\alpha\phi}}{dt} = 0$.

The basic relations for the perpendicular ion and cyclotron velocities are rederived in a different way in order to show that the velocity associated with the movement of the magnetic field follows directly from the gradient of the electrostatic potential. The two-fluid version of Ohm's law

$$\vec{E} + \vec{V}_i \times \vec{B} - \frac{\nabla \cdot \overleftrightarrow{P}_i}{ne} = \eta \vec{J} \qquad (2.13)$$

$$\vec{E} + \vec{V}_e \times \vec{B} + \frac{\nabla \cdot \vec{P}_e}{n e} = \eta \vec{J} \tag{2.14}$$

is used. These equations were, in effect, derived in Eqs.(2.10) and (2.11). Substituting the ion equation, Eq.(2.13) in the ϕ component of $\frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E}$ gives

$$\frac{\partial f}{\partial t} = - R^2 \nabla \cdot \left[f \left(\frac{\vec{V}_i}{R^2} - \frac{V_{i\phi}}{R^2} \frac{\vec{B}}{B_\phi} + \frac{\nabla \cdot \vec{P}_i \times \nabla \phi}{fne} + \eta \frac{\vec{J} \times \nabla \phi}{f} \right) \right] \tag{2.15}$$

If, instead of using Eq.(2.13) or Eq.(2.14) for $\vec{E}$, the equation

$$\vec{E} = \nabla U - \frac{\partial \vec{A}}{\partial t} \tag{2.16}$$

is used, where U is the electrostatic potential and $\vec{A}$ is the vector potential, then

$$\frac{\partial f}{\partial t} + R^2 \nabla \cdot \left(\frac{f\, u_p}{R^2} \right) = 0 \tag{2.17}$$

where

$$\vec{u}_p = R^2 \frac{(\vec{E} \times \nabla \phi)}{f} \tag{2.18}$$

Since the $\frac{\partial \vec{A}}{\partial t}$ part of $\vec{E}$ results in $\frac{\partial f}{\partial t}$ and the ∇U part contributes nothing. In general, if $\vec{u}_p = \frac{\vec{E} \times \vec{B}}{B^2}$

then

$$\frac{\partial}{\partial t} (h_\parallel B_\parallel) + h_\parallel{}^2 \nabla \cdot \left(\frac{B_\parallel \vec{u}_\perp}{h_\parallel} \right) = 0 \tag{2.19}$$

where $h_\parallel$ is the scale factor in the parallel direction and depends on the metric used. Assuming the dominant B field is in the ϕ direction, Eq.(2.17) is again obtained.

In Eq.(2.15), $\frac{\partial f}{\partial t}$ is of order ϵ^3, whereas the terms inside the square bracket are of order ϵ, so that, the electrostatic part of $\vec{u}_p$ can be identified with the latter. Thus, Eqs.(2.10) and (2.11) result with $\vec{E}$ replaced by ∇U. Equation (2.10) written in terms of the $\vec{B}$ field, becomes

$$\vec{V}_1 = V_{i\phi} \frac{\vec{B}}{B_\phi} + \left(\nabla U - \frac{\nabla \cdot \vec{P}_i}{ne} - n \vec{J} \right) \times \frac{\vec{B}}{B^2} \qquad (2.20)$$

and is well known in transport theory. In effect, the time derivative terms, inertia and $\frac{\partial \vec{A}}{\partial t}$, can be neglected in the perpendicular drift velocities. Thus, in the expression for $\vec{u}_p$, Eq.(2.18), $\vec{E}$ is replaced by ∇U, and becomes

$$\vec{u}_p \cong R^2 \frac{(\nabla U \times \nabla \phi)}{f} \qquad (2.21)$$

Since $\vec{u}_p$ has already been identified with the movement of the magnetic field, Eq.(2.17) can be written as

$$\frac{df}{dt} = - R^2 f \nabla \cdot \left(\frac{\vec{u}_p}{R^2} \right) \cong 0 \qquad (2.22)$$

The two-fluid generalization of "toroidal incompressibility", initially postulated by Potter et al[1,2], is now represented by Eqs.(2.21), (2.22), (2.10) and (2.11). They postulated that

$$\delta f \ll f \quad \text{so that} \quad \frac{1}{f} \frac{df}{dt} = 0 \qquad \text{in tokamaks} \qquad (2.23)$$

The generalization here, because of the inclusion of the additional terms, is less restrictive and more instructive.

In nonlinear MHD applications, the retention of the R^2 term, or its neglect, depends largely on the assumptions with respect to the metric used. In particular, for $R = R_0 + r\cos\Theta$ and for f constant, if the $r\cos\Theta$ term is neglected, then one obtains

$$\nabla \cdot \vec{u}_p = 0 \tag{2.24}$$

or that the plasma is incompressible ; if the $r\cos\Theta$ term is retained, then

$$\nabla \cdot \frac{\vec{u}_p}{R^2} = 0 \tag{2.25}$$

The nonlinear MHD toroidal papers that retain R^2, and those that do not, are divided. The factor R^2 introduces important toroidal effects which include mode coupling. In helical geometry, phenomena such as nonlinear coupling of modes of different helicity are not possible. Equation (2.25) results in greater asymmetry of magnetic islands than for example just including toroidal effects in the current. With R^2 in Eq.(2.25), the plasma is not incompressible, so that fast compressible wave motion must be included.

Equations (2.10) and (2.11) can be written in a more familiar form in terms of the parallel and perpendicular magnetic fields. For example, taking the $\vec{B}$ field in the ϕ direction gives

$$\vec{V}_i = V_{i\parallel}\frac{\vec{B}}{B} + \left[\nabla U - \frac{\nabla \cdot \overleftrightarrow{P}_i}{ne} - \eta\vec{J}\right] \times \frac{\vec{B}}{B^2} \tag{2.26}$$

$$\vec{V}_e = V_{e\parallel}\frac{\vec{B}}{B} + \left[\nabla U + \frac{\nabla \cdot \overleftrightarrow{P}_e}{ne} - \eta\vec{J}\right] \times \frac{\vec{B}}{B^2} \tag{2.27}$$

The velocity of the magnetic field $\vec{u}$ differs from either the drift velocity of the ions or that of the electrons. As has been observed by Spitzer [9] (for the moment let us consider the pressure as a scalar) the velocity induced by ∇p_i tends to be offset by ∇U , so that v_i is small.

Then, taking ∇p_e as small, the electrons tend to move with the magnetic field and at a poloidal drift frequency $\sim \frac{-\nabla p_i}{n\, er}$, equal to the negative of the conventional drift frequency. This suggests that neglecting dimagnetic effects may result in an overestimate of plasma velocity and perhaps magnetic field velocity.

The current density obtained from the two-fluid drift velocities using the stress tensor $\overleftrightarrow{P}$ expansion is identical with $J_\perp$ obtained from guiding center theory, as will be shown. This suggests that it should be possible to use guiding center theory to construct the parallel and perpendicular drift velocities required in the two-fluid model. The combination of the macroparticle guiding center theory and MHD theory is not new. In particular, macroparticle guiding center velocities have been used numerically [10] to compute the current density in the Grad-Shafranov equation. The difference here is that the vorticity equation takes the place of the Grad-Shafranov equation, which has the advantage that the electrostatic potential is calculated self-consistently.

In order to demonstrate the correspondance between the guiding center velocities and the two-fluid velocities, expressions for the former are included here. The transverse drift velocity from guiding center theory [11] is

$$\vec{V}_{\alpha\perp} = \frac{\vec{E} \times \vec{B}}{B^2} + W_{\alpha\parallel}{}^2 \frac{m_\alpha}{q_\alpha} \frac{\vec{B} \times (B\cdot\nabla)\vec{B}}{B^4} + \frac{W_{\alpha\perp}{}^2}{4} \frac{m_\alpha}{q_\alpha} \frac{\vec{B} \times \nabla B^2}{B^4} - \eta \frac{\vec{J} \times \vec{B}}{B^2} \qquad (2.28)$$

where W represents the gyrovelocity and the η term represents resistive diffusion.

Now forming $J_\perp$ from $V_{\alpha\perp}$ and adding the magnetization current $\vec{J}m_\perp = (\nabla \times \vec{M})_\perp$, where $\vec{M} = -\dfrac{P_\perp \vec{B}}{B^2}$, gives

$$\vec{J}_\perp = \frac{\vec{B}}{B^2} \times \left[\nabla_\perp P_\perp + (P_\parallel - P_\perp)(\vec{B}\cdot\nabla)\vec{B}\right] \quad . \tag{2.29}$$

This relation is identically obtained from two-fluid theory using Eqs.(2.13), (2.14), (2.06) and

$$\overleftrightarrow{P} = \overleftrightarrow{I} P_\perp + (P_\parallel - P_\perp) \frac{\vec{B}\vec{B}}{B^2} \tag{2.30}$$

In comparing Eqs.(2.13) and (2.28), we see that only the electrostatic term is included in $\vec{E}$, so that neither $\dfrac{\partial \vec{A}}{\partial t}$ or $\rho_\alpha \left(\dfrac{dV_\alpha}{dt}\right)_\perp$ enter into $V_{\alpha\perp}$.

The parallel drift velocity from guiding center theory [11] is

$$\rho_\alpha \left.\frac{dV_\alpha}{dt}\right|_\parallel = n_\alpha q_\alpha \frac{\vec{E}.\vec{B}}{B} - \frac{W_{\alpha\perp}{}^2 m_\alpha (\vec{B}.\nabla)\vec{B}}{B^2} \quad . \tag{2.31}$$

Now both the $\dfrac{\partial \vec{A}}{\partial t}$ term in $\vec{E}$, and $\rho_\alpha \left(\dfrac{dV_\alpha}{dt}\right)_\parallel$ enter into the equation.

In summary, the following points can be made relative to the introduction of the stress tensor and its relation to transport and guiding center theory :

1) Neoclassical diffusion, trapped particle effects, and macro-particle guiding center velocities will all not be calculated self consistently unless the pressure tensor is included. However, because of the complexity of the model , the stress tensor is replaced by the scalar pressure from this point on.

2) Although the form of the velocity equations are identical to those used in neoclassical theory [12], the solution of ∇ U differs. Here, ∇ U depends predominantly on the rate of change of vorticity. The neoclassical case only applies long after the islands have saturated, and the velocity settled down, in which case ∇ U is determined predominantly by parallel viscosity. Thus, transport in the transient phase (convective time scale) will be larger than in the neoclassical phase (resistive time scale).

B. Derivation of reduced toroidal equations

The drift velocity relations have already been derived in Eqs.(2.10) and (2.11). These relations do not depend on the precise form of the magnetic field, but assume that the dominant magnetic field is in the toroidal direction. The magnetic field used in the tokamak case is

$$\vec{B} = f \nabla \phi + \nabla \psi \times \nabla \phi. \tag{2.32}$$

The use an axisymmetric magnetic field in three dimensional geometry is not strictly correct, however, in the tokamak case the asymmetric field is of order ϵ^3 when compared to the vacuum toroidal field and so is neglected here. As already indicated, the velocity form depends on the structure of the magnetic field. For example, the toroidal plasma is not incompressible, but the toroidal incompressibility condition represented by Eq.(2.25), instead implies that the plasma cannot compress the toroidal field.

The poloidal component of the magnetic field is also required and can be represented as

$$\frac{\partial\psi}{\partial t} + \vec{u}_p.\nabla\psi = \eta\,\Delta^*\psi + \frac{\partial U}{\partial\phi} - R\,E_\phi\,, \tag{2.33}$$

where $\psi = RA_\phi$, and the term $\eta\,\Delta^*\psi$, with $\Delta^* = R^2\nabla.\left(\frac{1}{R^2}\right)\nabla$, is written separately from the other E_ϕ terms. The relations for the toroidal velocities then follow as

$$\rho_i\,\frac{dV_{i\phi}}{dt} = -\frac{1}{R}\frac{\partial p_i}{\partial\phi} - n_i\,e\,\vec{V}_{ip}.\nabla\psi + n_i\,e\,E_\phi\;, \tag{2.34}$$

$$\rho_e\,\frac{dV_{e\phi}}{dt} = -\frac{1}{R}\frac{\partial p_e}{\partial\phi} + n_e\,e\,\vec{V}_{ep}.\nabla\psi - n_e\,e\,E_\phi\;. \tag{2.35}$$

The term E_ϕ is small and, looking at the perfectly conducting shell case, it can be related to the induced voltage around the torus on the surface. Although E_ϕ is assumed constant here, in the two-fluid case, the parallel velocity and toroidal electric field are still coupled.

Using the following forms for density, pressure and temperature

$$\bar{\rho} = \frac{R^2\rho}{R_o^{\,2}} \qquad \bar{P} = \frac{R^2}{R_o^{\,2}}\,p^{1/\gamma} \qquad \bar{T} = \frac{\bar{P}}{\bar{\rho}} = \frac{p^{1/\gamma}}{\rho} \tag{2.36}$$

the two-fluid continuity equations follow from Eq.(2.01) as

$$\frac{d\bar{\rho}_i}{dt} = -\,R^2\bar{\rho}_i\,\nabla\cdot\frac{\vec{V}_{ip}}{R^2} \tag{2.37}$$

$$\frac{d\bar{\rho}_e}{dt} = -\,R^2\bar{\rho}_e\,\nabla\cdot\frac{\vec{V}_{ep}}{R^2} \tag{2.38}$$

Correspondingly, the two-fluid energy equations follow from Eqs.(2.03), (2.04) and (2.05) as

$$\frac{d\bar{p}_i}{dt} = -\,R^2\bar{p}_i\,\nabla\cdot\frac{\vec{V}_{ip}}{R^2} + \bar{Z}_i \tag{2.39}$$

$$\frac{d\bar{p}_e}{dt} = - R^2 \bar{p}_e \nabla . \frac{\vec{V}_{ep}}{R^2} + \bar{Z}_e \qquad (2.40)$$

where $\bar{Z}_\alpha = \frac{R^2}{R_0^2} \frac{p}{\gamma}^{\frac{1-\gamma}{\gamma}} Z_\alpha$.

In the ideal MHD tokamak case, then, it follows from Eq.(2.25) that $\bar{\rho}_\alpha$ and $\bar{p}_\alpha$ are adiabatic invariants of the motion.

It is convenient to use different expressions for the vorticity in the two-fluid and one-fluid cases. The resulting equations for two-fluids, as derived in Appendix A, are :

$$\frac{d}{dt}\left(\frac{\vec{W}_i}{\rho_i}\right) = \left(\frac{\vec{W}_i}{\rho_i} . \nabla\right) \vec{V}_i + \frac{1}{\rho_i} \nabla \times \vec{a}_i \qquad (2.41)$$

$$\frac{d}{dt}\left(\frac{\vec{W}_e}{\rho_e}\right) = \left(\frac{\vec{W}_e}{\rho_e} . \nabla\right) \vec{V}_e + \frac{1}{\rho_e} \nabla \times \vec{a}_e \qquad (2.42)$$

where $\vec{W}_i = \nabla \times \vec{V}_i + \frac{e}{m_i} \vec{B}$ $\qquad \vec{W}_e = \nabla \times \vec{V}_e - \frac{e}{m_e} \vec{B}$

and $\vec{a}_i = -\frac{e}{m_i} \left[\frac{\nabla p_i}{en_i} + \eta \vec{J} + \frac{m_i}{e} \upsilon \nabla \times \vec{\varphi}_i\right]$ $\qquad \vec{a}_e = \frac{e}{m_e} \left[\frac{\nabla p_e}{en_e} - \eta \vec{J}\right]$

and with $\vec{\varphi}_i = \nabla \times \vec{V}_i$ $\qquad \vec{\varphi}_e = \nabla \times \vec{V}_e$

Here, $\vec{W}_i$ and $\vec{W}_e$ are pseudo vorticities since they contain $\vec{B}$, but satisfy $\nabla . \vec{W} = 0$; $\vec{\varphi}_i$ and $\vec{\varphi}_e$ are true vorticities. In the absence of $\vec{a}$

$$\frac{d}{dt}\left(\frac{\vec{W}}{\rho}\right) = \left(\frac{\vec{W}}{\rho} . \nabla\right) \vec{V} \qquad (2.43)$$

which is variously called Walens equation, Helmholtz equation, or the d'Alembert - Euler vorticity equation, depending on whether hydrodynamics or ideal magnetohydrodynamics is being considered. For example, in hydrodynamics $\vec{B} = 0$, so that the ion equation becomes

$$\frac{d}{dt}\left(\frac{\vec{\varphi}}{\rho}\right) = \left(\frac{\vec{\varphi}}{\rho} . \nabla\right)\vec{V} + \frac{1}{\rho} \nabla \times \left[\upsilon \nabla \times \vec{\varphi} + \frac{\nabla p}{\rho}\right] , \qquad (2.44)$$

the standard fluid equation.

The two-fluid equations above are not the final reduced set suggested for the transport model. Because the parallel electron velocity is much faster than the parallel ion velocity, the former can be neglected in the transient time phase, and Eqs.(2.33) and (2.35) indicates that the electrons move with the magnetic field. Now, the time rate of change in vorticity of the electrons can be neglected, which states in effect that the plasma moves with the ions.

Although, $\nabla \cdot \frac{\vec{V}_e}{R^2} \cong \nabla \cdot \frac{\vec{u}_p}{R^2} \cong 0$, this still gives $\frac{d\vec{p}_e}{dt} \cong Z_e$, and the $\frac{d\vec{p}_e}{dt}$ equation must be retained and is one of the basic transport equations. Charge conservation is assumed so that $n_e = n_i$. Thus, the final set is essentially one-fluid except for the separate electron and ion pressure equations and is given in the next section.

III. NONLINEAR MHD MODEL

A. Basic one-fluid equations

The nonlinear reduced equations of single-fluid magnetohydrodynamics can be readily obtained from the two-fluid equations given in the last section, with the use of the basic one-fluid variables as follows :

$$\rho = n_e m_e + n_i m_i \tag{3.01}$$

$$\rho_q = e(n_i - n_e) = 0 \tag{3.02}$$

$$\vec{V} = \frac{\rho_e \vec{V}_e + \rho_i \vec{V}_i}{\rho} \tag{3.03}$$

$$p = p_e + p_i \tag{3.04}$$

$$\vec{J} = e\, n_i \vec{V}_i - e\, n_e \vec{V}_e \tag{3.05}$$

Equations (2.37), (2.38) give the one-fluid continuity equation

$$\frac{d\bar{\rho}}{dt} = - R^2 \bar{\rho}\, \nabla . \frac{\vec{V}_p}{R^2} \tag{3.06}$$

Correspondingly, Eqs.(2.39), (2.40) give the one-fluid energy equation

$$\frac{d\bar{p}}{dt} = - R^2 \bar{p}\, \nabla . \frac{\vec{V}_p}{R^2} + \bar{Z} \tag{3.07}$$

where $\bar{Z} = \frac{R^2}{R_o^2} \frac{p^{\frac{1-\gamma}{\gamma}}}{\gamma} Z$ with $Z = (\gamma-1)\,[\nabla.(\overleftrightarrow{K}\,\nabla T) + \eta\, J^2 + S_E]$. (3.08)

The expression for Z follows from Eqs.(2.04) and (2.05) where now the separate electron and ion transport differences no longer enter. The one-fluid drift velocity follows from Eqs.(2.10) and (2.11) as

$$\vec{V} = \frac{R^2}{f}\left[\nabla U - \frac{\nabla p_i}{n_i e} - \eta\,\vec{J}\right] \times \nabla\phi + V_\phi\,\frac{\vec{B}}{B_\phi} \qquad (3.09)$$

where scalar pressure is assumed and further that $\nabla p_e \ll \nabla p_i$.

The two equations that are unchanged going from two-fluids to one-fluid are the equations for $\vec{u}_p$ and $\frac{\partial\psi}{\partial t}$, Eqs.(2.21) and (2.33). The parallel velocity is easy derived from Eqs.(2.34) and (2.35) as

$$\rho\,\frac{dV_\phi}{dt} = -\frac{1}{R}\frac{\partial p}{\partial\phi} - \vec{J}_p \cdot \nabla\psi \qquad (3.10)$$

because of charge conservation, the E_ϕ term no longer appears.

The one-fluid limit of the two-fluid equations for vorticity, Eqs.(2.41) and (2.42), is obtained in the Appendix as

$$\frac{d}{dt}\left(\frac{\vec{W}_1}{\rho}\right) = \left(\frac{\vec{W}_1}{\rho}\cdot\nabla\right)\vec{V} + \frac{1}{\rho}\nabla \times \left[\frac{(\vec{J}\times\vec{B} - \nabla p + \upsilon\,(\nabla\times\vec{W})}{\rho}\right] \qquad (3.11)$$

where $\vec{W}_1 = \nabla \times \vec{V}$, and basically applies when $\bar{\rho} = $ constant. However, when $\rho \neq$ constant, a more useful expression, see Eq.(A12), is

$$\frac{d}{dt}\left(\frac{\vec{W}_3}{\rho}\right) = \left(\frac{\vec{W}_3}{\rho}\cdot\nabla\right)\vec{V} + \frac{1}{\rho}\nabla \times \left[\frac{\bar{\rho}}{\rho}\left(\vec{J}\times\vec{B} - \nabla p - \rho\,\frac{\nabla V^2}{2}\right)\right] \qquad (3.12)$$

where $\vec{W}_3 = \nabla \times \bar{\rho}\,\vec{V}$. When vorticity goes to zero, the acceleration must have a potential, which in fact is related to the local energy. In the equilibrium case the acceleration is zero and then

$$\vec{J}\times\vec{B} - \nabla p - \rho\,\frac{\nabla V^2}{2} = 0 \qquad (3.13)$$

This is sometimes called the modified Bernoulli equation and is essentially the equilibrium equation in the presence of rotation [13,14]. The rotation here is provided by the drift velocity. This indicates that the equilibriums used in the nonlinear calculations should include equilibrium drift flows.

In the low β case, the plasma moves along the magnetic field lines, and the vortex lines in turn move with the plasma. However, it is customary to define the vorticity in the ϕ direction. Then

$$\frac{d}{dt}\left(\frac{\vec{W}_\phi}{\rho}\right) = \frac{R}{\rho}\,\nabla\cdot\left[\frac{R^2}{R_o^{\,2}}\left(\vec{J}\times\vec{B} - \nabla p - \rho\,\frac{\nabla V^2}{2}\right)\times\nabla\phi\right] \qquad (3.14)$$

where $\vec{W}_\phi = (\nabla \times \bar{\rho}\,\vec{V})_\phi$,

and where it is assumed that the variation of $\vec{V}$ in the ϕ direction is zero. For $\bar{\rho}$ equal a constant, which is generally not the case, then Eq.(3.14) simplifies to

$$\frac{d}{dt}\left(\bar{\vec{W}}_\phi\right) = \frac{R^3}{R_o^{\,2}}\,\nabla\cdot\quad \frac{R^2}{R_o^{\,2}}\,\nabla\cdot\left[\frac{R^2}{R_o^{\,2}}\left(\vec{J}\times\vec{B} - \nabla p - \rho\,\frac{\nabla V^2}{2}\right)\times\nabla\phi\right] \qquad (3.15)$$

where $\bar{\vec{W}}_\phi = \frac{R^2}{R_o^{\,2}}\,\vec{W}_\phi$, and this is the vorticity equation of Refs 1-4.

A few additional remarks on the importance of including parallel velocity and diamagnetic effects are made here. The usual argument advanced by physicists working in nonlinear magnetohydrodynamics for the neglect of diamagnetic effects is that the numerical studies show that the saturated magnetic island widths are independent of diamagnetic effects. However, all these numerical studies [15,16] are, without exception, based on helical and not toroidal models. Yet, linear analysis [17] with toroidal models shows strong phase coupling between adjacent m modes ;

here m = poloidal mode number. Further, nonlinear analysis [4] with toroidal models demonstrate that saturated island widths depend strongly on toroidal effects.

In summary, the reasons for including diamagnetic effects are :

1) Since nonlinear saturated magnetic island size depends on toroidal effects, and since toroidal linear analysis confirms the importance of diamagnetic effects, the results of nonlinear helical codes are not conclusive with respect to the importance of diamagnetic effects on saturated island size.

2) Diamagnetic effects are needed to calculate first order flows. However, as already indicated, this is not sufficient if one is concerned about neoclassical transport, in which case the full stress tensor must be included.

The reasons for including parallel velocity can be summarized as follows :

1) With neutral beams, it is important to include toroidal drift velocity. In PLT [18], neutral beams result in a peak toroidal drift velocity of 10^7 cm/sec. or about 10 times the poloidal drift velocity. Once the neutral beams are shut off, the toroidal velocity becomes less than the poloidal velocity. However, the DITE [19] experiment suggests that neutral beams can produce steady state currents, in which case the toroidal drift velocity will always be larger than the poloidal drift velocity.

2) The ordering arguments, advanced by physicists working in nonlinear magnetohydrodynamics, for neglecting parallel velocity have been shown to be generally not valid. This inconsistency has also been pointed out by Green [20].

3) Another argument advanced for the neglect of the toroidal velocity, which is invalid, is that there is no longer a closed set of equations. In the more realistic situation, the exchange of energy between the plasma and magnetic field encompasses both toroidal and poloidal components, and not just poloidal components. In general, it is almost always possible to write the conservation equations in closed form.

An additional comment, related to neither parallel velocity nor *diamagnetic effects, is that the nonlinear magnetohydrodynamic model assumes* that charge separation is zero. This implies approximately that

$$\nabla^2 U \approx \rho_q = 0 \tag{3.16}$$

In general, in nonlinear magnetohydrodynamic numerical calculations, although the velocity is derivable from the electrostatic potential U, there is no numerical verification that U satisfies Poisson's equation approximately within any degree of accuracy.

In summary, the set of equations, that serve as a basis for the transport equations, are

$$\frac{\partial\psi}{\partial t} + \vec{u}_p \cdot \nabla\psi = \eta\, \Delta^* \dot{\psi} + \frac{\partial U}{\partial \phi} - R\, E_\phi \tag{3.17}$$

$$\frac{d}{dt}\left(\frac{\vec{W}}{\rho}\right) = \frac{R}{\rho}\, \nabla \cdot \left[\frac{R^2}{R_o^{\,2}} \left(\vec{J} \times \vec{B} - \nabla p - \rho \frac{\nabla V^2}{2}\right) \times \nabla\phi\right] \tag{3.18}$$

$$\frac{d\bar{\rho}}{dt} = -\, R^2\, \bar{\rho}\, \nabla \cdot \frac{\vec{V}_p}{R^2} \tag{3.19}$$

$$\frac{d\bar{p}}{dt} = -\, R^2\, \bar{p}\, \nabla \cdot \frac{\vec{V}_p}{R^2} + \tilde{Z} \tag{3.20}$$

$$\frac{d\bar{p}_e}{dt} = - R^2 \bar{p}_e \nabla \cdot \frac{\vec{V}_{ep}}{R^2} + \bar{Z}_e \tag{3.21}$$

$$\rho \frac{d\vec{V}_\phi}{dt} = - \frac{1}{R} \frac{\partial p}{\partial \phi} - \vec{J}_p \cdot \nabla\psi \tag{3.22}$$

where
$$\vec{V}_p = \left(\nabla U - \frac{\nabla p_i}{n_i e} - \eta \vec{J} \right) \times \frac{R^2 \nabla\phi}{f} + \frac{V_\phi \vec{B}_p}{B_\phi} \tag{3.23}$$

$$\vec{V}_{ep} = \left(\nabla U + \frac{\nabla p_e}{n_e e} - \eta \vec{J} \right) \times \frac{R^2 \nabla\phi}{f} \tag{3.24}$$

$$\vec{u}_p = \frac{R^2}{f} \left(\nabla U \times \nabla\phi \right) \tag{3.25}$$

$$\vec{W} = (\nabla \times \bar{\rho} \vec{V})_\phi \tag{3.26}$$

and the quantities without subscript represent the bulk plasma or ions.

There are six time-dependent equations here.
Although there are 10 equations listed, and there are 16 variables which include $\vec{J}$, f, η, $\overleftrightarrow{K}_i$, $\overleftrightarrow{K}_e$, Q_{ei}, there are in fact equations for all variables. the current follows from $\vec{J} = \nabla \times \vec{B}$, f satisfies $\frac{df}{dc} = 0$, and η is assumed to satisfy the Spitzer resistivity ($\eta = \text{const}/T^{3/2}$). There are also relations for $\overleftrightarrow{K}_i$, $\overleftrightarrow{K}_e$ and Q_{ei}, where the latter reflects the difference in T_e and T_i. In the complete transport equations, there are in fact more than 16 variables, but again the set can be made complete as it stands. The vorticity equation, Eq.(3.18), takes the place of the Grad-Shafranov equation.

In Sec.IV, where the transport equations are derived, it is only $\bar{\rho}_i$, $\bar{p}_i$ and $\bar{p}_e$ that are flux-surface averaged. However, the transport calculations require that ψ, W and $V\phi$ can also advanced in time to generate the new flux surfaces and velocities.

B. Comparison with helical model

The three-dimensional helical nonlinear magnetohydrodynamic model for tokamaks was extended to high beta ($\beta \sim \epsilon$) by Strauss [7]. This was an extension of his earlier work for low β ($\beta \sim \epsilon^2$) [21]. The equations of the one-fluid toroidal model here reduce identically to those of the helical model, if certain obvious differences are noted. This serves to confirm that the set of equations here are valid for high beta ($\beta \sim \epsilon$) tokamaks.

The relevant helical model equations from Ref.7 are

$$\vec{V} = \nabla U \times \hat{\phi} + V_z \hat{\phi} \tag{3.27}$$

$$\frac{d}{dt} \nabla_\perp^2 U = \vec{B}.\nabla \nabla_\perp^2 A + \frac{2}{R_0} \frac{\partial p}{\partial z} \tag{3.28}$$

$$\frac{\partial A}{\partial t} = \vec{B}.\nabla U \tag{3.29}$$

$$\frac{dp}{dt} = 0 \tag{3.30}$$

The equations in the helical model were derived for ideal magnetohydrodynamics, so that setting $\eta = 0$ in Eq.(3.09) and taking the helical limit $\frac{R^2}{R_0^2} = 1$ yields

$$\vec{V} = \left[\nabla U - \frac{\nabla p_i}{ne} \right] \times \frac{\hat{\phi}}{B_0} + V_\phi \frac{\vec{B}}{B_\phi} \tag{3.31}$$

Thus, the real differences between Eqs.(3.27) and (3.31) are the presence in the latter of $\frac{\nabla p_i}{ne}$, which cannot be neglected for $\beta \sim \epsilon$, and the $V_\Theta = \frac{V_\phi B_\Theta}{B_\phi}$ contribution which brings in the parallel velocity.

Since the density is assumed to be constant in the helical model, the toroidal equations (3.18), (3.17), (3.20) reduce identically to the corresponding helical equations (3.28), (3.29), (3.30) with the obvious differences. The velocity contribution is neglected in Eq.(3.28) the integration constant is not included in Eq.(3.29), and no transport is assumed in Eq.(3.30).

It needs to be emphasized that the conclusions drawn from toroidal as contrasted with helical tokamak studies may be quite different. The helical model has been predominantly used in nonlinear magnetohydrodynamic numerical simulations and as a result toroidal effects, which include asymmetry, mode coupling and saturated island size, have not been realistically modeled. This was in part due to the fact that it was never possible to numerically simulate magnetic islands with the original toroidal code [1,2]. Recently, the new nonlinear MHD code [4], employing different numerical techniques, has demonstrated large differences in saturated island size between toroidal and helical models.

The nonlinear helical model has been extended to stellerators by Strauss [22]. The reduced nonlinear helical equations obtained by him strongly suggest that if they were carried to next order, in particular, including in the drift velocity the stellerator helical field and the variation of the toroidal field, that the transport approach presented here could be applied to medium β ($\beta \sim \epsilon$) stellerators.

C. Comparison with compact torus model.

Recently, the reduced nonlinear low β magnetohydrodynamic equations were extended to compact tori. There are two versions of this model. In the first version [23], the magnetic field was taken to be $\vec{B} = \nabla\phi \times \nabla\psi + f\,\nabla\phi$ and the velocity assumed to be $\vec{V} = \nabla\phi \times \nabla U + \frac{\omega}{J}\nabla\phi$. In the second version [8,24], the magnetic field is the same, but the velocity was assumed to be

$$\rho\,\vec{V} = \nabla\phi \times \nabla U + \omega\nabla\phi + \nabla\Omega \qquad (3.32)$$

It is true that the velocity (or momentum) can in general be expressed as $\vec{V} = \nabla \times \vec{A} + \nabla\Phi$, the sum of a solenoidal part, $\nabla \times \vec{A}$ with $\nabla\cdot(\nabla \times \vec{A}) = 0$ and an irrotational part, $\nabla\Phi$ with $\nabla \times \nabla\Phi = 0$; however, this implies certain things about the physics of the situation. For example, the toroidal case previously presented here basically represents solenoidal flow (no irrotational part) of a plasma which is compressible. The compressibility condition arises from the form of the magnetic field. The compact torus velocity includes a solenoidal part, $\nabla\phi \times \nabla U + \omega\,\nabla\phi$, and an irrotational part, $\nabla\Omega$. The point is that while the formulas for the velocity in the compact torus case and the toroidal case agree in the incompressible case, they do not agree in the compressible case, which suggests that the velocity form in the compact torus, Eq.(3.32), needs to be carefully reexamined.

When the plasma is compressible, the plasma velocity tends to approach the ion sound velocity, which creates numerical problems. The usual numerical solution is to set the pressure equal to zero (low β). However, for high β tokamaks and reversed field pinches, the effects of plasma pressure must be included, and this also applies for high β compact tori.

Further, the zero-order magnetic field is not in equilibrium without zero-order mass flow, as Eq.(3.13) indicates, contrary to what was assumed in the compact torus model. Also, the argument that the velocity is always small $V \sim \frac{\eta}{a}$ when the boundary conditions change on the resistive time scale, and therefore that the $\vec{V}.\nabla\vec{V}$ term can be neglected, is *not always justified. In particular, with the m=1 oscillation*, which is considered part of the quasi-equilibrium, the external boundary conditions change very slowly, but the plasma close to the magnetic axis evolves very rapidly.

In general, for $\beta \sim \epsilon$ (or $\beta \sim \epsilon^2$), the velocity associated with the perpendicular movement of the magnetic field can be obtained from $\vec{u}_\perp = \frac{\vec{E} \times \vec{B}}{B^2}$ independent of $\frac{\partial \vec{A}}{\partial t}$, that is, it depends only on the electrostatic potential. The relation between $V_\perp$ and $u_\perp$ follows from the generalized Ohm law. The velocity parallel to the magnetic field, $v_\parallel$, follows from the parallel component of the momentum balance equation. The advantage of writing the nonlinear reduced equations in the form perpendicular and parallel to the magnetic field is that the fast Alfven wave is automatically eliminated, resulting in considerably greater stability in numerical solutions. Referring back to Eq.(2.19), setting $\nabla.(\frac{\vec{V}_\perp}{h_\parallel^2}) = 0$, is equivalent to $\frac{d}{dt}(h_\parallel B_\parallel) = 0$ which eliminates the fast Alfven wave. In the toroidal case here, using $\frac{df}{dt} = 0$ eliminates the toroidal Alfven wave, but the poloidal Alfven wave still remains.

Not withstanding the previous comments, it is still of interest to make a comparison of models, because the comparison illustrates the geometric problems associated with magnetic field and velocity representation, and the comparison indicates the possibility of generalizing the transport model here to include compact tori.

For the purposes of direct comparison, the second version of the compact torus model is used with both $\nabla \Omega$ and viscosity set equal to zero. The compact torus equation set then becomes :

$$\frac{\partial}{\partial t} \Delta^* U + R^2 \nabla \cdot \left[\frac{\Delta^* \psi}{R^2} (\nabla\phi \times \nabla\psi) + \frac{f}{R^2} \nabla\phi \times \nabla f \right] = 0 \qquad (3.33)$$

$$\frac{\partial \omega}{\partial t} + (\nabla\phi \times \nabla f) \cdot \nabla\psi = 0 \qquad (3.34)$$

$$\frac{\partial \rho}{\partial t} = S_\rho \qquad (3.35)$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{\rho} (\nabla\phi \times \nabla U \cdot \nabla\psi) = \eta \Delta^* \psi \qquad (3.36)$$

$$\frac{\partial f}{\partial t} + R^2 \nabla \cdot \left[\frac{f}{\rho R^2} (\nabla\phi \times \nabla U) - \frac{\omega}{\rho R^2} \nabla\phi \times \nabla\psi - \frac{\eta \nabla f}{R^2} \right] = 0 \qquad (3.37)$$

Since, this model assumes no convective terms, for example and is for low β, $\beta \sim \epsilon^2$, and is axisymmetric, the first four equations, Eqs.(3.33), (3.36), are identical with the corresponding toroidal set represented by Eqs.(3.18), (3.22), (3.19), (3.17), except for small differences. The principal difference between the compact torus equation set and the toroidal equation set, is the equation for f, Eq.(3.37).

In a compact torus, the toroidal field can be smaller than the poloidal field, so that the toroidal incompressibility condition of Eq.(2.17) no longer applies, but the more general one of Eq.(2.19) still does apply.

Thus, the condition $\vec{u}_\perp = \frac{\nabla U \times \vec{B}}{B}$ becomes different depending on the dominant B field. Therefore, it is better not to consider the poloidal B field as a perturbation on an exisymmetric torus, but rather to consider orbits and transport with the dominant poloidal field perturbed by toroidal curvature.

An approach [25], which was used to model the compact torus, and represents an alternative to the nonlinear MHD model here is to write the various nonlinear equations in the device coordinate system without trying to reduce them, or in particular eliminate the fast wave motion. Such a two-dimensional approach has been used successfully to simulate the early plasma development on screw and belt pinches [26], reversed-field pinches, and Tormac [27]. This model used classical resistivity and constant thermal conductivity and, since the temperatures in the experiments simulated were low, ~ 20 eV, it is reasonable that classical dissipation should give good agreement with experiment. However, because the Alfven wave has not been eliminated, the numerical analysis becomes unstable long before the experimental confinement time has been reached. Eliminating the Alfven wave should delay the numerical instability by at least a factor of 100 in time, depending of course on the appropriate sound velocity.

Comparing the primitive equations with the reduced set here, in the primitive set there will be 7 time-dependent equations corresponding to two magnetic fluxes, the density, the three velocity components and the pressure. In the reduced set there will be 5 time-dependent equations, assuming that the component of magnetic flux associated with the parallel magnetic field can be eliminated, in the tokamak case $\frac{df}{dt} \cong 0$, giving a condition on the perpendicular drift velocity. Then, the two perpendicular velocity components will be replaced by the time rate of change of vorticity along the magnetic field lines. The important point is not the reduction of the 7 primitive equations to 5 reduced equations, but the elimination of the Alfven wave.

IV. THREE DIMENSIONAL TRANSPORT MODEL

A. Discussion of different transport models

Two working transport models and one proposed transport model are discussed here. To facilitate the discussion, some of the previous equations are repeated in simplified form.

$$\rho \frac{d\vec{V}}{dt} + \nabla p = (\nabla \times \vec{B}) \times \vec{B} \qquad (4.01)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[\vec{V} \times \vec{B} - \eta \vec{J} - \frac{\nabla p}{ne} \right] \qquad (4.02)$$

$$\frac{\partial p}{\partial t} + \gamma \nabla . (p\vec{V}) = Z \qquad (4.03)$$

where $$Z = (\gamma-1) \left[\nabla . (\vec{K} \nabla T) + \eta J^2 + S_E \right]$$

and $$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{V}) = 0 \qquad (4.04)$$

These of course represent the 4 basic equations, 7 primitive equations, from which the five nonlinear reduced equations were derived in the previous section.

The first model discussed here is referred to as the

1) Standard transport model, because almost all the present transport codes are based on it. In this model [28], the inertia term in Eq.(4.01) is neglected so that the Grad-Shafranov equilibrium equation results. The transport model proceeds in two steps : first, start with some equilibrium, second, the transport calculation is carried through holding the flux-surfaces constant, but allowing ρ , p_i, p_e and B_p to change. Then, the first

step, the equilibrium calculation with the new variables, is repeated to generate the new flux surfaces and so forth. Transport computer codes for two-dimensional axisymmetric tokamaks have been developed using this model by General Atomic [5], Science Applications Incorporated[6], and the Oak Ridge National Laboratory [29]. Recently, Grad [30] has generalized the model to three dimensions.

Since the MHD constraint (equilibrium equation) is not satisfied during the transport phase, there is an inconsistency in the algorithm in this first model. The solution will be unaffected provided that the flux surface shapes change more slowly than the density, pressure and poloidal field, and provided that the MHD constraint is reimposed before strong violation of the constant shape assumption occurs. Numerical tests[5] have been made which slow that the algoritm does accurately describe the time evolution of plasma profiles in the general geometry, provided the geometry does not change too rapidly on the transport (resistive) time scale. Thus, the first model is appropriate only on the resistive time scale and in the absence of magnetic islands.

In an attempt to avoid the algorithm inconsistency, Taylor [31] and independently Pao [32] developed the transport approach, which will be called here for brevity, the (2) <u>time-differentiated transport model</u>. Since the inertia term is still neglected in this second model, one can take the time derivative of the momentum equation, and substitute the time variation of the pressure and magnetic field from Eqs.(4.03), (4.02), and obtain an equation which is independent of time as follows

$$\nabla \left[\gamma \, \nabla \cdot (p\vec{V}) - Z \right] + \left\{ \nabla \times \left[\nabla \times \left(\vec{V} \times \vec{B} - \eta \vec{J} - \frac{\nabla p}{ne} \right) \right] \right\} \times \vec{B} \qquad (4.05)$$

$$+ (\nabla \times \vec{B}) \times \left[\nabla \times \left(\vec{V} \times \vec{B} - \eta \vec{J} - \frac{\nabla p}{ne} \right) \right] = 0$$

Both Taylor and Pao showed that the general solution of this equation resulted in two arbitrary velocity functions related to the two components of velocity lying in the magnetic surface. They stated that these arbitrary functions could be resolved in terms of the electrostatic potential and parallel velocity.

For the case of a large-aspect-ratio circular tokamak, Pao [33], using an expansion in thermal conductivity ($K_\perp / K_\parallel$) showed that the tangential velocities are decoupled from the general diffusion problem. This, however, is a result of the particular derivation, and while the temperature may be uniform over a magnetic surface, the density is not and this is further substantiated by the equilibrium equation with rotation, Eq.(3.13). A model where the tangential velocities are disregarded is inconsistent with neoclassical theory. Nevertheless, a two-dimensional transport code, using the $K_\perp/K_\parallel$ expansion and where the tangential velocities are not considered, has recently been developed [34]. This second model is also appropriate only on the resistive time scale and in the absence of magnetic islands.

An alternative approach to the first two transport models, is the (3) reduced nonlinear MHD transport model discussed next. In this three-dimensional transport model, both the electrostatic potential and parallel velocity are calculated, so that there are no undetermined velocity functions, as in the second model. Further, to determine the development and evolution of magnetic islands correctly requires the inclusion of the inertia term in Eq.(4.01). Thus, a time scale, here called the convection time scale, is incorporated in this model that is much faster than the resistive time scale. This faster time scale permits neutral beam heating and flux conserving tokamak equilibria to be realistically evaluated.

As an example of the effect to time scales on flux-conserving equilibria studies, the helical model, described in Sec.III B, was used to study a low β plasma suddenly heated to high β and yielded the results that the plasma evolves essentially on the poloidal Alfven time scale as expected and that the final equilibria are unstable to ballooning modes. Alternatively, recent studies [35] of flux-conserving tokamak equilibria suddenly heated, however using a resistive time scale model, demonstrate that the plasma evolves on the resistive time scale and is generally stable. The boundary conditions are not slowly changing here, so that the resistive time scale definitely does not apply. What would be of interest is the same equilibrium suddently heated and modeled by two different approaches, in particular, the convective time scale approach with resistivity and the resistive time scale approach. Nevertheless, the previous studies demonstrate that not only will the final equilibria differ, but that the development in time will be much faster with the inertia term included.

B. Transport set of toroidal equations

From the reduced quasi one-fluid equations given in Sec.III, the transport equations can now be written directly. These equations include flux surface averages denoted by < >, which for a function A is defined by

$$< A > = \frac{1}{V'} \oiint A \frac{dS}{|\Delta\psi|} \tag{4.06}$$

with

$$V' = \frac{\partial V}{\partial \psi} = \oiint \frac{ds}{|\nabla\psi|}$$

Here, since the flux surfaces are three-dimensional, the integrals can not be simplified as in the axisymmetric case and an integral over dS, the elemental area on the flux surface ψ, is required.

Using the generalized gauss law, the following relation for a vector $\vec{A}$:

$$< \nabla.\vec{A}(\vec{X},t) > = \frac{1}{V'} \frac{\partial}{\partial\psi} \left[V' < \vec{A}(\vec{X},t).\nabla\psi > \right] \tag{4.07}$$

Since $\vec{B}.\nabla\psi \equiv 0$. $< \vec{B}.\nabla\psi > = 0$

Previously, discussion following Eq.(2.27), the rotation velocity was considered to be $-\frac{\nabla p_i}{ner}$. For this case, quasi neutrality (n_i=n_e) can be invoked, but inertia and viscosity do not vanish, and the diffusion is ambipolar only on the average over a magnetic surface. The flux surface average of the one-fluid particle density can now be written as

$$\int_{\psi_1}^{\psi_u} \left\{ \frac{\partial}{\partial t} \left(< n > V' \right) + \frac{\partial}{\partial \psi} \left[V' < n \, (\vec{V}-\vec{u}).\nabla\psi > \right] - < S_n > V' \right\} d\psi$$
$$- < n > V' \left. \frac{d\psi}{dt} \right|_{\psi_u} + < n > V' \left. \frac{d\psi}{dt} \right|_{\psi_1} = 0 \qquad (4.08)$$

This equation gives the particle flux relative to a ψ flux surface and since in general $\frac{d\psi}{dt} \neq 0$, these terms need to be included, where ψ_u and ψ_1 represent upper and lower bounds of the volume of integration. Thus, the bracketed term $\{ \quad \}$ is no longer zero, as is the case in usual transport models. There is allowance for a particle source S_n, even though it was not included originally.

The electron and ion energy transport equations follow respectively from Eqs.(3.21), (3.20) as

$$\int_{\psi_1}^{\psi_u} \left\{ \frac{3}{2} \frac{\partial}{\partial t} (< p_e > V') + \frac{\partial}{\partial \psi} \left[V' \left[< Q_e + \frac{5}{2} (\vec{V}_e - \vec{u}) p_e \right] .\nabla\psi > \right] \right.$$
$$\left. + \frac{\partial}{\partial \psi} \left[V' < \vec{u} \, p_e.\nabla\psi > \right] - < \vec{J}.\vec{E} > V' + < Q_{ei} > V' - < S_e > V' \right\} \qquad (4.09)$$
$$- \frac{3}{2} < p_e > V' \left. \frac{d\psi}{dt} \right|_{\psi_u} + \frac{3}{2} < p_e > V' \left. \frac{d\psi}{dt} \right|_{\psi_1} = 0$$

$$\int_{\psi_1}^{\psi_u} \left\{ \frac{3}{2} \frac{\partial}{\partial t} (< p_i > V') + \frac{\partial}{\partial \psi} \left[V' \left[< Q_i + \frac{5}{2} (\vec{V}_i - \vec{u}) p_i \right] .\nabla\psi > \right] \right.$$
$$\left. + \frac{\partial}{\partial \psi} \left[V' < \vec{u} \, p_i.\nabla\psi > \right] - < Q_{ei} > V' - < S_i > V' \right\} \qquad (4.10)$$
$$- \frac{3}{2} < p_i > V' \left. \frac{d\psi}{dt} \right|_{\psi_u} + \frac{3}{2} < p_i > V' \left. \frac{d\psi}{dt} \right|_{\psi_1} = 0$$

The tensor terms Q_e and Q_i here include only thermal conductivity, however, the more complete forms, as for example given by Braginskii [36] should be used. The term Q_{ei} allows for electron-ion thermalization and can be written as

$$Q_{ei} = \frac{3n}{\tau_i} \frac{m_e}{m_p} (T_e - T_i)$$

The last three time dependent equations needed are the equations for the poloidal flux ψ, the vorticity $\vec{W}$, and the parallel velocity $\vec{V}_\phi$, Eqs.(3.17), (3.18), (3.22). These permit the magnetic field and velocity to be advanced in time.

In contrast to the resistive time scale transport models (1) and (2), in this model, the movement of magnetic field and associated plasma velocities should be larger than the transport across the flux surfaces for a given time step δt. Thus, the flux surface average values of δ_{p_e}, δ_{p_i}, δ_ρ for each δt are smaller than the local values p_e, p_i, ρ, so that the local values can be altered by these average valves. However, this is no longer true following the transient phase when the magnetic islands have saturated. Then the transport across the flux surfaces is greater than the movement of the flux surfaces and this corresponds to the resistive time scale. All this suggests that the energy conservation equation, Eq.(A 14) and the rotating quasi-equilibrium equation, Eq.(3.13) need to be verified at different times during the transport calculation. Further, for long confinement times, after the movement of the flux surfaces and velocities have diminished, it will be clearly necessary to include the full pressure tensors in order to obtain agreement with neoclassical transport.

The different interpretations of plasma mass flow as given by the approach of Grad and Hogan [28], which requires the solution of a boundary valve, and that of Hazeltine and Hinton [12], where plasma flow is related to local gradients of plasma quantities, was resolved [34] as being due to differences in transport in the laboratory frame and the frame moving with the toroidal flux. In particular, it was argued that in the frame moving with the toroidal flux, the mass flow is a transport quantity equal to the flux with $\frac{\partial \vec{B}}{\partial t} = 0$.

Here, the resolution is different, and reflects the differences in the closure conditions of transport models (1) and (3). In the model here, in the laboratory frame, the first order perpendicular velocity of Hazeltine - Hinton identically results. Yet, the complete solution involves the fact that the velocity consists of a local part, related to space derivatives of plasma quantities, and a non-local part, related to space derivatives at the plasma boundary and an integral over space derivatives in the whole plasma domain. The reason the local part dominates in the laboratory frame is that the plasma does not have sufficient energy to compress the parallel magnetic field, Eq.(2.19), and correspondingly that inertia and $\frac{\partial \vec{A}}{\partial t}$ do not explicitly enter into the perpendicular drift velocity, in agreement with guiding center theory.

V. CONCLUSIONS

The principal conclusion of this paper is that it should in principle be possible to numerically compute transport in tokamak systems using a basic set of reduced nonlinear magnetohydrodynamic equations given by Eqs.(3.17) through (3.26) and a flux surfaced set of transport equations given by (4.09), (4.10) and (4.11). The model here allows the simulation of transport associated with magnetic islands and also the modeling of transport during the transient phase of experimental tokamak plasma discharges. It was indicated how the model might be extended to other toroidal geometries such as stellarators and compact tori. In essence, this paper represents only the first step in the formulation of a reduced nonlinear magnetohydrodynamic model for three-dimensional transport in toroidal systems.

An associated important conclusion is that the toroidal incompressibility condition can be generalized to other low β ($\beta < 1$) devices. The toroidal incompressibility condition, as extended here from the initial theory [1,2], physically states that the plasma does not have sufficient energy to compress the parallel magnetic field. More importantly, if the coordinate chosen has one coordinate that is parallel to the magnetic field at every point, then the fast Alfven waves can always be eliminated in the reduced nonlinear equations. The elimination of the Alfven waves is important for the stability of the numerical solutions [37]. As shown here, sound waves still persist but their elimination is not as important.

Using a coordinate system aligned with the magnetic field, the equations for the perpendicular and parallel plasma velocities, can be written independently of the device geometry, except for the boundary conditions. The resulting equations for $v_\perp$ and $\frac{dv_\parallel}{dt}$ were shown to be in basic agreement with guiding center theory.

Presently, the numerical computations are done only using conventional geometries, and then the tokamak has advantages because the dominant B field corresponds with the ϕ direction, so that it is easier to do the numerical computations. For other geometries, stellarators and compact toroids, the representation becomes considerably more complicated numerically, because the dominant magnetic field does not have a convenient geometric representation.

A recent important paper [38] discusses the general problem of evolution of equilibrium configurations with resistivity. There it was concluded that magnetic field diffusion is distinct from the diffusion of the plasma itself. This conclusion is valid if inertia can be neglected in the nonlinear regime.

It is known [39] that in order to describe the linear regime, or an m=1 mode, that inertia must be retained. However, analytically [40] and numerically [39], inertia can be neglected in the nonlinear regime of a single magnetic island if $m > 1$ for the helical axisymmetric case. In the axisymmetric case, Eq.(2.34) reduces to $\frac{d\psi}{dt} \approx \eta \nabla^* \psi$ so that $\frac{d}{dt} \sim \eta$. Then, in the helical case, Eq.(3.14) gives $\frac{dW}{dt} \sim \eta^2$ so that inertia can be neglected.

However, inertia cannot be neglected in the toroidal asymmetric case. Then, for example, the slow variation of U in the parallel direction, $\frac{\partial U}{\partial \phi}$, allows the electrons to obey the Boltzmann distribution and with quasi-neutrality

$$\frac{n}{n_0} = \frac{eU}{T_e} \, .$$

This puts a lower limit on ∇U which applies on the resistive time scale. Further, the vorticity cannot be set equal to zero, not only because of $\frac{\partial U}{\partial \phi}$ but because of the other toroidal terms, until the $m = 2$ magnetic island saturates. For JET[41], the nonlinear saturation time is about one second, while the resistive skin time is about 20 seconds. Thus, in JET, as in other present and future experiments, the plasma and magnetic field will still be evolving at the end of the pulse.

It has been observed [38] that the evolution of equilibria could be followed with MHD equations including inertia terms. The reason given for not adopting this procedure is that Alfvén waves and sound waves greatly complicate the calculations and that the introduction of an artificially strong dissipation does not work. This argument, however, does not recognize that it is possible to eliminate the fast Alfven wave, and also the recent success of a reduced nonlinear MHD code [4] in producing saturated magnetic islands on a long time scale. Although density variation, parallel velocity and diamagnetic effects were not included in the code, the most difficult part of the numerical development has succeeded, and indicates that the development of the numerical code of the transport model presented here could be just a matter of time.

The inclusion of diamagnetic effects essentially introduces the possibility of a motion of the magnetic configuration with respect to the ions. The elimination of all fast wave motion such as sound waves is not desirable since this motion is needed to correctly simulate reconnection of magnetic field lines at a singular layer.

The recent successful nonlinear MHD code [3,4] uses finite elements and Eulerian coordinates. The earlier less successful code [1,2] used finite differences and Lagrangian coordinates and was never able to reproduce magnetic islands. The advantages of finite elements in handling complex geometries and being able to incorporate differential type boundary conditions easily, coupled with recent extensive development, [42, 43] indicates that computationally this may be the best way to proceed.

A basic problem that remains is how to generate the poloidal flux surfaces. Magnetic surfaces when they exist, are solutions of $\vec{B}.\nabla\psi = 0$. In general, ψ, defining the poloidal field in toroidal geometry, is not a flux function in contrast to the helical flux ψ^* in a helical configuration. Only approximate expressions can be given to describe magnetic surfaces in the presence of finite amplitude modes with different helicities m/n, and for stongly interacting modes, large islands, the configuration of magnetic surfaces can only be obtained by following individual field lines.[44] It may be possible to use multi-dimensional cubic-spline interpolation to compute these ψ surfaces in a manner already used with guiding-center equations. [45]

Another, less significant, problem is to understand the difference between the theorectical analysis [46,47] of poloidal damping time and the corresponding nonlinear MHD numerical results. [48] Theoretical model analysis yields the result that the dominant damping mechanism for poloidal rotation is parallel ion viscosity, so that the damping rate is about equal to the ion-ion collision frequency. The numerical resu... dicate a much slower damping rate ($\sim 10^{-3}\ \nu_{ii}$). Although parallel viscosity is not included in the numerical model, this is not the complete answer, since the numerical poloidal rotation velocity is about as large as the sound speed, which disagrees with a basic assumption in the theoretical model.

The transport model here might be useful in explaining anomalous transport. Experiments indicate that particle and energy transport coefficients in high temperature plasmas are anomalous and not understood and thus that it is justified to model transport by a one-dimensional cylindrically symmetric model for the plasma column. The view here is different, and it is expected that the inclusion of magnetic islands, the transient time phase, and differences in the local electron and ion velocities, should account for at least part of the anomalous transport.

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APPENDIX

Here, it is first proved that the two-fluid vorticity equations, Eqs. (2.41) and (2.42), in the one-fluid limit yield Eq. (3.11), $\frac{d}{dt}(\frac{\vec{W}_1}{\rho})$ where $\vec{W}_i = \nabla \times \vec{V}$. Since, in general $\bar{\rho}$ is not constant, a more useful form of vorticity is $\vec{W}_3 = \nabla \times \bar{\rho}\,\vec{V}$, and the time rate of change of this vorticity is then derived.

Starting with

$$\rho\,\vec{W}_1 = \rho_i\,\vec{W}_i + \rho_e\,\vec{W}_e \tag{A1}$$

where $\vec{W}_i = \nabla \times \vec{V}_i + \frac{e}{m_i}\vec{B}$ and $\vec{W}_e = \nabla \times \vec{V}_e - \frac{e}{m_e}\vec{B}$
the $\vec{B}$ field drops out in the one-fluid limit, which is the desired result. Now using Eqs. (2.41) (2.42) gives

$$\rho\,\frac{\partial \vec{W}_i}{\partial t} - \nabla \times \left[\rho_i\,(\vec{V}_i \times \vec{W}_i) + \rho_e\,(\vec{V}_e \times \vec{W}_e)\right] + \nabla\rho_i \times (\vec{V}_i \times \vec{W}_i)$$
$$+ \nabla\rho_e \times (\vec{V}_e \times \vec{W}_e) = -\rho\,\nabla \times (\frac{\nabla p}{\rho}) + \rho_i\,\nabla \times \left\{\frac{\nu}{\rho}\left[\nabla \times (\nabla \times \vec{V}_i)\right]\right\} \tag{A2}$$

where the resistive terms cancel out. Since with the $\vec{W}_e$ terms only the $\vec{B}$ part contributes, this gives

$$\rho\,\frac{\partial \vec{W}_1}{\partial t} - \rho\,\nabla \times (\vec{V} \times \vec{W}_1) = \rho\,\nabla \times \left[\frac{\vec{J} \times \vec{B} - \nabla p + \nu\,(\nabla \times \vec{W}_1)}{\rho}\right] \tag{A3}$$

where Eq. (3.05) was used. Now, since $\nabla.\vec{W}_1 = 0$,

$$\nabla \times \left[\vec{V} \times \vec{W}_1\right] = -\vec{W}_1\,\nabla.\vec{V} + (\vec{W}_1.\nabla)\vec{V} - (\vec{V}.\nabla)\vec{W}_1 \tag{A4}$$

and this gives

$$\frac{d}{dt}\left(\frac{\vec{W}_1}{\rho}\right) + \left(\frac{\vec{W}_1}{\rho}.\nabla\right)\vec{V} = \frac{1}{\rho}\nabla \times \left[\frac{\vec{J} \times \vec{B} - \nabla p + \nu\,(\nabla \times \vec{W}_1)}{\rho}\right] \tag{A5}$$

In hydrodynamics it is useful to use $\vec{W} = \nabla \times \vec{V}$ since the fluid is incompressible. In magnetohydrodynamics it is not so useful since the plasma is compressible. Instead, consider the three forms of vorticity :

$\vec{W}_1 = \nabla \times \vec{V}$ useful in incompressible case $\vec{F} = m\,\vec{a}$

$\vec{W}_2 = \nabla \times \rho\vec{V}$ def in terms of linear momentum $\vec{F} = \frac{d}{dt}(\rho\vec{V})$

$\vec{W}_3 = \nabla \times \bar{\rho}\vec{V}$ $\qquad \vec{\bar{F}} = \frac{d}{dt}(\rho\vec{V})$

The linear momentum, $\rho\vec{V}$, is a property of the flowing substance that resists changes in motion. The form $\vec{W}_3$ is of interest, because in some cases $\frac{d\bar{\rho}}{dt} = 0$, and was the form used in the original and recent nonlinear MHD toroidàl models. Start with

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}.\nabla)\vec{V}\right] = \vec{J} \times \vec{B} - \nabla.\vec{P} \tag{A6}$$

Multiply by $\frac{R^2}{R_o^{\,2}}$ and take curl

$$\frac{\partial}{\partial t}(\nabla \times \bar{\rho}\vec{V}) + \nabla \times \left[\bar{\rho}(\vec{V}.\nabla)\vec{V}\right] = \nabla \times \left\{\frac{R^2}{R_o^{\,2}}\left[\vec{J} \times \vec{B} - \nabla.\vec{P}\right]\right\} \tag{A7}$$

with

$$(\bar{\rho}\,\vec{V}.\nabla)\vec{V} = \nabla\bar{\rho}V^2 - \vec{V} \times \vec{W}_3 - \bar{\rho}\,\vec{V} \times (\nabla \times \vec{V}) - (\vec{V}.\nabla)\,\bar{\rho}\,\vec{V} \tag{A8}$$

this gives

$$\frac{\partial \vec{W}_3}{\partial t} - \nabla \times (\vec{V} \times \vec{W}_3) - \nabla \times \left[\bar{\rho}\, \vec{V} \times (\nabla \times \vec{V}) + (\vec{V}.\nabla)\, \bar{\rho}\, \vec{V} \right]$$
$$= \nabla \times \left\{ \frac{R^2}{R_o^2} \left[\vec{J} \times \vec{B} - \nabla.\vec{P} \right] \right\} \qquad , \tag{A9}$$

or

$$\frac{d}{dt} \left(\frac{\vec{W}_3}{\rho}\right) = \frac{\vec{W}_3}{\rho}.\nabla\, \vec{V} + \frac{1}{\rho} \nabla \times \left\{ \frac{R^2}{R_o^2} \left[\vec{J} \times \vec{B} - \nabla.\vec{P} \right] + \bar{\rho}\, \vec{V} \times (\nabla \times \vec{V}) + (\vec{V}.\nabla) \bar{\rho}\, \vec{V} \right\} \tag{A10}$$

When $\bar{\rho}$ is constant, Eq. (A10) reduces to Eq. (A5). Because of the curl operator in Eq. (A10), it is useful to write the last two terms there in a different equivalent form. In particular

$$\bar{\rho}\, \vec{V} \times (\nabla \times \vec{V}) + (\vec{V}.\nabla)\, \bar{\rho}\, \vec{V} = \nabla\, (\bar{\rho}\, V^2) - \bar{\rho}\, \frac{\nabla V^2}{2} \tag{A11}$$

which gives

$$\frac{d}{dt} \left(\frac{\vec{W}_3}{\rho}\right) = \frac{\vec{W}_3}{\rho}.\nabla\ \ \vec{V} + \frac{1}{\rho} \nabla \times \left\{ \frac{\bar{\rho}}{\rho} \left[\vec{J} \times \vec{B} - \nabla.\vec{P} - \rho\, \frac{\nabla V^2}{2} \right] \right\} \tag{A12}$$

and this is the desired final result.

The energy conservation equation can be obtained as follows : Multiply Eq.(A12) by $\frac{R}{f}\, U^*$, for the case $\vec{V}$ independent of ϕ direction, where

$$U^* = U - \frac{m}{e} \int \frac{dp}{\rho} \tag{A13}$$

and where it is assumed that $\rho = \rho(p)$; multiply Eq.(3.17) by $\frac{J\phi}{R}$; multiply Eq.(3.20) by $\gamma p/(\gamma - 1)\ \bar{p}$; and finally multiply Eq.(3.22) by $V\phi$. Adding these four adjusted equations and integrating over volume gives

$$\frac{d}{dt} \int_V dV \left[\frac{\bar{\rho} |\nabla_p U^*|^2}{2 B_o^2} + \frac{\rho V_\phi^2}{2} + \frac{|\nabla\psi|^2}{2 R^2} + \frac{p}{\gamma - 1} \right]$$

$$= \int_V dV [S_E] + \int_S d\vec{S} . [E_\phi \nabla\psi + K_\perp \nabla T] \qquad (A14)$$

This is the equation for the time evolution of the total energy, where now diamagnetic effects and parallel velocity are included.

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ERRATA

Report DRFC - STGI EUR-CEA-FC-1076

THREE DIMENSIONAL TRANSPORT MODEL FOR TOROIDAL PLASMAS

by

C. COPENHAVER

title page, please note misspelling of name :

abstract page, C. COPENHAVER instead of C. COPENHAUER

page 10, 5th line from the bottom should be written :

.... In general, if $\vec{u}_\perp = \frac{\vec{E} \times \vec{B}}{B^2}$

page 20, 4th line from the bottom should be written :

However, when $\bar{\rho} \neq$ constant, ...

page 21, eq. (3.15) written correctly is

$$\frac{d}{dt}(\vec{W}\phi) = \frac{R^3}{R_0^2} \nabla . \left[\frac{R^2}{R_0^2} \left(\vec{J} \times \vec{B} - \nabla p - \rho \frac{\nabla V^2}{2} \right) \times \nabla \phi \right] \qquad (3.15)$$

page 29, 9th line from the bottom should be written :

Since, this model assumes no convective terms, for example $\vec{V} . \nabla p = 0$,

page 30, top 3 lines should be written :

Thus, the condition $U_\perp = \frac{\nabla U \times \vec{B}}{B^2}$ becomes different depending on the dominant B field. Therefore, it is better not to consider the poloidal B field as a perturbation on an axisymmetric torus,...

page 37, equations (4.09) and (4.10) written correctly are

$$\int_{\psi_1}^{\psi_u} d\psi \left\{ \frac{3}{2}\frac{\partial}{\partial t}(\langle p_e\rangle V') + \frac{\partial}{\partial\psi}\left[V'\langle \left[Q_e + \frac{5}{2}(\vec{V}_e - \vec{u})p_e\right]\cdot\nabla\psi\rangle\right] + \frac{\partial}{\partial\psi}\left[V'\langle \vec{u}\, p_e \cdot \nabla\psi\rangle\right] - \langle \vec{J}\cdot\vec{E}\rangle V' + \langle Q_{ei}\rangle V' - \langle S_e\rangle V' \right\}$$

$$- \frac{3}{2}\langle p_e\rangle V' \frac{d\psi}{dt}\Big|_{\psi_u} + \frac{3}{2}\langle p_e\rangle V' \frac{d\psi}{dt}\Big|_{\psi_1} = 0 \qquad (4.09)$$

$$\int_{\psi_1}^{\psi_u} d\psi \left\{ \frac{3}{2}\frac{\partial}{\partial t}(\langle p_i\rangle V') + \frac{\partial}{\partial\psi}\left[V'\langle \left[Q_i + \frac{5}{2}(\vec{V}_i - \vec{u})\, p_i\right]\cdot\nabla\psi\rangle\right] + \frac{\partial}{\partial\psi}\left[V'\langle \vec{u}\, p_i\cdot\nabla\psi\right] - \langle Q_{ei}\rangle V' - \langle S_i\rangle V' \right\}$$

$$- \frac{3}{2}\langle p_i\rangle V' \frac{d\psi}{dt}\Big|_{\psi_u} + \frac{3}{2}\langle p_i\rangle V' \frac{d\psi}{dt}\Big|_{\psi_1} = 0 \qquad (4.10)$$

page 46, 2nd and 3rd lines from the top should be written :

.... Eq. (3.11), $\frac{d}{dt}\left(\frac{\vec{W}_1}{\rho}\right)$, where $\vec{W}_1 = \nabla \times \vec{V}$.

page 46, equations (A1) and (A2) written correctly are

$$\rho \vec{W}_1 = \rho_i \vec{W}_i + \rho_e \vec{W}_e \qquad (A1)$$

$$\rho \frac{d\vec{W}_1}{dt} - \nabla \times \left[\rho_i (\vec{V}_i \times \vec{W}_i) + \rho_e (\vec{V}_e \times \vec{W}_e)\right] + \nabla\rho_i \times (\vec{V}_i \times \vec{W}_i)$$

$$+ \nabla\rho_e \times (\vec{V}_e \times \vec{W}_e) = - \rho \nabla \times \left(\frac{\nabla \rho}{p}\right) + \rho_i \nabla \times \left\{ \frac{\nu}{\rho}\left[\nabla \times (\nabla \times \vec{V}_i)\right]\right\} \qquad (A2)$$

page 47, 8th line from top should be written

$$\vec{W}_3 = \nabla \times \bar{\rho}\, \vec{V} \qquad \vec{F} = \frac{d}{dt}(\bar{\rho}\, \vec{V})$$

The report has been sent out without review of the final version. The above corrections should be made.

C. Copenhaver