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STATIONARY SCATTERING THEORY

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INTRODUCTION

We present a complementary approach to the time dependent scattering theory described by B. Enss for one-body Schrödinger operators. Roughly speaking the stationary theory is concerned with those objects you read about in textbooks on quantum theory like scattering waves and amplitudes. The starting point of the old theory is not here an asymptotic condition for large times, as for wave-operators of V. Enss lecture, but rather for large distances. It is known as "Sommerfeld radiation condition" and leads, when incorporated in the Schrödinger equation as a "Cauchy condition at infinity", to the celebrated Lippman-Schwinger equation. In the more recent abstract stationary theory some generalized form of the Lippman-Schwinger equation plays the basic role ; solving this equation leads to a linear map between generalized eigenfunctions of the perturbed and unperturbed operators. This map is the "section" at fixed energy of the wave-operator from the time dependent theory. Although the radiation condition does not appear explicitely in this formulation it can be shown to hold a posteriori in a variety of situations, thus restoring the link with physical theories. A general approach to the radia-

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tion condition for a large class of Partial Differential operators is described in the work of S. Agmon and L. Hörmander[1] ; we will mention some of their results here.

In these lectures I will describe an abstract framework for the stationary theory. It is strongly inspired from the Kato-Kuroda theory[2] and also related to the more recent two Hilbert space theories of T. Kato[3], M. Schecter[4], or I. Segal[5] . As we will see it allows to incorporate many of the technical progress of these last years. Among them are S. Agmon's "elliptic" a priori estimate method and the geometric methods ; these last have appeared as particularly useful for many investigations on the bound state problem for N-body Hamiltonians[6,7,8]. They also seem to be promising for the scattering problem.

THE STANDARD FORMULATION OF STATIONARY SCATTERING THEORY FOR THE ONE-BODY QUANTUM PROBLEMS

The basic ansatz of orthodox textbooks on quantum scattering in R^n by a local potential $V(X)$ vanishing at infinity is the existence of a family of solutions $u_+^{(k)}(X)$ of the reduced Schrödinger equation :

$$(-\Delta+V)\, u_+^{(k)}(x)=0$$

which can be decomposed as

$$u_+^{(k)}(x)=u_0^{(k)}(x)+v_+^{(k)}(x) \tag{1}$$

where $v_+^{(k)}(X)$ is a "scattered wave" and $u_0^{(k)}(X)=\exp(ikx)$
By this one means essentially that $v^{(k)}(x)$ has the asymptotic form

$$v_+^{(k)}(x)\approx |x|^{\frac{1-n}{2}}\exp(i|k||x|)\, f(|k|,\omega) \tag{2}$$

where w is the angular variable of the particle. This is known as "Sommerfeld Radiation condition" which can be more rigorously stated as

$$\lim_{R\to\infty}\int_{|x|=R}\left|\frac{\partial v_+^{(k)}}{\partial |x|}-i|k|\,v_+^{(k)}\right|^2 d\sigma=0 \tag{3}$$

The function f is called the "Scattering amplitude" and from it one gets the scattering cross-section

$$\sigma(k)=\int_S |f(k,\omega)|^2\, d\omega$$

where S is the unit sphere in R^n.

It has been known for a long time that solutions in class

C^2 of the Helmoltz equation $(-\Delta - k^2)u = 0$ outside a bounded obstacle can be decomposed in a unique way into a solution of $(-\Delta - k^2)u_0 = 0$ in whole space and a function satisfying radiation condition (3). For scattering by a potential with non-compact support the work of T. Ikebe[9] represents one of the most complete investigation of (1) which can be accomplished by integral equation methods (see also Reed-Simon[10]). Using Fredholm theory in a suitable Banach space containing plane waves Ikebe shows existence of a complete set of solutions $u_+^{(k)}(x)$ satisfying (1) and (2) provided the potential V decays faster than $|x|^{-2-\varepsilon}$, $\varepsilon > 0$, at infinity. Such solutions satisfy the integral equation and the Cauchy conditions (1) and (2) at infinity ; it is known as the "Lippman-Schwinger equation" and reads :

$$(4) \qquad u_+^{(k)}(x) = u_0^{(k)}(x) - \int G_k^+(x-y)\, V(y)\, u_+^{(k)}(y)\, dy$$

Here $G_k^+(x)$ is the fundamental solution of $(-\Delta - k^2)G_k^+ = \delta$ satisfying (3) ; e.g. for $N = 3$ one has $G_k^+(x) = (4\pi|x|)^{-1}\exp i|k||x|$

It appears that the limitation on the allowed decay of the potential in comparison to the $|x|^{-1-\varepsilon}$ decay allowed by Enss' analysis is linked to the inhomogeneity $u_0^{(k)}(x) = \exp(ikx)$ in (4). In order that the kernel $G_k^+(x-y)V(y)$ defines a compact operator between suitable Banach spaces containing plane waves it is necessary to impose such restrictions. One can expect that with different inhomogeneities $u_0^{(k)}(x)$ having some decay at infinity, for example spherical waves, one should be able to enlarge the class of potentials for which solutions of (4) exist. This possibility will be described in these lectures.

As a preparation for the next sections let us notice that (4) can be rewritten in a more abstract operator formalism. In fact $G_k^+(x-y)$ is the boundary value as $\varepsilon \to 0^+$ of the resolvent kernel $\langle x, (-\Delta - (k^2+i\varepsilon))^{-1} y\rangle$; so at least formaly (5) reads

$$(5) \qquad u_+^{(k)} = u_0^{(k)} - \lim_{\varepsilon \downarrow 0^+} (-\Delta - (k^2+i\varepsilon))^{-1} V u_+^{(k)}$$

The second resolvent equation gives the formal solution :

$$(6) \qquad u_+^{(k)} = u_0^{(k)} - \lim_{\varepsilon \downarrow 0^+} (-\Delta + V - (k^2+i\varepsilon))^{-1} V u_0^{(k)}$$

In order to give a meaning to (5) and (6) one has to specify in which type of topology the $\varepsilon = 0$ limits are taken. Even if $Vu_0^{(k)}$ is in L^2 these limits certainly don't exist in L^2 since in general for potentials tending to zero at infinity k^2 is in the essential spectrum of $-\Delta + V$. The next paragraph will be concerned with a general analysis of these boundary values of resolvents.

THE LIMITING ABSORPTION PRINCIPLE

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. If λ is in $\sigma(A)$ it is well-known that the resolvents $(A-(\lambda\pm i\varepsilon))^{-1}$, $\varepsilon\in\mathbb{R}\setminus\{0\}$, have no weak limit in $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. This follows from the uniform boundedness theorem and the estimate $\|(A-(\lambda\pm i\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{H})} = \varepsilon^{-1}$. Nevertheless the following question can make sense :

Problem.

Let $\lambda\in\mathbb{R}$; is it possible that for some Banach space $\mathfrak{X}$, continuously and densely contained in $\mathcal{H}$ (which we will abbreviate by $\mathfrak{X}\hookrightarrow\mathcal{H}$). the limits

(7) $$R^{\pm}(\lambda)=\lim_{\varepsilon\downarrow 0^+}(A-(\lambda\pm i\varepsilon))^{-1}$$

exist in some topology (weak, strong or norm) of $\mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$, the algebra of bounded linear maps from $\mathfrak{X}$ to $\mathfrak{X}^*$?

Notice that since

(8) $$\mathfrak{X}\hookrightarrow\mathcal{H}\hookrightarrow\mathfrak{X}^*$$

and $(A-(\lambda\pm i\varepsilon))^{-1}\in\mathcal{L}(\mathcal{H})$, $\varepsilon>0$, one has a fortiori $(A-(\lambda\pm i\varepsilon))^{-1}\in\mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$; so it makes sense to talk about limits in $\mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$.

In the following we will denote by the same symbol $\langle\cdot,\cdot\rangle$ the scalar product in $\mathcal{H}$ and the duality between $\mathfrak{X}$ and $\mathfrak{X}^*$

Definition 1

If (7) holds for some Banach space $\mathfrak{X}$ we will say that the Limiting Absorption Principle (L.A.P.) holds for A at λ in $\mathfrak{X}$.

Remarks

1) If $\lambda\in\varsigma(A)$, then $(A-(\lambda\pm i\varepsilon))^{-1}$ converges in norm in $\mathcal{L}(\mathcal{H})$ hence a fortiori in $\mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$ for an arbitrary $\mathfrak{X}$ satisfying (8).

2) If $\lambda\in\sigma_p(A)$, the point spectrum of A, then the L.A.P. holds for no $\mathfrak{X}$ at λ.

This follows from :

$$(8') \quad \lim_{\varepsilon \downarrow 0^+} \pm i\varepsilon \langle \varphi, (A-(\lambda \pm i\varepsilon))^{-1}\psi \rangle = \langle \varphi, E(\{\lambda\})\psi \rangle$$

where $E(.)$ denotes the spectral family of A. If the L.A.P. holds at λ in $\mathfrak{X}$ then $\langle \varphi, E(\{\lambda\})\psi \rangle = 0$, $\forall\ \varphi, \psi \in \mathfrak{X}$. By the density of $\mathfrak{X}$ in $\mathcal{H}$ this would imply $E(\{\lambda\}) = 0$ contradicting $\lambda \in \sigma_p(A)$.

From this last remark follows that if the L.A.P. holds in $\mathfrak{X}$ for all $\lambda \in I$, where I is some open interval in $\mathbb{R}$, then the spectral measure $E((-\infty,\lambda]) = E_\lambda$ is continuous on I. This can be improved as follows :

Proposition 1

Let $I \subset \sigma(A)$ be a bounded open interval such that the L.A.P. holds for A in $\mathfrak{X}$ on I. Then $E(I)\mathcal{H} \subset \mathcal{H}_{ac}(A)$ where $\mathcal{H}_{ac}(A)$ is the subspace of absolute continuity for A.

Proof : Let $\varphi \in \mathfrak{X}$, then by the spectral theorem one has

$$F(z) = \langle \varphi, (A-z)^{-1}\varphi \rangle = \int \frac{1}{\lambda - z}\, d\langle \varphi, E_\lambda \varphi \rangle$$

By a classical result of de La Vallée-Poussin the set

$$B = \left\{ \lambda \in \mathbb{R} \ ,\ F^+(\lambda) = \lim_{\varepsilon \downarrow 0^+} F(\lambda + i\varepsilon) < \infty \right\}$$

is such that the restriction of $d\langle \varphi, E_\lambda \varphi \rangle$ to B is absolutely continuous with respect to the Lebesgue measure. So in particular $E(I)\varphi \in \mathcal{H}_{ac}(A)$, for all $\varphi \in \mathfrak{X}$. Since $\mathfrak{X}$ is dense in $\mathcal{H}$ the conclusion follows.

For the following it is essential to recall the connection between the boundary values $F^+(\lambda)$, $\lambda \in I$, and the Radon-Nikodym derivative of the spectral measure :

$$(9) \quad \frac{d}{d\lambda}\langle \varphi, E_\lambda \varphi \rangle = \frac{1}{\pi}\, \mathrm{Im} \langle \varphi, R^+(\lambda)\varphi \rangle \quad , \varphi \in \mathfrak{X}.$$

This suggests the

Definition 2

Let $\varphi \in \mathfrak{X}$ and assume the L.A.P. holds for A at λ in $\mathfrak{X}$ with $\lambda \in \sigma_c(A)$ (the continuous spectrum of A). The element of $\mathfrak{X}^*$ given by

$$(10) \quad \varphi(\lambda) = (R^+(\lambda) - R^-(\lambda))\varphi$$

will be called <u>the spectral trace of φ at λ with respect to A.</u>

Notice the mapping $\varphi \in \mathcal{X} \to \varphi(\lambda) \in \mathcal{X}^*$ is continuous. Now let

$$N_\lambda = \{\varphi \in \mathcal{X} \ , \mathrm{Im} < \varphi, R^+(\lambda)\varphi > = 0\}$$

Then

Proposition 2

N_λ is a closed subspace of $\mathcal{X}$ consisting of these $\varphi \in \mathcal{X}$ such that $\varphi(\lambda) = 0$

Proof : Let us consider the sesquilinear form on $\mathcal{X} \times \mathcal{X}$

(11) $a(\lambda; \psi, \varphi) = i^{-1} < \psi, (R^+(\lambda) - R^-(\lambda))\varphi >$

Notice that sesquilinearity follows from self-adjointness of A. Now a is continuous on $\mathcal{X} \times \mathcal{X}$ since $R^\pm(\lambda) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ it is also positive since

$$(12) \quad a(\lambda; \varphi, \varphi) = \lim_{\varepsilon \downarrow 0^+} i^{-1} < \varphi, [(A-(\lambda+i\varepsilon))^{-1} - (A-(\lambda-i\varepsilon))^{-1}]\varphi >$$

$$= \lim_{\varepsilon \downarrow 0^+} < \varphi, \frac{2\varepsilon}{(A-\lambda)^2+\varepsilon^2}\varphi >$$

So the kernel of a is a closed subspace of $\mathcal{X}$, since this kernel is precisely N_λ the first assertion of the Proposition 2 follows. Now by Cauchy-Schwartz inequality one has $<\psi, [R^+(\lambda) - R^-(\lambda)]\varphi> = 0$ for all $\psi \in \mathcal{X}$ if and only if $\varphi(\lambda) = 0$ which concludes the proof.

The left annihilator of N_λ , denoted by $\mathcal{E}_\lambda$, is defined as

$$\mathcal{E}_\lambda = \{u \in \mathcal{X}^*, \ <u, \varphi> = 0 \quad \text{for all} \quad \varphi \in N_\lambda\}$$

Proposition 3

Assume L.A.P. holds for A at λ in $\mathcal{X}$. Then
i) $\mathcal{E}_\lambda$ is a closed subspace of $\mathcal{X}^*$
ii) If $\varphi \in \mathcal{X}$ then $\varphi(\lambda) \in \mathcal{E}_\lambda$

Proof : i) is a standard result and ii) follows from

$$<\psi, [R^+(\lambda) - R^-(\lambda)]\varphi> = - <[R^+(\lambda) - R^-(\lambda)]\psi, \varphi>$$

and Proposition 2.

In general $\mathcal{E}_\lambda$ contains strictly the set of traces. Notice however the following

Proposition 4

Assume $\mathfrak{X}$ is reflexive. Then $\varphi \in \mathfrak{X} \longrightarrow \varphi(\lambda) \in \mathcal{E}_\lambda$ has a dense image in $\mathcal{E}_\lambda$.

Proof : Let $F_\lambda = \{ \varphi(\lambda), \varphi \in \mathfrak{X} \}$ and assume the closure of F_λ for the $\mathfrak{X}^*$ topology is not dense in $\mathcal{E}_\lambda$. By Hahn-Banach theorem there exists $\psi \in \mathfrak{X}^{**} = \mathfrak{X}$ such that $\langle u, \psi \rangle \neq 0$ and $\langle \varphi(\lambda), \psi \rangle = 0 \ \forall \varphi \in \mathfrak{X}$. In particular one has $\langle \psi(\lambda), \psi \rangle = \operatorname{Im} \langle \psi, R^+(\lambda) \psi \rangle = 0$ which implies $\psi \in N_\lambda$ and accordingly $\langle u, \psi \rangle = 0$. a contradiction.

Remark : It is easy to show that if $u \in \mathcal{E}_\lambda$ then

$$\langle u, (A-\lambda) v \rangle = 0$$

for all $v \in \mathcal{D}(A)$ such that $(A-\lambda) v \in \mathfrak{X}$. When A is a self-adjoint realisation of a differential operator P(D,x) this allows to interpret $\mathcal{E}_\lambda$ as a space of weak solutions in $\mathfrak{X}^*$ of $(P(D,x)-\lambda)u = 0$. For example, when A is the Laplacian on $L^2(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$ is the Besov space introduced below a result of S. Agmon and L. Hörmander states that for $\lambda > 0$, $\mathcal{E}_\lambda$ is exactly the set of distributions u whose Fourier transforms $\hat{u}(\xi)$ have support on $\xi^2 = \lambda$ and are square-integrable on this sphere. In this case they also show that $\mathcal{E}_\lambda$ is exactly the set of traces $\varphi(\lambda)$, $\varphi \in B(\mathbb{R}^n)$, thus improving the general result of Proposition 4.

THE L.A.P. AND SPECTRAL THEORY

Let us look more closely at the connection between the notion of spectral trace $\varphi(\lambda)$, $\varphi \in \mathfrak{X}$, and the spectral decomposition for A. For this we introduce $h(\lambda)$ as the closure of $\{ \varphi(\lambda), \varphi \in \mathfrak{X} \}$ for the norm associated to the scalar product :

$$(13) \quad \langle \varphi(\lambda), \psi(\lambda) \rangle_{h(\lambda)} = (2i\pi)^{-1} \langle \varphi, [R^+(\lambda) - R^-(\lambda)] \psi \rangle$$

By the arguments used in the proof of Proposition 2 it is easy to check that the above set of traces is a pre-Hilbert space.

We want to construct a unitary operator U from $\mathcal{H}_{ac}(A)$ to an Hilbert integral

$$\mathfrak{h} = \int^{\oplus} h(\lambda)\, d\lambda$$

mapping A_0 onto the diagonal operator of multiplication by in λ in $\mathfrak{h}$. For this we make the following assumption :

There exists a family $(I_n)_{n\in\mathbb{N}}$ of disjoint open intervals such that

i) $\overline{\bigcup_n I_n} = \mathbb{R}$

ii) The L.A.P. holds on I_n . $\forall n \in \mathbb{N}$, in $\mathfrak{X}$.

This implies that $\mathcal{H} = \mathcal{H}_{ac}(A) \oplus \mathcal{H}_p(A)$ where $\mathcal{H}_p(A)$ is the closed subspace spanned by eigenvectors of A . This follows in particular from Proposition 1 which implies that the singular spectrum of A is denumerable and is contained in the union of the end-points of the I_n's. Without restricting generality we can assume $\mathcal{H}_p(A) = \{0\}$ and $\sigma(A)$ consists of the closure of one single (possibly infinite) open interval I.

We refer to J. Dixmier[12] for the theory of Hilbert integrals. What is needed is a denumerable set of vector fields $\lambda \in \bar{I} \to \varphi_n(\lambda) \in h(\lambda)$. $n \in N$, such that :

(14i) The mappings $\lambda \in \bar{I} \to \langle \varphi_n(\lambda), \varphi_m(\lambda)\rangle_{h(\lambda)}$ are measurable $\forall$ n,m .

(14ii) $\int_I \|\varphi_n(\lambda)\|^2_{h(\lambda)}\, d\lambda < \infty$

(14iii) $\{\varphi_n(\lambda)\ ,\ n \in \mathbb{N}\}$ is a total set in $h(\lambda)$.

It is enough to define $\varphi_n(\cdot)$ on I. For this we consider a basis $(\varphi_n)_n$ of $\mathfrak{X}$. Since $\langle \varphi_n(\lambda), \varphi_m(\lambda)\rangle_{h(\lambda)} = \frac{d}{d\lambda}\langle \varphi_n, E_\lambda \varphi_m\rangle$ by (9) and (13) properties (14i) and (14ii) hold. The continuity of the mapping $\varphi \in \mathfrak{X} \to \varphi(\lambda) \in h(\lambda)$ implies in addition (14iii).

Now the mapping $\varphi \in \mathfrak{X} \to \int^{\oplus}_{\sigma(A)} \varphi(\lambda)\, d\lambda$ is isometric for the norm induced by $\mathcal{H}$ on $\mathfrak{X}$ and the norm on $\mathfrak{h}$:

$$\left\|\int^{\oplus}_{\sigma(A)} \varphi(\lambda)\, d\lambda\right\|^2 = \int_{\sigma(A)} \|\varphi(\lambda)\|^2_{h(\lambda)}\, d\lambda$$

So it can be continued to a unitary mapping U from $\mathcal{H}$ to h . Obviously U diagonalises A. Then the spectral trace defined by

(10) on $\mathfrak{X}$ can be extended to all of $\mathcal{H}_{ac}(A)$ in the sense that to $\varphi \in \mathcal{H}_{ac}(A)$ there corresponds by U a square integrable vector field $\lambda \in \bigcup_n I_n \longrightarrow \varphi(\lambda) \in h(\lambda)$. In the following we will still call it the spectral trace of φ (although for a general $\varphi \in \mathcal{H}$ it is only defined almost everywhere).

To conclude this section let us establish a useful relation between $\mathcal{E}_\lambda$ and $h(\lambda)$:

Proposition 5

There exists a positive finite constant C such that $\forall\, \varphi \in \mathfrak{X}$

(15) $\|\varphi(\lambda)\|_{h(\lambda)} \geqslant C \|\varphi(\lambda)\|_{\mathfrak{X}^*}$

Accordingly

(16) $h(\lambda) \subset \mathcal{E}_\lambda$

Proof : One has

$$
\begin{aligned}
\|\varphi(\lambda)\|_{h(\lambda)} &= (2\pi)^{-1} \sup_{\psi \in \mathfrak{X}} \left\langle \frac{\psi(\lambda)}{\|\psi(\lambda)\|_{h(\lambda)}}, \varphi(\lambda) \right\rangle_{h(\lambda)} \\
&= \sup_{\psi \in \mathfrak{X}} \left| \left\langle \frac{\psi}{\|\psi(\lambda)\|_{h(\lambda)}}, \varphi(\lambda) \right\rangle \right| \\
&\geqslant C \sup_{\psi \in \mathfrak{X}} \left| \left\langle \frac{\psi}{\|\psi\|_{\mathfrak{X}}}, \varphi(\lambda) \right\rangle \right| \\
&= C \|\varphi(\lambda)\|_{\mathfrak{X}^*}
\end{aligned}
$$

In the last equality we have used the boundedness of the map $\psi \in \mathfrak{X} \longrightarrow \psi(\lambda) \in h(\lambda)$ due to (13).

EXAMPLES

As the first and simplest example let us consider the Laplacian $A_0 = -\Delta$ on $L^2(\mathbb{R}^n)$ with its natural domain of self-adjointness $\mathcal{H}_2(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n),\ D^\alpha u \in L^2(\mathbb{R}^n),\ |\alpha| \leq 2\}$. The spectral decomposition of A_0 is well-known ; A_0 has absolutely continuous spectrum consisting of $\mathbb{R}^+$. An explicit spectral representation is easy to construct using Fourier transform $\mathcal{F}$. For $\lambda > 0$ and $\varphi_0 \in C^\infty(\mathbb{R}^n)$ define $\varphi_0(\lambda) \in L^2(S)$, with S the unit sphere in $\mathbb{R}^n$, by

(17) $\varphi_0(\lambda, w) = \lambda^{\frac{n-1}{2}} (\mathcal{F}\varphi)(\sqrt{\lambda} \cdot w)$

By Parseval equality one has

$$\|\varphi\|^2_{L^2(\mathbb{R}^n)} = \int \|\varphi_o(\lambda)\|^2_{L^2(S)}\, d\lambda$$

On the other hand it is well-known (see e.g. Lions-Magenes[13]) that the mapping $\varphi \in C_o^\infty(\mathbb{R}^n) \longrightarrow \varphi_o(\lambda) \in L^2(S)$ is continuous for the norm :

$$\|\varphi\|^2_{L^2_s} = \int (1+|x|^2)^s\, |\varphi(x)|^2 dx$$

provided $s > \frac{1}{2}$. This property suggests the choice $\mathcal{H} = L^2_s(\mathbb{R}^n)$, $s > \frac{1}{2}$. This is done by S. Agmon[14] with the following result :

<u>Theorem 1</u>

For all $\lambda \in \mathbb{R}\setminus\{0\}$ the limits

$$R_o^\pm(\lambda) = \lim_{\varepsilon \downarrow 0^+} (-\Delta - (\lambda \pm i\varepsilon))^{-1}$$

exist in the norm topology of $\mathcal{L}(L^2_s, L^2_{-s})$, $s > \frac{1}{2}$. Furthermore the family of operators

$$R_o^\pm(z) = \begin{cases} (-\Delta - z)^{-1} & , \ \mathrm{Im}\, z \gtrless 0 \\ R^\pm(\lambda) & , \ z = \lambda \pm i0 \quad , \lambda \in \mathbb{R} \end{cases}$$

is norm continuous in $\mathcal{L}(L^2_s, L^2_{-s})$. The details of the proof can be found in Reed-Simon[10]

In the Agmon's L^2_s theory[14] the mappings $\varphi \in L^2_s(\mathbb{R}^n) \to \varphi_o(\lambda) \in \mathcal{E}_\lambda$ are not surjective and the inclusion (17) is strict. This follows from the fact (see Lions-Magenes[13]) that the traces $\varphi_o(\lambda)$, $\varphi \in L^2_s(\mathbb{R}^n)$ are in the Sobolev-space $\mathcal{H}_{s-\frac{1}{2}}(S)$ which is strictly contained in $L^2(S)$ hence in $\mathcal{E}_\lambda$ by Proposition 5. Furthermore[1] the L^2_{-s} topology is strictly weaker than the $L^2(S)$ topology on the set of distributions whose Fourier transform have support on S.

One has a feeling that the "best" intermediate space $\mathcal{H}$ should optimize Propositions 4 and 5. This optimal space should be some inductive limit of spaces $L^2_s(\mathbb{R}^n)$, $s > \frac{1}{2}$ when s tends to $\frac{1}{2}$. It is not $L^2_{\frac{1}{2}}(\mathbb{R})$ for which the spectral trace is not continuous but something very close to it namely the Besov type space B defined as follows :

$$B(\mathbb{R}^n) = \left\{ u \ , \ \|u\|^2_{B(\mathbb{R}^n)} = \sum_{j=1}^{\infty} 2^{j-1} \int_{2^{j-2} < |x| < 2^{j-1}} |u(x)|^2 dx < \infty \right\} \tag{18}$$

Although this definition is not very intuitive from a physical point of view, the dual space norm is a lot more appealing if one likes to think of scattering theory from the point of view of the radiation condition (2) :

$$\|u\|^2_{B^*(\mathbb{R}^n)} = \sup_{R>1} \left(\frac{1}{R} \int_{|x|<R} |u(x)|^2 dx \right)$$

The following results are proved by S. Agmon and L. Hörmander[1].

<u>Theorem 2</u>

i) The L.A.P. holds for A_0 on $\mathbb{R} \setminus \{0\}$.

ii) For any $\lambda > 0$ one has

$$L^2(S) = \{ \varphi_0(\lambda) \, , \, \varphi \in B(\mathbb{R}^n) \}$$

and

$$\| \varphi_0(\lambda) \|_{L^2(S)} = C \| \varphi_0(\lambda) \|_{B^*(\mathbb{R}^n)}$$

for some finite constant C .

That $B(\mathbb{R}^n)$ certainly is the best choice not only mathematicaly but also from a physical point of view follows from the fact that the radiation condition finds in this framework the following natural formulation[1] :

<u>Theorem 3</u>

Let $u \in B^*(\mathbb{R}^n)$ be such that $u = R_0^{\pm}(\lambda) f$ for some $\in B(\mathbb{R}^n)$
Then

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|<R} | u(x) - |x|^{\frac{1-n}{2}} f^{\pm}(\sqrt{\lambda}, \tfrac{x}{|x|}) \exp(\pm i\sqrt{\lambda}|x|) |^2 dx = 0$$

where

$$f^{\pm}(\sqrt{\lambda}, \omega) = c \lambda^{-1} \varphi_0(\lambda, \pm \omega)$$

with $\varphi_0(\lambda)$ given by (17) and

$$c = (\pi/2)^{1/2} \exp(\pm i\pi(n-3)/4)$$

Taking (5) into account this last result obviously shows that for a large class of potentials V and for free waves $u_0^{(k)}$, the scattered waves $\psi_+^{(k)}(x)$ in (1) will satisfy Sommerfeld radiation condition. This concludes this presentation of the L.A.P. for the Laplacian.

In view of our later discussion of the three-body problem let us state without proof the following general result[15] ;

Proposition 6

Let A_1 and A_2 be self-adjoint operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Assume the L.A.P. holds for A_1 (resp. A_2) on $R \setminus \Sigma_1$ (resp. $R \setminus \Sigma_2$) on $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) where Σ_1 (resp. Σ_2) is a closed set. Then L.A.P. holds for

$$A = A_1 \otimes I_{\mathcal{H}_2} + I_{\mathcal{H}_1} \otimes A_2$$

in $\mathfrak{X} = \mathfrak{X}_1 \otimes \mathfrak{X}_2$ on $R \setminus (\Sigma_1 + \Sigma_2)$.

PERTURBATIVE APPROACH TO THE LIMITING ABSORPTION PRINCIPLE

We now develop a perturbative framework allowing to prove the L.A.P. for a self-adjoint operator A obtained by a perturbation of another operator A for which the L.A.P. is known to hold. This approach uses a natural abstract generalization of the Lippman-Schwinger equation and leads in a very natural way to the stationary theory of wave-operators.

So let A_0 and A be self-adjoint operators on separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ respectively. In order to compare states in these spaces we need an "identification operator" $J \in \mathcal{L}(\mathcal{H}_0, \mathcal{H})$, the space of bounded linear mappings from $\mathcal{H}_0$ to $\mathcal{H}$. We want the identification of states in $\mathcal{H}$ to be unique ; this is satisfied under the condition :

$(H_1\text{-i})$ $\quad JJ^* = 1_{\mathcal{H}}$

To avoid unessential domain problems we require furthermore

$(H_1\text{-ii})$ $\quad J\,\mathcal{D}(A_0) \subset \mathcal{D}(A)$

The perturbation V is then defined as

$$V = AJ - JA$$

so that under condition $(H_1\text{-ii})$ one has $\mathcal{D}(V) \supset \mathcal{D}(A_0)$. Notice that under condition (19) the "effective perturbation" VJ^* is symmetric.

Now let $\mathfrak{X}_0 \hookrightarrow \mathcal{H}_0$ and $\mathfrak{X} \hookrightarrow \mathcal{H}$ be reflexive Banach spaces. If $z \in \rho(A_0)$ then the restriction of $(A_0-z)^{-1}$ to $\mathfrak{X}_0$ considered as a mapping from $\mathfrak{X}_0$ to $\mathfrak{X}_0^*$ is a bounded operator which we will denote by $R_0(z)$; we define the operator $R(z) \in \mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$ in the same way. Our basic assumptions will be :

(H_2-i) $J\mathfrak{X}_0 = \mathfrak{X}$

(H_2-ii) J extends to a bounded linear map from $\mathfrak{X}_0^*$ to $\mathfrak{X}^*$ (still denoted by J).

(H_3) Let $K_0(z)$, $z \in \rho(A_0)$, be the linear map from $\mathfrak{X}_0$ to $\mathfrak{X}$ defined by

$$\begin{cases} \mathcal{D}(K_0(z)) = \{\varphi_0 \in \mathfrak{X}_0 \, , \, V(A_0-z)^{-1}\varphi_0 \in \mathfrak{X}\} \\ K_0(z)\varphi_0 = V(A_0-z)^{-1}\varphi_0 \end{cases}$$

Then $K_0(z) \in \mathcal{K}(\mathfrak{X}_0,\mathfrak{X})$, the space of compact mappings from $\mathfrak{X}_0$ to $\mathfrak{X}$.

(H_4) Let $\lambda \in \sigma(A_0)$ be such that the L.A.P. is true for A_0 at λ in $\mathfrak{X}_0$; then there exist $K_0^{\pm}(\lambda) \in \mathcal{K}(\mathfrak{X}_0,\mathfrak{X})$ such that

$$K_0^{\pm}(\lambda) = \text{norm} \lim_{\varepsilon \downarrow 0^+} K_0(\lambda \pm i\varepsilon)$$

(H_5) $u \in \mathfrak{X}^*$ and $(J^* + K_0^{\pm}(\lambda)^*)\, u = 0$ imply $u \in \mathcal{H}$ and $(A-\lambda)\, u = 0$

<u>Remarks</u> :

Under assumptions (H_2-i) J considered as a map from $\mathfrak{X}_0$ to $\mathfrak{X}$ is a semi-Fredholm operator [11] with zero deficiency. Since $K_0^{\pm}(\lambda)$ is compact, $J + K_0^{\pm}(\lambda)$ also is semi-Fredholm with finite deficiency. Assumption (H_5) allows to control the situations where this deficiency is non zero : this happens only if λ is an eigenvalue of A .

The starting point of this perturbative analysis is the following generalization of the resolvent equation

$$(20)\quad (A-z)^{-1} J = J(A_0-z)^{-1} - (A-z)^{-1} V (A_0-z)^{-1}$$

leading to

$$(21)\quad R(z)\, J = J R_0(z) - R(z)\, K_0(z)$$

The first main consequence of our assumptions is :

Theorem 4

Let the L.A.P. hold for A_0 at λ in $\mathcal{X}_0$. Then under assumptions (H_1) --- (H_5) it holds for A at λ in $\mathcal{X}$ unless $\lambda \in \sigma_p(A)$. In this case λ is an eigenvalue of finite multiplicity.

Proof : Assume first that $J+K_0^{\pm}(\lambda)$ has zero deficiency, i.e. the range of $J+K_0^{\pm}(\lambda)$ is all of $\mathcal{X}$. Then by (H_4) there exists a family $\Gamma(\lambda \pm i\varepsilon)$ of left pseudo-inverses for $J+K_0(\lambda \pm i\varepsilon)$ satisfying

$$(J+K_0(\lambda\pm i\varepsilon))\,\Gamma(\lambda\pm i\varepsilon) = 1_{\mathcal{X}}$$

and converging in norm to a left pseudo-inverse $\Gamma^{\pm}(\lambda)$ for $J+K_0^{\pm}(\lambda)$. Then

$$R(\lambda\pm i\varepsilon) = J R_0(\lambda\pm i\varepsilon)\,\Gamma(\lambda\pm i\varepsilon)$$

converges as ε tends to zero in the same topology as $R_0(\lambda\pm i\varepsilon)$ so that L.A.P. holds for A at λ.

Assume now that $J+K_0^{\pm}(\lambda)$ has non zero deficiency ; then by (H_5) λ is an eigenvalue of A. Let $\Gamma^{\pm}(\lambda)$ be a left pseudo-inverse for $J+K_0^{\pm}(\lambda)$ such that

$$(J+K_0^{\pm}(\lambda))\,\Gamma^{\pm}(\lambda) = P^{\pm}(\lambda)$$

where $P^{\pm}(\lambda)$ is a projection operator on the range of $J+K_0^{\pm}(\lambda)$; since this range has finite codimension it follows that $1-P^{\pm}(\lambda)$ is a finite rank operator. As before consider a sequence $\Gamma(\lambda\pm i\varepsilon) \in \mathcal{L}(\mathcal{X}, \mathcal{X}_0)$ of left pseudo-inverses for $J+K_0(\lambda\pm i\varepsilon)$ such that $\|\Gamma(\lambda\pm i\varepsilon) - \Gamma^{\pm}(\lambda)\| \to 0$ as $\varepsilon \to 0^+$. If $P(\lambda\pm i\varepsilon)$ is defined by

$$(22)\quad (J+K_0(\lambda\pm i\varepsilon))\,\Gamma(\lambda\pm i\varepsilon) = P(\lambda\pm i\varepsilon)$$

then one gets multiplying (21) by $\Gamma(\lambda\pm i\varepsilon)$ on the left:

$$(23)\quad R(\lambda\pm i\varepsilon)\,P(\lambda\pm i\varepsilon) = J R_0(\lambda\pm i\varepsilon)\,\Gamma(\lambda\pm i\varepsilon)$$

Now let $\psi \in \mathcal{X}$ belong to the range of $P^{\pm}(\lambda)$. Then $\forall \varphi \in \mathcal{X}$:

$$\lim_{\varepsilon\downarrow 0^+} \langle \varepsilon R(\lambda\pm i\varepsilon)\varphi, \psi\rangle = \lim_{\varepsilon\downarrow 0^+} \langle \varepsilon R(\lambda\pm i\varepsilon)\varphi, P(\lambda\pm i\varepsilon)\psi\rangle$$

In this equality we used the fact that $\varepsilon R(\lambda\pm i\varepsilon)\varphi$ is bounded in $\mathcal{H}$ hence in $\mathcal{X}^*$ and that by (22) $\|P(\lambda\pm i\varepsilon)-P^{\pm}(\lambda)\|$ tends to zero . Finally it follows from (b) and (23) that

$$(24)\quad \langle \varphi, E(\{\lambda\})\varphi \rangle = \pm \lim_{\varepsilon \downarrow 0^+} \varepsilon \langle R_0(\lambda \pm i\varepsilon) J^* \varphi, \Gamma(\lambda \pm i\varepsilon)\varphi \rangle$$

$$= 0$$

since $J^*\varphi \in \mathcal{H}_0$ by (H_2-i) and $\Gamma(\lambda \pm i\varepsilon)\psi$ converges in $\mathcal{H}_0$ to $\Gamma^{\pm}(\lambda)\psi$.
Since ψ belongs to a subspace of finite codimension in $\mathcal{H}$ then by a density argument it follows that $E\{\lambda\}$ is a finite rank projection operator ; this concludes the proof.

Remark

The operators $(I - P^{\pm}(\lambda)^*)$ are projections onto $N(J^* + K_0^{\pm}(\lambda)^*)$ the null space of $J^* + K_0^{\pm}(\lambda)^*$. Now (24) and a density argument show that $P^{\pm}(\lambda)^* E(\{\lambda\}) = 0$; hence $(I - P^{\pm}(\lambda)^*)\varphi = \varphi$ if $\varphi \in E(\{\lambda\})\mathcal{H}$. This implies that $E(\{\lambda\})\mathcal{H} \subset N(J^* + K_0^{\pm}(\lambda)^*)$
Since assumption (H_5) gives the converse inclusion one gets

$$(25)\quad N(J^* + K_0^{\pm}(\lambda)^*) = E(\{\lambda\})\mathcal{H}$$

Assuming now that λ is not an eigenvalue of A we want to construct mappings between some spaces of generalized eigenfunctions for A_0 and A with eigenvalue λ. Remember that for potential scattering such a mapping is given formaly by (6) which provides solutions of the Lippman-Schwinger equation (5). The next result shows that any generalized solution of $(A - \lambda)u = 0$ in $\mathcal{E}_\lambda$ can be identified to the sum of a solution of $(A_0 - \lambda)u_0 = 0$, $u_0 \in \mathcal{E}_\lambda^0$, and an "ingoing (or outgoing) state".

We define N_λ^0 and $\mathcal{E}_\lambda^0$ by (11) and (12) with A_0 instead of A .

Proposition 7

Under assumptions (H_1) --- (H_5) and if $\lambda \notin \sigma_p(A)$ one has

$$(J^* + K_0^{\pm}(\lambda)^*)\mathcal{E}_\lambda \subset \mathcal{E}_\lambda^0$$

Proof : By Proposition 4 it is enough to show that $(J^* + K_0^{\pm}(\lambda)^*)\varphi(\lambda) \in \mathcal{E}_\lambda^0$ if $\varphi \in \mathcal{X}$. This holds by (12) if $\langle \varphi_0, (J^* + K_0^{\pm}(\lambda)^*\varphi(\lambda)\rangle = 0 \;\; \forall \varphi_0 \in N_\lambda^0$. Now

$$\langle \varphi_0, J^* + K_0^{\pm}(\lambda)^* \varphi(\lambda)\rangle = \pm \lim_{\varepsilon \downarrow 0^+} 2i\varepsilon \langle \varphi_0, (J^* + (A_0 - (\lambda \pm i\varepsilon))^{-1} V^*) (A - (\lambda \pm i\varepsilon))^{-1} (A - (\lambda \mp i\varepsilon))^{-1} \varphi\rangle$$

By the second resolvent equation the right member of thi equality is also equal to $\pm \lim_{\epsilon \downarrow 0+} 2i\epsilon < R_0(\lambda \pm i\epsilon)\varphi_0, J^* R(\lambda \pm i\epsilon)\varphi >$
Then it follows from Schwartz inequality that

$$(26) \quad |<\varphi_0, (J^* + K_0^{\pm}(\lambda))\, \varphi(\lambda)>| \le \lim_{\epsilon \downarrow 0+} (2\epsilon \| R_0(\lambda \pm i\epsilon)\varphi_0\| \, \|R(\lambda \pm i\epsilon)\varphi\| = 2\pi \|\varphi_0(\lambda)\|_{h_0(\lambda)} \|\varphi(\lambda)\|_{h(\lambda)}$$

This completes the proof since $\varphi_0(\lambda) = 0$.

One can formulate a more intrinsic version of Proposition 7 which is a postcriori independent of the choice of auxiliary Banach spaces $\mathfrak{X}_0$ and $\mathfrak{X}$:

Proposition 8

Assume L.A.P. holds for A_0 at λ in $\mathfrak{X}_0$. Then under ssumptions (H_1)---(H_5) and if $\lambda \notin \sigma_p(A)$ there exist $Z_\pm(\lambda) \in \mathcal{L}(h(\lambda), h_0(\lambda))$ and $W_\pm(\lambda) \in \mathcal{L}(h_0(\lambda), h(\lambda))$ such that $\forall \varphi \in \mathfrak{X}$;

i) $\quad Z_\pm(\lambda)\, \varphi(\lambda) = -(J^* + K_1^{\pm}(\lambda)^*)\, \varphi(\lambda)$

ii) $\quad Z_\pm(\lambda) = W_\pm(\lambda)^*$

iii) $\quad W_\pm(\lambda)$ has dense range in $h(\lambda)$.

Proof : We define $W_\pm(\lambda)$ on the set of traces $\{\varphi_0(\lambda), \varphi_0 \in \mathfrak{X}_0\}$ by : .

$$(27) \quad W_\pm(\lambda)\varphi_0(\lambda) = (R^+(\lambda) - R^-(\lambda))(J + K_0^{\pm}(\lambda))\varphi_0$$

Then if $\varphi \in \mathfrak{X}$ one has

$$(28) \quad 2\pi <\varphi(\lambda), W_\pm(\lambda)\varphi_0(\lambda)>_{h(\lambda)} = -<(J^* + K_0^{\pm}(\lambda)^*)\, \varphi(\lambda), \varphi_0>$$

Then it follows from (26) that $W_\pm(\lambda)$ extends to a bounded operator from $h_0(\lambda)$ to $h(\lambda)$ such that $\|W_\pm(\lambda)\| \le 1$.
Properties i) and ii) follow from (28). Finally by Definition (27), $W_\pm(\lambda)\, \varphi(\lambda)$ is the spectral trace at λ of $(J + K_0^{\pm}(\lambda))\varphi_0$; since the null space of $J^* + K_0^{\pm}(\lambda)^*$ is zero the range of the semi-Fredholm operator $J + K_0^{\pm}(\lambda)$ is all of $\mathfrak{X}$. Since the set of spectral traces is dense in $h(\lambda)$ by construction property iii) follows.

Remarks

1) One does not expect in general $W_\pm(\lambda)$ to be isometric. In fact an easy calculation shows that $\|W_\pm(\lambda)\varphi_0(\lambda)\|^2_{h(\lambda)} =$

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon \pi^{-1} \| J(A_0 - (\lambda \pm i\varepsilon))^{-1} \varphi_0 \|^2 \leq \lim_{\varepsilon \downarrow 0^+} \varepsilon \pi^{-1} \| (A_0 - (\lambda \pm i\varepsilon))^{-1} \varphi_0 \|^2 = \| \varphi_0(\lambda) \|^2_{h_0(\lambda)}$$

On the other hand we will see later (see last remark) that $Z_\pm(\lambda)$ are isometric. Accordingly by ii) $W_\pm(\lambda)$ are in fact partially isometric.

2) The projection operators

$$P_\pm(\lambda) = Z_\pm(\lambda) W_\pm(\lambda)$$

satisfy .

$$\| W_\pm(\lambda) P_\pm(\lambda) \varphi_0(\lambda) \|_{h(\lambda)} = \| P_\pm(\lambda) \varphi_0(\lambda) \|_{h_0(\lambda)}$$

By the preceding remark one has

$$\| P^\pm(\lambda) \varphi_0(\lambda) \|^2_{h(\lambda)} = \lim_{\varepsilon \downarrow 0^+} \varepsilon \pi^{-1} \| J(A_0 - (\lambda \pm i\varepsilon))^{-1} \varphi_0 \|^2$$

One can show that $P_\pm(\lambda)$ are the traces on the spectral subspace $h_0(\lambda)$ of the limits (if they exist !) :

$$P_\pm = s\text{-}\lim_{t \to \pm\infty} e^{iA_0 t} J^* J \, e^{-iA_0 t}$$

Notice that $P_\pm$ commute with A_0 and that by $(H_1\text{-}i)$ $P_\pm$ are projection operators.

CONNECTION WITH TIME DEPENDENT SCATTERING THEORY

We now establish the connection between this abstract stationary construction and the time dependent approach to scattering. In the two Hilbert space formalism the wave-operators are defined in the following way, extending the definition given by V. Enss in his lectures for potential scattering :

$$W_\pm(A_0, A; J) = s\text{-}\lim_{t \to \pm\infty} e^{iAt} J e^{-iA_0 t} P_{ac}(A_0)$$

where $P_{ac}(A_0)$ denotes the orthogonal projection operator on the absolutely continuous part of A_0.

It is not difficult to show using Abel's limit that if $W_\pm(A_0, A; J)$ exist they admit the following integral representation :

$$(29) \quad \langle \varphi, W_\pm(A_0, A; J)\varphi_0 \rangle = \langle \varphi, J\varphi_0 \rangle \pm i \lim_{\varepsilon \downarrow 0^+} \int_0^{\pm\infty} dt \, e^{-\varepsilon|t|} \langle e^{-iAt}\varphi, V e^{-iA_0 t}\varphi_0 \rangle$$

for all $\varphi \in \mathcal{H}$ and $\varphi_0 \in \mathcal{H}_{0ac}(A_0) \cap \mathcal{D}(A_0)$. Notice that since by $(H_1\text{-}ii)$ V is A_0-bounded the integrand in (29) is bounded by $Ce^{-\varepsilon|t|}$ for some constant C depending only on $\|\varphi\|_{\mathcal{H}}$ and the A_0-graph norm of φ_0 .

The representation (29) is essential for the proof of the main result of this section. Before stating it we need the following :

Definition 3

Let $I \subset R$ be some closed interval. We will say that I is Regular (with respect to $A_0, A; \mathfrak{X}_0, \mathfrak{X}$) if the following properties hold :

i) L.A.P. holds for A_0 in $\mathfrak{X}_0$ on I .

ii) Let $I_\pm = \{z \in \mathbb{C}, z = \lambda \pm i\varepsilon, \varepsilon > 0, \lambda \in I\}$, then $R_0(z)$ is uniformly bounded in $\mathcal{L}(\mathfrak{X}_0, \mathfrak{X}_0^*)$ on $I_\pm$.

iii) $K_0(z)$ is continuous for the norm topology on $I_\pm$ with continuous boundary values on both sides of $I_\pm$.

We will need the following :

Lemma 1

Assume (H_1) --- (H_5) are satisfied and let I be a bounded regular interval. If $\psi_0 \in \mathfrak{X}_0$ and $\varphi_0 = E_0(I)\psi_0$ one has:

i) $\varphi_0(\lambda) = \psi_0(\lambda) \quad \forall \lambda \in I$

ii) $V e^{-iA_0 t}\varphi_0 \in \mathfrak{X} \quad \forall t \in \mathbb{R}$

iii) $\langle u, V e^{-iA_0 t}\varphi_0\rangle = \int d\lambda \exp(-i\lambda t)\langle u, (K_0^+(\lambda) - K_0^-(\lambda))\psi_0\rangle$
$\forall u \in \mathfrak{X}^*$

Proof : By the functional calculus one has $\varphi_0 = \int_I^{\oplus} \psi_0(\lambda)\, d\lambda$ which implies i). Consider now the $\mathfrak{X}$-valued integral $\Psi_t = \int_I d\lambda \exp(-i\lambda t)(K_0^+(\lambda) - K_0^-(\lambda))\psi_0$. By iii) of Def. 3 one has $\|\Psi_t\|_{\mathfrak{X}} \leq \int_I d\lambda \|(K_0^+(\lambda) - K_0^-(\lambda))\psi_0\|_{\mathfrak{X}}$. The proof of iii) is an elementary exercise in functional calculus when u belongs to the dense domain of V^* i.e. $V^* u \in \mathfrak{X}_0$; for general $u \in \mathfrak{X}^*$ it follows by a continuity argument. Obviously iii) implies $\Psi_t = V e^{-iA_0 t}\varphi_0$ hence ii).

We can now state the

Theorem 5

Assume $W_\pm(A_0, A; J)$ exist and (H_1)---(H_5) are satisfied. Then for any bounded regular interval I such that $I \cap \sigma_p(A) = \emptyset$ one has $\forall \varphi \in \mathcal{H}$ and $\forall \varphi_0 \in E_0(I)\mathcal{H}_0$:

(30) $$\langle \varphi, W_\pm(A_0,A;J)\,\varphi_0\rangle = \int_I d\lambda\, \langle \varphi(\lambda), W_\pm(\lambda)\varphi_0(\lambda)\rangle_{h(\lambda)}$$

Proof : Since $E_0(I)\mathfrak{X}_0$ is dense in $E_0(I)\mathcal{H}_0$ and $W_\pm(\lambda)$, $\overline{W_\pm}(A_0,A;J)$ are bounded, it is enough to show (30) with $\varphi_0 = E_0(I)\psi_0$ for some $\psi_0 \in \mathfrak{X}_0$ and with $\varphi \in \mathfrak{X}$. By the lemma above one has

$$\int_0^{\pm\infty} e^{-\varepsilon|t|}\langle e^{-iAt}\varphi, V e^{-iA_0t}\varphi_0\rangle$$
$$= (2i\pi)^{-1}\int_0^{\pm\infty} dt\, d\lambda \,\langle e^{-i(A-(\lambda\pm i\varepsilon))t}\varphi, (K_0^+(\lambda)-K_0^-(\lambda))\psi_0\rangle$$

The integrand on the r.h.s. of this equality is in $L^1(\mathbb{R}^\pm \times I)$; in fact it is bounded by $Ce^{-\varepsilon|t|}\,\|\varphi\|\,\|\psi_0\|_{\mathfrak{X}_0}$ with $C = \sup_{\lambda\in I}\|K_0^\pm(\lambda)\|$. So by Fubini's theorem we can integrate first on λ obtaining by the functional calculus:

$$\int_0^{\pm\infty} dt\, e^{-\varepsilon|t|}\langle e^{-iAt}\varphi, V e^{-iA_0t}\varphi_0\rangle$$
$$= \mp(2i\pi)^{-1}\int_I d\lambda \,\langle R(\lambda\pm i\varepsilon)\varphi, (K_0^+(\lambda)-K_0^-(\lambda))\psi_0\rangle$$

Now since I is regular it follows from (21) and the arguments used in the proof of Theorem 4 that R(z) also is uniformly bounded on I . Since I is compact one obtains by Riemann-Lebesgue Lemma :

$$\lim_{\varepsilon\downarrow 0^+}\int_I d\lambda\,\langle R(\lambda\pm i\varepsilon)\varphi, (K_0^+(\lambda)-K_0^-(\lambda))\psi_0\rangle$$
$$= \int_I d\lambda\,\langle R^\pm(\lambda)\varphi, (K_0^+(\lambda)-K_0^-(\lambda))\psi_0\rangle$$

whence finally by (29) :

$$\langle \varphi, W_\pm(A_0,A;J)\varphi_0\rangle$$
$$= -(2i\pi)^{-1}\int_I d\lambda \langle [(R_0^+(\lambda)-R_0^-(\lambda))J^* - (K_0^+(\lambda)-K_0^-(\lambda))^* R^\pm(\lambda)]\varphi, \psi_0\rangle$$

It is easy to show using (21) in the limit $z = \lambda \pm i0$ that

(31) $$(R_0^+(\lambda)-R_0^-(\lambda))J^* - (K_0^+(\lambda)-K_0^-(\lambda))^* R^\pm(\lambda)$$
$$= (J^*+K_0^\pm(\lambda))^*(R^+(\lambda)-R^-(\lambda))$$

so that the proof is completed with the help of Proposition 8.

ASYMPTOTIC COMPLETENESS

The Theorem 5 combined with Proposition 8 provides a very powerful way to prove asymptotic completeness of wave-operators. We will need the following extra assumptions:

(H_6-i) There exists a denumerable family of disjoint open intervals $(I_n)_{n\in\mathbb{N}}$ such $\bigcup I_n = \mathbb{R}$ and the L.A.P. holds for A_0^n on each I_n in $\mathcal{X}_0^n$.

(H_6-ii) Any closed interval I such that $I \subset I_n$ for some n is regular.

Let us show first

Proposition 9

Under assumptions (H_1) --- $(H_5)(H_6)$ the L.A.P. holds for A on each I_n except possibly at some isolated points which can accumulate only at ∂I_n and are eigenvalues of A with finite multiplicity.

Proof : By Theorem 4 it is enough to show that $\sigma_p(A) \cap I_n$ consists of isolated eigenvalues. Now $\lambda \in \sigma_p(A) \cap I_n$ if and only if the deficiency of $J + K_0^I(\lambda)$ is non-zero. By the continuity of $K_0^I(\lambda)$ on I_n following from (H_6-ii) and Fredholm operator theory [11,16], this happens either on all of I or at some isolated points of I. The first possibility being obviously excluded one gets the stated property.

Theorem 6

If $W_\pm(A_0,A;J)$ exist and (H_1) --- (H_6) hold one has

$$W_\pm(A_0,A;J)\mathcal{H}_{ac}(A_0) = \mathcal{H}_{ac}(A)$$

Proof : Assume φ is in the orthogonal complement of $W_\pm(A_0,A;J)\mathcal{H}_{ac}(A_0)$, then $\forall \varphi_0 \in \mathcal{H}_0$:

$$\langle \varphi, W_\pm(A_0,A;J)\,E_0(I)\varphi_0\rangle = 0$$

for any regular interval I since then $E_0(I)\mathcal{H}_0 \subset \mathcal{H}_{ac}(A_0)$ by Proposition 1. In particular if $I \cap \sigma_p(A) = \emptyset$ one has by (30) :

$$\langle \varphi, W_\pm(A_0,A;J)\,f(A_0)E_0(I)\varphi_0\rangle$$
$$= \int_I d\lambda\, f(\lambda)\langle \varphi(\lambda), W_\pm(\lambda)\varphi_0(\lambda)\rangle_{h(\lambda)} = 0$$

for all $f \in L^\infty(I)$ and $\varphi_0 \in E_0(I)\mathcal{H}_0$. Then almost everywhere on I :

$$\langle \varphi(\lambda), W_+(\lambda)\varphi_0(\lambda)\rangle_{h(\lambda)} = 0$$

Taking a basis $(\varphi_{0,n})_n$ of $\mathcal{H}_0$ one has $\langle \varphi(\lambda), W_\pm(\lambda)\varphi_{0,n}(\lambda)\rangle_{h(\lambda)} = 0$ for all n almost everywhere. Since $(\varphi_{0,n}(\lambda))_n$ is a total set in $h_0(\lambda)$ and by Proposition 8 iii) it follows that $\varphi(\lambda) = 0$ almost everywhere on I and accordingly $E(I)\varphi = 0$ since the spectral measure of A is absolutely continuous on I by Proposition 1.

Now by assumption $(H_6$ ii) this implies $E(I_n \setminus \sigma_p(A))\varphi = 0$ for all n ; so $E(\cdot)\varphi$ has support on $(\bigcup_n \partial I_n) \cup \sigma_p(A)$ and accordingly $\varphi \in \mathcal{H}_{ac}^\perp(A)$

Remarks

1) In general the assumptions (H_1) --- (H_6) don't imply existence of wave-operators but only of the weak Abel limits

$$w\lim_{\varepsilon\to 0^+}\int_0^{\pm\infty} dt\, e^{-\varepsilon|t|} e^{iAt} J e^{-iA_0 t} P_{ac}(A_0)$$

2) Let I be a regular interval such that $I \cap \sigma_p(A) = \emptyset$. Then one can show that asymptotic completeness implies existence of the strong limits :

$$W_\pm^*(A_0, A; J)E(I) = s\text{-}\lim_{t\to\pm\infty} e^{iA_0 t} J e^{-iAt} E(I)$$

Accordingly if $\varphi \in E(I)\mathcal{H}$ one has

$$(32) \qquad \|W_\pm^*(A_0, A; J)\varphi\| = \lim_{t\to\pm\infty} \|J^* e^{-iAt}\varphi\| = \|\varphi\|$$

On the other hand Theorem 5 and Proposition 8 ii) imply :

$$\|W_\pm^*(A_0, A; J)\varphi\|^2 = \int_I d\lambda \langle Z_\pm(\lambda)\varphi(\lambda), W_\pm^*(A_0,A;J)\varphi\rangle$$

$$= \int_I d\lambda\, \|Z_\pm(\lambda)\varphi(\lambda)\|^2_{h_0(\lambda)} = \int_I d\lambda \|\varphi(\lambda)\|^2_{h(\lambda)}$$

Since this remains true when one replaces φ by $f(A)\varphi$, $f \in L_\infty(I)$, one obtains finally by (32)

$$\|Z_\pm(\lambda)\varphi(\lambda)\|^2_{h_0(\lambda)} = \|\varphi(\lambda)\|^2_{h(\lambda)}$$

for all $\varphi \in \mathcal{H}$. This shows that $Z_\pm(\lambda)$ is isometric for all $\lambda \in I$.

EXAMPLES

Potential Scattering.

Let us come back to the situation described by V. Enss ; it corresponds to $A_o = -\Delta$ on $\mathcal{H} = L^2(\mathbb{R}^n)$ and $A = -\Delta + V$ where V is a real multiplicative operator on $\mathcal{H} = \mathcal{H}_o = L^2(\mathbb{R}^n)$; here we take $J = 1$. We assume for simplicity $\mathcal{D}(A) = \mathcal{D}(A_o)$ in accordance with $(H_1\text{-ii})$. Following S. Agmon[6] we define :

Definition 4

V is a short range potential if $(1+|X|^2)^{\frac{1+\varepsilon}{2}}\, V(X)$ is a compact mapping from $\mathcal{H}_2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for some $\varepsilon > 0$.

Now the mapping $u(x) \longrightarrow (1+|X|^2)^{-\frac{1+\varepsilon}{2}}\, u(x)$ is bounded from $\mathcal{H}_{m,s}(\mathbb{R}^n)$ to $\mathcal{H}_{m,s+1+\varepsilon}(\mathbb{R}^n)$ for all $m \in \mathbb{N}$, $s \in \mathbb{R}$ where

$$\mathcal{H}_{m,s}(\mathbb{R}^n) = \left\{ u \in L^2_{loc}(\mathbb{R}^n) \ , (1+|X|^2)^{s/2} D^\alpha u \in L^2(\mathbb{R}^n), |\alpha| \leq m \right\}$$

Then for all $s \in \mathbb{R}$ multiplication by V also is a compact mapping from $\mathcal{H}_{2,s}(\mathbb{R}^n)$ to $L^2_{s+1+s}(\mathbb{R}^n)$.

The short range class contains potentials bounded at large distances by $C(1+|X|^2)^{\frac{-1-\varepsilon}{2}}$ for some $C, \varepsilon > 0$ and satisfying locally a regularity assumption $V \in L^p_{loc}(\mathbb{R}^n)$ with $p = 2$ for $n \leq 3$ and $p > \frac{n}{2}$ for $n \geqslant 4$. This regularity assumption guarantees $\mathcal{D}(A) = \mathcal{D}(A_o)$.

Now with the choice

$$\mathcal{X}_o = \mathcal{X} = L^2_{\frac{1+\varepsilon}{2}}(\mathbb{R}^n)$$

assumptions (H_2) and (H_3) are satisfied. To control (H_4) notice that

$$V(A_o - (\lambda \pm i\varepsilon))^{-1} = V(A_o+1)^{-1} + (\lambda - 1 \pm i\varepsilon) V(A_o+1)^{-1} (A_o - (\lambda \pm i\varepsilon))^{-1}$$

By Theorem 1 and compactness of $V(A_o+1)^{-1}$ as a mapping from $\mathcal{X}_o^*$ to $\mathcal{X}_o$ one gets easily in view of (H_4) and (H_6) :

Proposition 10

Any closed interval $I \subset \mathbb{R} \setminus \{0\}$ is regular.

It remains to verify (H_5) ; for this we need the following results :

<u>Lemma 2</u>

Let $\lambda \neq 0$; then for all $s > \frac{1}{2}$ there exists a finite positive constant C such that

$$\| R_0^{\pm}(\lambda) f \|_{\mathcal{H}_{2,s-1}} \leq C \| f \|_{L^2_s}$$

for all

$$f \in N^0_{\lambda,s} = \left\{ f \in L^2_s(\mathbb{R}^n) \, , \, \mathrm{Im} \langle f, R_0^{\pm}(\lambda) f \rangle = 0 \right\}$$

<u>Lemma 3</u>

Assume $(I + K_0^{\pm}(\lambda)) f = 0$ for some $f \in L^2_s(\mathbb{R}^n)$ with $s \geq \frac{1+\varepsilon}{2}$; then $f \in N^0_{\lambda,s}$

<u>Proof</u> : Assume $(I + K_0^{\pm}(\lambda)) f = 0$, $f \in L^2_s(\mathbb{R}^n)$ and let $u = R_0^{\pm}(\lambda) f$, $u \in \mathcal{H}_{2,-s}(\mathbb{R}^n)$.

Then $f = -Vu$ and

$$\begin{aligned} \mathrm{Im} \langle f, R_0^{\pm}(\lambda) f \rangle &= \mathrm{Im} \langle f, u \rangle \\ &= -\mathrm{Im} \langle Vu, u \rangle \\ &= \lim_{\varepsilon \downarrow 0^+} \langle V(H_0 - (\lambda \pm i\varepsilon))^{-1} f, (H_0 - (\lambda \pm i\varepsilon))^{-1} f \rangle \end{aligned}$$

$= 0$ since V is symmetric.

Then we have :

<u>Proposition 11</u>

The null space of $I + K_0^{\pm}(\lambda)^*$ is contained in $\mathcal{H}_{2,s}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

<u>Proof</u> : By the Fredholm alternative the null spaces of $I + K_0^{\pm}(\lambda)$ and $I + K_0^{\pm}(\lambda)^*$ have the same dimension. One can easily construct explicitely a bijective map between these spaces by showing that if $u = R_0^{\pm}(\lambda) f$ one has :

$$(I + K_0^{\pm}(\lambda)) f = 0 \iff (I + K_0^{\pm}(\lambda)^*) u = 0 \tag{33}$$

Then the Proposition will follow from Lemmas 2 and 3 if we show that the null space of $I + K_0^{\pm}(\lambda)$ is contained in $L^2_s(\mathbb{R}^n)$ for all s . Now by Lemma 2 $f \in L^2_{\frac{1+\varepsilon}{2}}$ implies $u \in \mathcal{H}_{2,-\frac{1}{2}+\frac{\varepsilon}{2}}$ hence $f = -Vu \in L^2_{\frac{1+2\varepsilon}{2}}$. Repeating this argument n times gives $f \in L^2_{\frac{1+n\varepsilon}{2}}$ for all n ; this concludes the proof.

Corollary :

If $(I + K_0^{\pm}(\lambda)^{*}) u = 0$ for some $\lambda \neq 0$; then $u \in \mathcal{H}_{2,s}(\mathbb{R}^n)$ for all $s \in R$ and $(A-\lambda) u = 0$.

Proof : It remains to show that u is an eigenvector of A. For all $g \in C_0^{\infty}(\mathbb{R}^n)$ one has :

$$\langle (-\Delta-\lambda) g, u \rangle = \lim_{\varepsilon \downarrow 0^+} \langle (-\Delta-\lambda) g, (-\Delta-(\lambda \pm i\varepsilon))^{-1} f \rangle$$
$$= \langle g, f \rangle$$

so that $(-\Delta-\lambda) u = f$ in the sense of distributions. But this equality also holds in $L^2(\mathbb{R}^n)$ since $u \in \mathcal{H}_{2,s}(\mathbb{R}^n)$ for all s . Finally $f = -Vu$ concludes the proof.

Now all assumptions (H_1) --- (H_6) are satisfied so that we have :

Theorem 7

Let V be a short-range potential. Then :

i) $W_{\pm}(A_0, A; J)$ exist and are complete.

ii) The continuous singular part of A is empty.

iii) The point spectrum of A outside zero consists of isolated eigenvalues with finite multiplicity. The corresponding eigenfunctions are in $\mathcal{H}_{2,s}(\mathbb{R}^n)$ for all $s \in R$.

Proof : We refer to V. Enss' lectures for the proof that wave-operators exist. Completeness follow then from Theorem 6. Assertion ii) and the first part of iii) follow from Propositions 1 and 9 ; (in fact (H_6-i) holds with the two sets $I_1 = \mathbb{R}^- \setminus \{0\}$ and $I_2 = \mathbb{R}^+ \setminus \{0\}$). Decay properties follow from (25) and the above Corollary.

The Three-Body Problem.

Let us consider now a three-particle quantum system with local pair interactions $V_{ij}(X_i - X_j)$, $i = j$, where X_i denotes the position of particle i. Assuming that these interactions vanish at infinity there are four different types of "asymptotic freedom" corresponding respectively to three infinitely separated particles not interacting anymore or one cluster $\alpha = (ij)$ of two particles not interacting with k , $(ijk) = (123)$. We introduce an associated partition of unity with the help of C^{∞} functions J_D, where D labels the above four cluster decompositions. We require the following properties :

(35.i) J_D is homogeneous of degree 0.

(35.ii) $\sum_D J_D^2 = 1$

(35.iii) For some positive constant a, $\mathrm{Supp}\, J_D \subset \{|x_i - x_j| > a|x|\}$ if i,j belong to different clusters of D. Here $|X|$ denotes the total length of the vector X representing the three-particle configuration in the center of mass system.

The unperturbed operator A_0 is chosen as

$$A_0 = \oplus H_D$$

acting on $\mathcal{H}_0 = \bigoplus_D L^2(\mathbb{R}^6)$ where H_D is obtained from the Hamiltonian

$$A = H_0 + \sum_{1 \le i < j \le 3} V_{ij}$$

by removing interactions between particles belonging to different clusters of D. The kinetic energy operator H_0 (in the center of mass system) is a six-dimensional Laplacian with coefficients depending on the masses of the particles. Using Jacobi coordinates one can show that each two cluster Hamiltonian H_D has the structure described in Proposition 6.

$$H_D = \frac{P_k^2}{2\mu_k} \otimes I_{L^2(x_i - x_j)} + I_{L^2(y_k)} \otimes h_\alpha \tag{34}$$

where (y_k, p_k) are the relative position and momentum operators for particle k with respect to the center of mass of (i,j) with $(i,j,k) = (1,2,3)$. The operator h_α is the two-particle Hamiltonian for the pair α with interaction V_{ij}. If we assume that two-particle interactions are short-range, h_α is known by the preceeding example to satisfy the L.A.P. outside the set

$$\Sigma_\alpha = \{0\} \cup \sigma_p(h_\alpha)$$

in $L^2_{\frac{1+\varepsilon}{2}}(\mathbb{R}^3)$ for some $\varepsilon > 0$. Since $P_k^2/2\mu_k$ also does outside zero it follows from Proposition 6 that H_D satisfies the L.A.P. in $L^2_{\frac{1+\varepsilon}{2}}(\mathbb{R}^6)$ outside Σ_α. Finally A_0 satisfies the L.A.P. in

$$\mathcal{X}_0 = \bigoplus_D L^2_{\frac{1+\varepsilon}{2}}(\mathbb{R}^6)$$

in $\mathbb{R} \setminus \Sigma$ where

$$\Sigma = \{0\} \cup \left(\bigcup_\alpha \Sigma_\alpha\right)$$

Now the Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}^6)$ and we choose

$\mathcal{H} = L^2_{\frac{1+\varepsilon}{2}}(\mathbb{R}^6)$. We construct the identification map as follows : let $\varphi_0 = \underset{D}{\oplus} \varphi_D$ be an arbitrary element of $\mathcal{H}_0$. Then

$$J\varphi_0 = \sum_D J_D \varphi_D$$

It is easy to show that if $\varphi \in \mathcal{H}$ then $J^* \varphi = \underset{D}{\oplus} J_D \varphi$

By (35.ii) one has then $JJ^* = 1$ and obviously $J\mathcal{H}_0 = \mathcal{H}$.
The perturbation V is given by

$$(36) \quad V(\underset{D}{\oplus}\varphi_D) = \sum_D J_D V_D \varphi_D + \sum_D [H_0, J_D]\varphi_D$$

Assumption (35.iii) implies that the terms $J_D V_D$ decay at infinity like $|X|^{-1-\varepsilon}$. Homogeneity of J_D implies on the other hand that $[H_0, J_D]$ is a differential operator of order one with coefficients tending to zero at infinity like $|X|^{-1}$. Accordingly this term cannot be treated by the usual method for short-range potentials since it is not a compact mapping from $\mathcal{H}_{2,-s}(\mathbb{R}^6)$ to $L^2_s(\mathbb{R}^6)$ for some $s > \frac{1}{2}$.

However one can expect that V falls into the class of perturbations satisfying the general assumptions (H_4) and (H_6). Let us motivate this hope by the following remarks.

1) The "effective interaction" VJ^* is short-range. This was already mentioned for the local multiplicative term in V in the r.h.s. of (36). Furthermore one has

$$\sum_D [H_0, J_D] J_D = - \sum_D J_D [H_0, J_D]$$

by (35.ii). Accordingly

$$(37) \quad \sum_D [H_0, J_D] J_D = \frac{1}{2} \sum_D [[H_0, J_D], J_D]$$

$$= \frac{1}{2} \sum_D |\nabla J_D|^2$$

where ∇ denotes the 6-dimensional gradient modulo some kinematical factors. By (37) the second term on the r.h.s. of (36) gives a contribution to the effective interaction which decays like $|X|^{-2}$. However we have chosen in these lectures to not present the abstract stationary theory in terms of the effective interaction so we will not develop this argument any longer.

2) The main reason why $K_0^{\pm}(\lambda)$ might satisfy (H_4) although V is not short-range is the radiation condition satisfied

by ingoing or outgoing states $R_o^{\pm}(\lambda) f$, $f \in B(\mathbb{R}^{\ell})$. One can easily convince oneself by looking at Agmon-Hörmander's form of the radiation condition (Theorem 3) or at Theorem 6.3 of their paper that $[H_o, J_b]$ acting on states satisfying a radiation condition improves the decay a little bit more than just $|x|^{-1}$; this suggests strongly working in Besov space (18) instead of $L_s^{\ell}(\mathbb{R})$, $s > \frac{1}{2}$.

3) To control the "Fredholm alternative" with the help of (H_s) in potential scattering a very useful tool was provided by the trace result of Lemma 2 . It turns out that a trace property can be proved inductively for operators A of the type described in Proposition 6 provided it holds for A_1 and A_2. This brings another essential tool for an analysis of the N-body problem along lines identical to those of potential scattering modulo the geometrical methods.

To conclude this short presentation of the geometrical ideas in many particle scattering problems, let me mention the recent work of P. Perry, I. Sigal, and B. Simon[17]. These authors show absence of continuous singular spectrum and discreteness of eigenvalues by Mourre's method[18]. The interested reader can find there more technical details on the geometrical approach.

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