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NOTES ON GAUGE THEORY AND GRAVITATION+

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Abstract

In order to investigate whether A. Einstein's general relativity theory (GRT) fits into the general scheme of a gauge theory, first the concept of a (classical) gauge theory is outlined in an introductionary spacetime approach. Having thus fixed the notation and the main properties of gauge fields, GRT is examined to find out what the gauge potentials and the corresponding gauge group might be. In this way, we are led to the possibility of interpreting GRT as a gauge theory of the 4-dimensional translation group $T(4) = (\mathbb{R}^4,+)$, and where the gauge potentials are incorporated in a T(4)-invariant way via orthonormal anholonomic basis 1-forms. To include also the spin aspect we just indicate a natural extension of GRT by gauging also the Lorentzgroup, whereby a Riemann-Cartan spacetime (U_4-spacetime) comes into play. As usual in our papers, the calculus of exterior forms is used throughout. As, however, our notation has got up for itself a little bit during the course of time, a short overview is given in Sec. 2.

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1. Introduction

1.1 What is a gauge theory? Nowadays, classical (= nonquantized) gauge theory is formulated in terms of principal fiber bundels with connection and associated vectorbundles [1]. But the use of these modern concepts, although highly desired, sometimes thrusts aside the origin and the underlying idea of what is called a "gauge procedure" or "gauge principle". Therefore, in Sec. 3, we revive the idea of introducing gauge fields (potentials) as (exterior) compensating fields in order to regain invariance of any field theory under their symmetry group after the group action has become spacetime dependent ("local gauge transformation", the transformation of a field under a symmetry group with spacetime dependent parameters [2]). In this way, gauge fields are closely linked to the invariance properties of the corresponding Lagrangian, and as a consequence, are incorporated in certain identities arising from that invariance. All this together amounts in a list of certain properties gauge fields have to obey and by which it is summarized what we want to call a (classical) "gauge theory" (see end of Sec. 3).

1.2 What has gauge theory to do with gravitation? The best theory of gravitation still available is Einstein's general relativity theory (GRT), no doubt. It is well-tested enough, simple enough (compared to the richness of its consequences) and beautiful enough (in particular because of its geometric-mathematical structure) to let them both feel enthusiastic, the pragmatician and the esthetical idealist. Nevertheless, there are still some essential problems in this theory:

(1) GRT contains singularities, which cause some problems in understanding the physical consequences of high densities of matter as well as the role of a singularity on a manifold at all (which are contrary concepts by definition).

(2) GRT still resists quantization already in the linearized case, not to speak of the full theory because of its nonlinear character.

(3) GRT still resists unification with other physical theories, in particular because of the problems mentioned in (1) and (2).

To overcome these defects, one is well advised to look at other physical theories, which, to some extend, submit a unification and quantization (including renormalization) scheme, arising from what we called before a "gauge theory" [3]. According to this one expects that, once GRT is reconsidered and reformulated as a gauge theory similar to the theories of weak, electromagnetic and strong forces, gravity will be mollified to fit into the general scheme, thus leading to a unified theory of all known physical forces.

But all these hopes and expectations are, however, very speculative at the moment and as far as I can see, there is no immediate evidence for solving all problems merely by formulating gravity as an (perhaps extended) gauge theoretical version of GRT. Therefore I want to stress another aspect which is very close at hand: As mentioned above, (classical) gauge theory has experienced a considerable amount of geometrization the last decade, where the notion of fiberbundles has taken place in it. Nowadays, gauge theory is as imbued with geometrical ideas as GRT, so it is simply desirable to combine these theories.

1.3 How do we proceed? Several attempts were made to formulate any gravity theory as a gauge theory of a certain group, but most of them were guided by what one wants and not what one already has, thus overlooking sometimes the best theory of gravitation already available: GRT. Here, we try a different approach. Starting in Sec. 3 with a simple introduction into the gauge idea in general and ending up with the main properties of gauge fields, we go on and look at GRT, analyse the basic variables and investigate whether they may be interpreted as gauge fields and what the corresponding gauge group might be (Sec. 4). For that purpose, the formulation of gauge theory at the same footing as GRT (over spacetime - no bundle viewpoint) happens to be the most appropriate one.

As it turns out, we shall interpret GRT as a gauge theory of the 4-dimensional translation group $T(4) = (\mathbb{R}^4,+)$, where general coordinate transformations are interpreted as local (= spacetime dependent) translations. To include also the spin aspect, we just line out the further gauge procedure for the Lorentzgroup, thus arriving at a Riemann-Cartan (U_4-) spacetime as underlying manifold (Sec. 5). Finally, the main results and aspects are summarized in Sec. 6.

1.4 What is assumed? In general, we assume spacetime M^4 to be a 4-dimensional connected paracompact C^∞-manifold, endowed with an indefinite metric of signature 1 (hyperbolic manifold, nomenclature follows [4]). For physical applications, matter fields are represented by vector (spinor-) valued p-forms ϕ over spacetime ($0 \leq p \leq 4$; see Sec. 2), whose behaviour is determined by field equations (of at most second order) which are the Euler-Lagrange equations of an action integral. Therefore we assume that the variational principle of classical field theory as well as some basic facts of conventional GRT are known to the reader (see [5], [6]). The reader is further assumed to be familiar with the calculus of exterior forms which is used throughout for the sake of simplicity and best suitability to the problem. As sometimes our point of view will depart slightly from usual exterior calculus, we summarize the characteristic features of our notation in Sec. 2.

Most of the calculations are omitted and left as an exercise. The main reference for detailed calculations is Ref. [7], where the results of this paper are contained as a by-product.

2. Exterior Forms

2.1 Let M^n be a n-dimensional manifold and $E_p(M^n) := E_p(M^n,\mathbb{R})$ the module of (scalar valued) exterior p-forms over $E_o(M^n) := C^\infty(M^n;\mathbb{R})$, the set of smooth mappings from M^n to the real numbers $\mathbb{R}$. The direct sum of $E_p(M^n)$, $p = 0,1,\ldots,n$, is then widened to a (graded) algebra by the componentwise extended exterior product

$$\wedge : E_p \times E_q \to E_{p+q} \qquad \forall\, p,q \in [0,n] \;, \quad p + q \leq n$$

which is defined here in the following construction of a p-basis $\{e^{i_1\ldots i_p}\}$, $i_k = 1,\ldots,n\ \forall\, k$, of E_p out of a 1-basis $\{e^i\}$, $i = 1,\ldots,n$, of E_1:

$$e^{i_1\ldots i_p} := e^{i_1} \wedge \ldots \wedge e^{i_p} := p!\; e^{[i_1} \otimes \ldots \otimes e^{i_p]} \tag{2.1}$$

where $\otimes$ denotes tensor product and square brackets antisymmetrization as usual. Thus any p-form $\phi \in E_p$ can be written as

$$\phi = \phi_{|i_1 \ldots i_p|}\, e^{i_1 \ldots i_p} , \qquad \phi_{i_1 \ldots i_p} = \phi_{[i_1 \ldots i_p]} \tag{2.2}$$

where vertical bars demand summation over $i_1 < i_2 < \ldots < i_p$. From (2.1/2) it follows the well known commutation law

$$\phi \wedge \psi = (-)^{pq}\, \psi \wedge \phi \quad \text{if } \phi \in E_p,\ \psi \in E_q . \tag{2.3}$$

Recall, the exterior product of a p-form ϕ and a function f is simply $f \wedge \phi := f \cdot \phi$. Nevertheless, the wedge sign will be used sometimes in this case either. Differentiation is represented by the exterior derivative $d: E_p \to E_{p+1}\ \forall\, p \in [0,n-1]$, which is uniquely defined by the properties

(1) d is $\mathbb{R}$-linear

(2) $d(\phi \wedge \psi) = d\phi \wedge \psi + (-)^p\, \phi \wedge d\psi \qquad \forall\, \phi \in E_p$

(3) $d \circ d \equiv 0$ (2.4)

(4) df = ordinary differential of $f \in E_0$

(5) d is local (domain of ϕ = domain of $d\phi\ \forall\, \phi$).

2.2 If the manifold M^n admits a pseudo-Riemannian metric g, we use the latter to define a scalar product in $E_1(m,\mathbb{R})\ \forall\, m \in M^n$, simply defined by $(<,>: E_1 \times E_1 \to E_0)$

$$<e^i, e^j> := g^{ij} \in E_0 , \qquad \forall\, i,j = 1,\ldots,n , \tag{2.5}$$

where g^{ij} are the inverse of the components g_{ij} of g with respect to the basis $\{e^i\}$: $g = g_{ij}\, e^i \otimes e^j$. As the metric g provides an isomorphism between vector fields and 1-forms, we define a dual 1-form basis $\{e_j\}$ to $\{e^k\}$ with respect to g by

$$e_i := g_{ij}\, e^j \in E_1 . \tag{2.6}$$

We call it "dual" because it is the unique 1-form basis corresponding to the dual vector basis $\{\xi_i\}$ $(e^i(\xi_j) = \delta^i{}_j)$ by $e_i = g(\xi_i, \cdot)$ and therefore $<e_i, e^j> = \delta_i{}^j$.

Note: Basis and dual basis with respect to g are both 1-forms but the latter is closely linked to (and represents therefore) the metric g (or vice versa).

There is a natural extension of <,> introduced by the inner product (or contraction) $i\colon E_1 \times E_p \to E_{p-1}$ $(E_{-1} := \emptyset)$, $i(\alpha)\phi := i(\alpha,\phi)$, with the defining properties $(\alpha \in E_1)$:

$$\begin{aligned} &(1) \quad i(\alpha)(\phi \wedge \psi) = i(\alpha)\phi \wedge \psi + (-)^p \phi \wedge i(\alpha)\psi \qquad \forall \phi \in E_p \\ &(2) \quad i(\alpha)f = 0 \qquad \forall f \in E_o \\ &(3) \quad i(\alpha)e^i = \alpha^i \qquad (\alpha = \alpha^i e_i = \alpha_i e^i) \ . \end{aligned} \tag{2.7}$$

Exterior derivative and contraction define the Lie derivative $\ell\colon E_1 \times E_p \to$ $\to E_p$, $\ell(\alpha,\phi) =: \ell(\alpha)\phi$, where

$$\ell(\alpha) := i(\alpha)d + di(\alpha) \qquad \forall \alpha \in E_1 \ . \tag{2.8}$$

Note: $\ell(\alpha)\phi \equiv \ell(\tilde{\alpha})\phi$, where $\tilde{\alpha}$ is the corresponding vector field to $\alpha \in E_1$ (i.e. $\alpha =: g(\tilde{\alpha},\cdot)$).

2.3 Let M^n be orientable, i.e. there exists a continuous nonvanishing $\varepsilon \in E_n(M^n)$. We define the metric volume element by

$$\varepsilon := \sqrt{(-)^s g} \quad e^{1\ldots n} \in E_n \tag{2.9}$$

where $g = \det(g_{ij}) = \det(<e_i,e_j>)$ and s = signature of g. The inner product together with ε provide a linear isomorphism $*\colon E_p \to E_{n-p}$ (Hodge-star-operator), defined for the basis as follows

$$*e^{i_1\ldots i_p} := i(e^{i_p}) \ldots i(e^{i_1})\varepsilon \ . \tag{2.10}$$

Just as ε provides a volume element for M^n, any p-form ϕ defines a p-dimensional volume element and hence may be integrated over a p-dimensional compact submanifold N of M^n, the corresponding integral is denoted by $\int_N \phi$ (see [4] for details).

2.4 Vector-valued forms. Let V denote a m-dimensional vector space and $E_p(M^n,\mathbb{R}) =: E_p(M^n)$ the module of scalar ($\mathbb{R}$-) valued p-forms, as above. Elements of $E_p(M^n,V) := E_p(M^n) \otimes V$ will be called V-valued p-forms over M^n. To clarify the notation, recall that elements of $\overset{p}{\otimes} T^*_m M^n \otimes V$ may be interpreted as p-linear mappings of $\overset{p}{\otimes} T_m M^n$ into V, $\forall\, m \in M^n$ (usual notation, see [4]). Thus V-valued p-forms are thought as (row) vectors in V with $\mathbb{R}$-valued p-forms as components with respect to a basis of V: $E_p(M^n,V) \ni \phi = (\phi^1,\dots,\phi^m) = \phi^A \otimes E_A$, $\phi^A \in E_p$, $\forall\, A$, $\{E_A\}$ basis of V. If $V = \mathbb{R}$, $E_p(M^n, V = \mathbb{R}) = E_p(M^n)$ as before.

Exterior derivative, inner product, etc., of V-valued p-forms are defined "componentwise" (i.e. $d\phi = d\phi^A \otimes E_A$, a.s.o.). Let V, V' be vector spaces of dimension m and m', respectively. There is, however, no exterior product of V (V'-)-valued p-forms, as long as there is no product of vectors in V, V' or V' × V defined. The various definitions of that products depend on the special case under consideration.

Examples. ($\{E_A\}$ basis of V)

(1) $V' = \mathbb{R}$. $\alpha \in E_p(M^n, V'=\mathbb{R}) =: E_p$, $\phi \in E_q(M^n,V) =: E_q \otimes V$.

Define $\wedge : E_p \times (E_q \otimes V) \to E_{p+q} \otimes V$

by $\alpha \wedge \phi := \alpha \wedge \phi^A \otimes E_A$.

(2) $V' = V$ and V endowed with the structure of a Lie algebra,

$\phi \in E_p(M^n,V) =: E_p \otimes V$, $\psi \in E_q(M^n,V) =: E_q \otimes V$.

Define $\wedge : (E_p \otimes V) \times (E_q \otimes V) \to E_{p+q} \otimes V$

by $\phi \wedge \psi := \phi^A \wedge \phi^B \otimes [E_A, E_B]$.

(3) $V' = \mathfrak{g}$ = Lie-algebra of a Lie group G acting on V via a representation,

$A \in E_p(M^n,\mathfrak{g}) =: E_p \otimes \mathfrak{g}$, $\phi \in E_q(M^n,V) =: E_q \otimes V$.

Define $\wedge : (E_p \otimes \mathfrak{g}) \times (E_q \otimes V) \to E_{p+q} \otimes V$

by $A \wedge \phi := A^a \wedge \hat{X}_a \phi = A^a \wedge \phi^A \otimes \hat{X}_a E_A$,

where $\hat{X}_a$ are the generators of G in the V-representation, $a = 1,\dots,\dim G$.

(4) $V' = V = Gl(m,\mathbb{R})$. Define $\phi \wedge \psi$ by matrix multiplication in $Gl(m,\mathbb{R})$.

2.5 Algebraic differentiation. Frequently, we will use "differentiation" with respect to p-forms, say $\partial/\partial\phi$, where $\phi \in E_p$. This is to understand as the coefficient (on the right) of $\delta\phi$ in a suitable expansion of any functional of ϕ which is to be "differentiated". Take e.g. $L(\phi) := \phi$, then $\delta L = \delta\phi \wedge \partial L/\partial\phi = \delta\phi$, so $\partial L/\partial\phi = 1$. In general, take $L = L(\phi) \in E_r$ and $K(\phi) \in E_s$ and $\phi \in E_p$, then obviously $\partial L/\partial\phi \in E_{r-p}$ and $\partial K/\partial\phi \in E_{s-p}$ and

$$\delta(L \wedge K) = \delta L \wedge K + L \wedge \delta K = \delta\phi \wedge [\partial L/\partial\phi \wedge K + (-)^{pr}\, L \wedge \partial K/\partial\phi]$$

because of (2.3). Thus we have

$$\frac{\partial}{\partial\phi}(L \wedge K) = \frac{\partial L}{\partial\phi} \wedge K + (-)^{pr}\, L \wedge \frac{\partial K}{\partial\phi} \qquad (2.11)$$

for any $L \in E_r$, $\phi \in E_p$. The use and utility of this notation will become obvious in the next section.

2.6 Lagrangians. Let M^4 = spacetime. An action integral for the Lagrangian formulation of any classical field theory, leading to field equations of at most second order, is of the type

$$I(A) = \int_{A \subseteq M^4} L(\phi, d\phi) \qquad (2.12)$$

where ϕ are vector (spinor-, Lie-algebra-) valued p-forms in general, describing a physical system; $A \subseteq M^4$ a compact 4-manifold and the Lagrangian $L(\phi,d\phi)$ a $\mathbb{R}$-valued 4-form. We are in particular interested in the symmetry properties of the corresponding Euler-Lagrange expression

$$*T := \frac{\delta L}{\delta\phi} := \frac{\partial L}{\partial\phi} - (-)^p\, d\, \frac{\partial L}{\partial d\phi} \qquad (2.13)$$

under the action of a certain Lie group G. As is well known, these symmetry properties may be deduced from the symmetry properties of (2.11), that is, the form of the field equations $*T = 0$ is left unchanged in A if

$$\delta L \equiv 0 \quad \text{in } A \qquad (2.14)$$

(δ = infinitesimal action of G). Condition (2.14) may be weakened to $\delta L \equiv$ exact functional of $\delta\phi$, but we ignore that delicacy for simplicity.
Note: As $\phi \in E_p \otimes V$ may be viewed as a row matrix $\phi = (\phi^1,\ldots,\phi^m)$, $m = \dim V$, $\partial/\partial\phi$ corresponds to a column matrix and thus $\delta\phi \wedge \partial L/\partial\phi \in E_4(M^4,\mathbb{R})$.

Examples. For ϕ and L

(1) $\phi = x \in E_0(M^4 \supseteq U,\mathbb{R}^4)$ chart mapping

$d\phi = dx \in E_1(M^4 \supseteq U,\mathbb{R}^4)$ coordinate basis (needed in general to construct L, e.g. by $*$-operation)

(2) $\phi \in E_0(M^4,\mathbb{R})$ scalar field (massive, neutral)

$$L(dx,\phi,d\phi) = \frac{1}{2}\,[d\phi \wedge {*}d\phi + m^2\,\phi \wedge {*}\phi] \quad .$$

(3) $\psi \in E_0(M^4,S)$ Dirac-field; ($\gamma = \gamma_\mu dx^\mu$, γ_μ Dirac matrices)

$$L(dx,\phi,d\phi) = \frac{i}{2}\,(\bar\psi\, {*}\gamma \wedge d\psi + d\bar\psi \wedge {*}\gamma\psi) - m\bar\psi \wedge {*}\psi \quad .$$

(4) $\psi \in E_1(M^4,S)$ Rarita-Schwinger (spin 3/2)-field

$$L(dx,\psi,d\psi) = \frac{i}{2}\,(\bar\psi \wedge \gamma_5\gamma \wedge d\psi - d\bar\psi \wedge \gamma_5\gamma \wedge \psi) - \frac{m}{2}\,\bar\psi \wedge \gamma_5\gamma \wedge \gamma \wedge \psi \quad .$$

3. The Gauge Idea

3.1 Global gauge invariance. Consider a Lagrangian 4-form of type $L = L(\phi,d\phi)$ which is invariant under the infinitesimal action δ of a Lie group G, i.e.

$$\delta L \equiv 0 \quad , \qquad L = L(\phi,d\phi) \quad , \tag{3.1}$$

where the vector (spinor-, ..) valued p-forms ϕ provide a representation of G (see examples above). Carrying out the variation (3.1) explicitly

$$\delta L \equiv \delta\phi \wedge \frac{\partial L}{\partial\phi} + \delta d\phi \wedge \frac{\partial L}{\partial d\phi} \equiv 0$$

immediately leads to the identity (use $\delta d = d\delta$)

$$- \delta\phi \wedge {}^*\mathcal{T} \equiv d{}^*j \qquad . \tag{3.2}$$

where ${}^*\mathcal{T}$ corresponds to the Euler-Lagrange expression (2.12) and ${}^*j$ to a weakly (i.e. modulo field equations ${}^*\mathcal{T} = 0$) conserved current, that is

$$d{}^*j = 0 \quad \text{if } {}^*\mathcal{T} = 0 \ , \qquad {}^*j := \delta\phi \wedge \frac{\partial L}{\partial d\phi} \quad . \tag{3.3}$$

Let $\hat{X}_a$, $a = 1,\ldots, q := \dim G$ denote the generators of G in the ϕ-representation and ε^a, $a = 1,\ldots,q$, <u>constant</u> infinitesimal group parameters, then

$$\delta\phi = \varepsilon^a \hat{X}_a \phi \tag{3.4}$$

("<u>global</u>" action of G because ε^a constant) and (3.2) becomes

$$- \hat{X}_a \phi \wedge {}^*\mathcal{T} \equiv d{}^*j_a \quad , \tag{3.5}$$

which yields q (= dim G) weakly conserved currents

$$d{}^*j_a = 0 \ , \qquad {}^*j_a := \hat{X}_a\phi \wedge \frac{\partial L}{\partial d\phi} \tag{3.6}$$

in the case ${}^*\mathcal{T} = 0$.

<u>3.2 Local gauge invariance</u>. Suppose now $\varepsilon^a = \varepsilon^a(x)$ $\forall$ a, i.e. ε^a spacetime dependent ("<u>local</u>" action of G, G is now called "gauged"), then (3.3/4) merely leads to the trivial case

$$\varepsilon^a d{}^*j_a + d\varepsilon^a \wedge {}^*j_a \equiv 0 \iff d{}^*j_a = {}^*j_a = 0$$

because of the arbitrariness of ε^a, $d\varepsilon^a$ $\forall$ a. To circumvent this awkward situation, we introduce q new auxiliary variables A^a, $a = 1,\ldots,q := \dim G$ to compensate the q undesired terms $\sim d\varepsilon^a$. To this end, start again with another Lagrangian

$$L' = L'(\phi, d\phi, A^a) \tag{3.7}$$

where the auxiliary variables A^a are incorporated as exterior fields, and demand invariance under local action of G, i.e.

$$\delta L' \equiv \delta\phi \wedge {}^{*}T' + d^{*}j' + \delta A^a \wedge \frac{\partial L'}{\partial A^a} \equiv \tag{3.8}$$

$$\equiv \varepsilon^a \hat{X}_a\phi \wedge {}^{*}T' + \varepsilon^a d^{*}j'_a + d\varepsilon^a \wedge {}^{*}j'_a + \delta A^a \wedge \frac{\partial L'}{\partial A^a} \equiv 0$$

where the primed expressions have an analogous meaning as above. As the new fields A^a are introduced to cancel the inhomogeneous terms $d\varepsilon^a \wedge {}^{*}j^a$, we try the simplest of all possibilities ("minimal procedure"), namely

$${}^{*}j'_a := \hat{X}_a\phi \wedge \frac{\partial L'}{\partial d\phi} \equiv \frac{\partial L'}{\partial A^a} \tag{3.9}$$

and

$$\delta A^a \equiv - d\varepsilon^a \tag{3.10}$$

whereby a symmetry condition similar to (3.5) is obtained. Note that by (3.10), the A^a have to be 1-forms $\forall$ a, the kind of their coupling to the ϕ-field is, however, restricted by (3.9) only.

To trace this particular coupling, we observe that ${}^{*}j'_a$ has to be invariant under the group action $\forall$ a (i.e. $\delta^{*}j'_a \equiv 0 \;\forall$ a), as $d\phi$ (resp. $\partial L'/\partial d\phi$) transforms contragrediently to A^a (resp. $\partial L'/\partial A^a$), as far as the inhomogeneous parts are concerned:

$$\delta d\phi \equiv d\delta\phi = d\varepsilon^a \wedge \hat{X}_a\phi + \varepsilon^a\hat{X}_a d\phi \quad . \tag{3.11}$$

To proceed, define $A \wedge \phi := A^a \wedge \hat{X}_a\phi$ (by which A is interpreted as Lie-algebra valued 1-form, see Sec. 2.4), Example (3): $A \in E_1(M^4,g)$) thus getting $\hat{X}_a\phi \equiv \partial(A \wedge \phi)/\partial A^a$ and therefore

$$\frac{\partial L'}{\partial A^a} \equiv \frac{\partial(A \wedge \phi)}{\partial A^a} \wedge \frac{\partial L'}{\partial d\phi} \quad \Leftrightarrow \quad \frac{\partial L'}{\partial d\phi} \equiv \frac{\partial L'}{\partial(A \wedge \phi)} \quad . \tag{3.12}$$

The last identity simply says that $d\phi$ and $A \wedge \phi$ have to appear in L' only through the combination

$$D\phi := d\varphi + A \wedge \phi \tag{3.13}$$

called "(G-) covariant exterior derivative" of ϕ. The denotation is justified because, by invariance of "j_a', $D\phi$ has to transform in the same way as ϕ does, namely

$$\delta D\phi = \varepsilon^a \hat{X}_a D\phi \quad . \tag{3.14}$$

Performing the variation of $A \wedge \phi$ under the group action δ

$$\delta(A \wedge \phi) = \delta A^a \wedge \hat{X}_a\phi + A \wedge \delta\phi = \delta A^a \wedge \hat{X}_a\phi + \varepsilon^a[f^c{}_{ba}A^b \wedge \hat{X}_c\phi + \hat{X}_a(A \wedge \phi)]$$

($f^c{}_{ba}$ = structure constants of G) and using (3.13) we see, however, that the simple transformation property of the A^a in (3.10) will not suffice in general to guarantee (3.14), but has to be replaced by

$$\delta A^a := - d\varepsilon^a - f^a{}_{bc}\, A^b\, \varepsilon^c =: - D\varepsilon^a \tag{3.15}$$

which meets (3.10) in the case of an abelian group. The q (= dim G) compensating 1-forms A^a obeying the local gauge transformation (3.15) under "their" gauge group G are called (G-) gauge potentials.

Because of (3.15), the gauge potentials may be transformed to zero by local gauge transformations, at least locally. But then, locally where $A^a = 0\ \forall\ a$, $L'(\phi,d\phi,A^a)$ and the $L(\phi,d\phi)$ we started with have to coincide, which yields in general

$$L'(\phi,d\phi,A^a) = L(\phi,D\phi) \tag{3.16}$$

as the coupling of the gauge potentials to the ϕ's has to appear via the (G-) covariant derivative (3.13). Thus local gauge invariance is obtained merely by replacing the ordinary exterior derivative d by the covariant one ("principle of minimal coupling").

Insertion of (3.15) into (3.8) finally leads to the identity

$$- \hat{X}_a \phi \wedge {}^*T \equiv d{}^*j_a - f^c{}_{ba} A^b \wedge {}^*j_c =: D{}^*j_a \qquad (3.17)$$

where the primes are omitted according to (3.16).

Remarks

(1) Consider $E_1(M^4, g) \ni A = A^a \otimes X_a$, where $\{X_a\}$ = basis of g, a = 1,...,q. Then $\delta A = \delta A^a \otimes X_a + A^a \otimes \delta X_a$ and therefore $(\delta A)^a = \delta A^a - f^a{}_{bc} A^b \varepsilon^c$ if δ corresponds to G. In this more precise notation of (3.15), we see that the choice (3.10) was not really wrong as long as we were not prepared to look at the A^a as forming the components of one object, the Lie-algebra valued 1-form $A \in E_1(M^4, g)$. The need for this was introduced by (3.14) as a consequence of invariance of (3.9) under G. But the latter simply implies that ${}^*j_a$ in addition are components of a Lie-algebra valued 3-form ${}^*j \in E_3(M^4, g)$ (not to be confused with ${}^*j$ in (3.2/3)), transforming like an "isovector", i.e. $\delta{}^*j = \delta{}^*j^a \otimes X_a + {}^*j^a \otimes \delta X_a = \varepsilon^b\, {}^*j^a\, f^c{}_{ba} \otimes X_c \Leftrightarrow (\delta{}^*j)^a = f^a{}_{bc}\, \varepsilon^b\, {}^*j^c$ $(\delta{}^*j^a \equiv 0$ as above; the group indices are rised and lowered by the invariant group metric of G). Thus (3.17) is merely a component version of a G-covariant identity of the corresponding isovectors ${}^*j \in E_3(M^4, g)$, $D{}^*j \in E_4(M^4, g)$, etc.

<u>Note</u> that in the case of an abelian group $(\delta A)^a = \delta A^a$ and (3.9/10) is sufficient to start the gauge procedure. We will use this fact in Sec. 4 to gauge the translation group T(4).

(2) Although supererogatory, we want to give another argument for (3.12). Use (3.9/10) to get

$$(-)^p\, \delta\phi \wedge A^a \wedge \hat{X}_a \frac{\partial L'}{\partial d\phi} + \delta(A \wedge \phi) \wedge \frac{\partial L'}{\partial d\phi} - \delta A^a \wedge \frac{\partial L'}{\partial A^a} \equiv 0 \; .$$

Comparison with

$$\delta L' \equiv \delta\phi \wedge \frac{\partial L'}{\partial \phi} + \delta d\phi \wedge \frac{\partial L'}{\partial d\phi} + \delta A^a \wedge \frac{\partial L'}{\partial A^a} \equiv 0$$

yields

$$\delta\phi \wedge \left[\frac{\partial L'}{\partial \phi} + (-)^p A^a \wedge \hat{X}_a \frac{\partial L'}{\partial d\phi}\right] + \delta D\phi \wedge \frac{\partial L'}{\partial d\phi} \equiv 0 \; .$$

Read off the coefficients of $\delta\phi$ and $\delta D\phi$ according to Sec. 2.5 and obtain $\partial L'/\partial d\phi \equiv \partial L'/\partial D\phi$ as in (3.12), but moreover

$$\left.\frac{\partial L'}{\partial \phi}\right|_{ex} := \frac{\partial L'}{\partial \phi} + (-)^p A^a \wedge \hat{X}_a \frac{\partial L'}{\partial d\phi}$$

($|ex$ indicates that ϕ in $D\phi$ is held fixed), whereby

$$*T' := \frac{\partial L'}{\partial \phi} - (-)^p d \frac{\partial L'}{\partial d\phi} \equiv \left.\frac{\partial L'}{\partial \phi}\right|_{ex} - (-)^p D \frac{\partial L'}{\partial d\phi} \tag{3.18}$$

which reveals the G-invariance of the Euler-Lagrange expression $*T'$, as expected.

(3) Observe that the "minimal coupling" was a result of the "minimal procedure" to obtain local gauge invariance, i.e. to obtain $d\varepsilon^a \wedge *j'_a + \delta A^a \wedge \partial L'/\partial A^a \equiv 0$, in (3.8) by (3.9/10). In this sense the term "minimal" is legitimate, because one can think of attaining local gauge invariance by other exterior fields than gauge potentials, which may result in a more complicated kind of coupling. However, if the compensating fields are to be gauge potentials, i.e. the "minimal procedure" is performed, than there is no other coupling than "minimal coupling".

3.3 Lagrangians for the gauge fields. The gauge fields A^a demand, interpreted as physical (exterior) fields, an action principle for their own. According to the general scheme for Lagrangians leading to field equations of at most second order, we assume $L_0 = L_0(e, A^a, dA^a)$, where any basis $\{e^i\} \in E_1(M^4, \mathbb{R})$ ($i = 0,1,2,3$, or equivalently, $e \in E_1(M^4, \mathbb{R}^4)$) is cited explicitly in L_0, as they may, in general, provide a representation of the gauge group (e.g. orthonormal basis for the Lorentzgroup), but not necessarily as gauge fields. Recall that any 1-form basis will be needed to construct a 4-form L_0, e.g. by $*$-operation.

Of course, L_0 has to be invariant under the corresponding local gauge group, a constraint which restricts the various possibilities of L_0 considerably. To see this, perform the same variation procedure as in the case of the ϕ-fields

$$\delta L_o \equiv \delta e \wedge \frac{\partial L_o}{\partial e}\Big|_{ex} + \delta A^a \wedge \frac{\partial L_o}{\partial A^a} + \delta dA^a \wedge \frac{\partial L_o}{\partial dA^a} \equiv 0 \quad , \tag{3.19}$$

use δA^a in (3.15) and $\delta e := \varepsilon^a(\hat{Y}_a e)$, where $\hat{Y}_a$ are the generators of G in the e-representation, to get the following identities

$$\frac{\partial L_o}{\partial A^a} \equiv \frac{1}{2}\,\frac{\partial (A \wedge A)^b}{\partial A^a} \wedge \frac{\partial L_o}{\partial dA^b} \quad , \tag{3.20}$$

$$\hat{Y}_c e \wedge \frac{\partial L_o}{\partial e}\Big|_{ex} \equiv f^a{}_{bc}(A^b \wedge \frac{\partial L_o}{\partial A^a} + dA^b \wedge \frac{\partial L_o}{\partial dA^a}) \tag{3.21}$$

where $(A \wedge A)^a := f^a{}_{bc}\, A^b \wedge A^c$. By (3.20) we see that the gauge potentials have to appear in L_o only via the combination

$$F^a := dA^a + \frac{1}{2}\,(A \wedge A)^a \tag{3.22}$$

the <u>(G-) gauge field strengths</u>, $a = 1,\dots, q = \dim G$, which transform homogeneously under G, i.e. like an isovector $(\delta F)^a = f^a{}_{bc}\varepsilon^b F^c$. The use of (3.20) in (3.21) finally yields

$$- f^a{}_{bc} F^b \wedge \frac{\partial L_o}{\partial F^a} \equiv D{*}J_c \equiv - (\hat{Y}_c e)^j \wedge {*}t_{(o)j} \tag{3.23}$$

where

$${*}J_a := \frac{\partial L_o}{\partial A^a} + d\,\frac{\partial L_o}{\partial dA^a} \equiv D\,\frac{\partial L_o}{\partial dA^a} \tag{3.24}$$

are the Euler-Lagrange expressions for the (free) gauge potentials (compare (3.18)), $D{*}J_c$ is defined analogously to (3.17) and ${*}t_{(o)j} := \partial L_o/\partial e^j|_{ex}$ turn out to be the gauge invariant canonical energy/momentum currents (see [7]; $\partial/\partial e|_{ex}$ means that the e to construct $A^a = A^a{}_i e^i \in E_1$ are held fixed, that is, the 1-forms A^a are treated as independent variables, and not their components $A^a{}_i \in E_o$).

<u>Note</u> that in the case $\hat{Y}_a \equiv 0\ \forall\, a$, the right hand side of (3.23) vanishes identically. Because of $f_{abc} = f_{[abc]}$ this implies $\partial L/\partial F^a \sim F_a$ or ${*}F_a$ and L_o has to be squared in the field strengths. A standard example for this is the "Yang-Mills"-like version $L_{YM} = \frac{1}{2} F^a \wedge {*}F_a$. However, in general

$(\hat{Y}_a \neq 0)$, there is no need for the free Lagrangian to be "Yang-Mills"-like, that means, to be squared in the field strengths.

Example: Take $G \to SO(3,1)$, $A^a \to \omega^{ij} = \omega^{[ij]}$, $F^a \to \Omega^{ij} = \Omega^{[ij]} := d\omega^{ij} + \omega^i{}_k \wedge \omega^k{}_j$, $\{e^i\}$ orthonormal basis, $\hat{Y}_a \to \hat{Y}_{ij} = \hat{Y}_{[ij]}$, where $(\hat{Y}_{ij}e)^k = \delta^{km}_{ij}\, e_m$ and indices are moved by $\eta = \mathrm{diag}\,(-1,1,1,1)$. The analogue of (3.23) is

$$\frac{1}{2} D{}^{*}J_{ij} \equiv e_{[i} \wedge {}^{*}t_{(o)j]} \tag{3.25}$$

where $\frac{1}{2}\,{}^{*}J_{ij} = \frac{1}{2}\,\delta L_o/\delta\omega^{ij}$ is the usual spin current of the gauge fields, and (3.25) relates the SO(3,1)-covariant divergence of J_{ij} to the anti-symmetric part of the canonical energy/momentum current ${}^{*}t_{(o)}$ in the well known way. The identities (3.25), however, do not restrict L_o to the simple "Yang-Mills"-like version $\sim \frac{1}{2}\,\Omega^{ij} \wedge {}^{*}\Omega_{ij}$, but allow a variety of Lagrangians:

$$
\begin{aligned}
L_o^{(o)} &:= \frac{1}{2}\,\Omega^{ij} \wedge {}^{*}e_{ij} \\
L_o^{(1)} &:= \frac{1}{2}\,\Omega^{ij} \wedge {}^{*}\Omega_{ij} \\
L_o^{(2)} &:= \frac{1}{2}\,\Omega^{ij} \wedge e_j \wedge {}^{*}(\Omega_{ik} \wedge e^k) \\
L_o^{(3)} &:= \frac{1}{2}\,\Omega^{ij} \wedge e^k \wedge {}^{*}(\Omega_{ik} \wedge e_j) \\
L_o^{(4)} &:= \frac{1}{2}\,\Omega^{ij} \wedge e_{ij} \wedge {}^{*}(\Omega^{km} \wedge e_{km}) \\
L_o^{(5)} &:= \frac{1}{2}\,\Omega^{ij} \wedge e_{jk} \wedge {}^{*}(\Omega^{km} \wedge e_{mi}) \\
L_o^{(6)} &:= \frac{1}{2}\,\Omega^{ij} \wedge e^{km} \wedge {}^{*}(\Omega_{km} \wedge e_{ij}) \quad .
\end{aligned}
\tag{3.26}
$$

Note in particular that there is also a Lagrangian linear in the gauge field strengths, $L_o^{(o)}$.

3.4 Gauge theory defined. After having defined gauge potentials, gauge field strengths and gauge groups, we are now ready to summarize their main properties, thus declaring also what we want to call a (classical) gauge theory.

In general, we are dealing with two kinds of objects. On the one hand, we have certain V-valued p-forms over spacetime, $\phi \in E_p(M^4,V)$, $p \in [0,4]$, where V is a vector space a Lie group G acts on via a representation $\rho: G \to Gl(V)$. The ϕ's we call G-fields, because they provide a linear representation space by their image space V. ("ϕ-representation" of G). Observe that G-fields are more general than the usual "matter-fields", as they may include chart mappings $x \in E_0(M^4,\mathbb{R}^4)$ or basis 1-forms $e \in E_1(M^4,\mathbb{R}^4)$ as well. The behaviour of the matter-corresponding G-fields we assume to be determined by field equations which are the Euler-Lagrange equations of an action integral. Invariance of the corresponding Lagrangian under (infinitesimal) global (i.e. spacetime-independent) action of G amounts in several identities, as seen in Sec. 3.1.

The second type of objects we are dealing with are the gauge potentials A^a, $a = 1,\ldots, q = \dim G$, which were introduced as a certain kind of exterior fields to ensure also local (i.e. spacetime-dependent) invariance under G in a minimal way. By construction of that "minimal procedure", the A^a have the following main properties:

(a) A^a are $q := \dim G$ independent 1-forms ($a = 1,\ldots,q$), independent in the sense that we need q different A^a to compensate the q inhomogeneous terms $\sim d\varepsilon^a$ separately, see Sec. 3.2.

(b) A^a, $\forall$ a, transform inhomogeneously under the local action of "their" group G, which is called a gauge group from now on. The action of G (in particular in (3.15), which reveals that A^a make up a Lie-algebra valued 1-form $A \in E_1(M^4,\mathfrak{g})$, see remark (1) in Sec. 3.2) is thus renamed local (global) gauge transformation, the corresponding invariance local (global) gauge invariance.

(c) A^a, $\forall$ a, may be transformed to zero locally, by local gauge transformation according to (3.15).

(d) Any Lagrangian L_0 for the pure gauge potentials of the type $L_0 = L_0(e,A^a,dA^a)$, has to include the A^a only through the combination

$F^a = dA^a + \frac{1}{2}(A \wedge A)^a$, the gauge field strengths, in order to preserve local gauge invariance of L_o, see Sec. 3.3.

(e) The corresponding Euler-Lagrange expressions $^{*}J_a := \delta L_o/\delta A^a$ have to obey certain identities by reason of local gauge invariance of L_o again, see (3.23).

(f) The coupling of the gauge potentials to the G-fields ϕ is performed minimally, $L'(\phi, d\phi, A^a) = L(\phi, D\phi)$, see Sec. 3.2.

(g) The "charge" currents $^{*}j_a := \delta L/\delta A^a$ have to obey certain identities by local gauge invariance of $L(\phi, D\phi)$, see (3.17).

To combine G-fields and gauge potentials within one action principle, we first introduce a coupling constant κ by $A =: \kappa A'$, wherefore $D\phi = d\phi + \kappa A' \wedge \phi$ and the strength of the coupling is measured by κ. However, as a matter of convenience, we absorb κ again, but are aware that now the gauge potentials A' have to appear in the free Lagrangian $L_o = L_o(e, A', dA')$ via $F' := dA' + \frac{\kappa}{2} A' \wedge A'$. The replacement of F' by $F = \kappa F'$ then results in a factor $\kappa^{-\zeta}$ in front of $L_o = L_o(e, F)$, where ζ corresponds to the polynomial degree of L_o with respect to F ($r = 1$ if L_o is linear in F, etc.). The combined Lagrangian for G-fields and gauge potentials now looks like

$$L(e, \phi, d\phi, A, dA) = L_m(\phi, D\phi) + \frac{1}{\kappa^\zeta} L_o(e, F) \qquad (3.27)$$

in an obvious notation. We define a (G-) gauge theory to be any field theory based on (3.27), where the ϕ's are G-fields and $F = (F^a)$ are the (G-) field strengths of the (G-) gauge potentials $A = (A^a)$ (both frequently called (G-) gauge fields) obeying (a) - (g). Note that in this way, a gauge theory is a theory of interacting fields, the G-fields and the (G-) gauge fields. If, however, we want to consider the behaviour of free (uncoupled) gauge fields alone, we ought to refer to a pure (G-) gauge theory in this case, but this notation is handled very loosely, in general.

4. Gravitation

4.1 GRT as a gauge theory. As mentioned in the introduction, we try to interpret Einstein's general relativity theory (GRT) as a gauge theory. To this end, we have to look for gauge potentials in the theory, which obey, by construction, the conditions (a) - (g) of Sec. 3.4. In searching these potentials we first have to reformulate conventional (pure) GRT in terms of 1-forms according to the fact that gauge potentials have to be 1-forms. Secondly, these 1-forms have to be interpreted as forming one Lie-algebra-valued 1-form, whereby the corresponding gauge group comes into play. After having found both, the gauge fields and the gauge group, we shall proceed asking whether and in which way, this gauge theory may be extended in order to overcome, at some time or other, the problems of conventional GRT.

4.2 GRT in terms of exterior forms. Looking at the usual Lagrangian density in GRT

$$L_E = \frac{1}{2}\sqrt{-g}\,R \tag{4.1}$$

(usual terminology [6], 8π times gravitational constant =: velocity of light =: 1) is rather deplorable because there are no 1-forms at the first glance. Everything is made up of tensorial functions $g_{\mu\nu}$ (tensorial 0-form, $(g_{\mu\nu}) \in E_0(V^4, \mathbb{R}^4 \otimes \mathbb{R}^4)$, V^4 = Riemannian spacetime) and their derivatives only ($R = R(g,\Gamma,\partial\Gamma)$, $\Gamma = \Gamma(g,\partial g)$). But if we pass to the Lagrangian 4-form

$$\mathcal{L}_E = \frac{1}{2}\sqrt{-g}\,R\,d^4x = \frac{1}{2}\sqrt{-g}\,R\,dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{4.2}$$

we see that there are 1-forms indeed, dx^μ, which we can use to define

$$e_\mu := g_{\mu\nu}\,e^\nu \quad , \qquad e^\nu := dx^\nu \quad . \tag{4.3}$$

Using the equations of Cartan

$$de_\mu + \kappa^\nu{}_\mu \wedge e_\nu = 0 \tag{4.4a}$$

$$d\kappa^\nu{}_\mu + \kappa^\nu{}_\sigma \wedge \kappa^\sigma{}_\mu = R^\nu{}_\mu \tag{4.4b}$$

($\kappa^{\nu}{}_{\mu}$ the Levi-Civita connection 1-forms with respect to the basis $\{e_{\mu}\}$, its components being the usual Christoffel symbols $\kappa^{\nu}{}_{\mu} = \Gamma^{\nu}{}_{\mu\sigma} e^{\sigma}$; $R^{\nu}{}_{\mu} = R^{\nu}{}_{\mu|\sigma\tau|} e^{\sigma\tau}$, the Riemannian curvature 2-forms, its components forming the Riemann-Christoffel tensor), we are able to rewrite (4.2) like this

$$L_E = \frac{1}{2} R^{\mu}{}_{\nu} \wedge {}^{*}e_{\mu}{}^{\nu} \quad . \tag{4.5}$$

As L_E is obviously invariant with respect to mere change of the basis, we prefer an orthonormal, anholonomic basis $\{e^i\}$, $(\langle e^i, e^j \rangle) = (\eta^{ij}) = \mathrm{diag}\,(-1,1,1,1)$, $i,j = 0,1,2,3$, where the metric is represented by (greek indices refer to a holonomic basis $\{e^{\mu} = dx^{\mu}\}$, latin indices to an anholonomic one)

$$\begin{aligned} g &= e_{\nu} \otimes e^{\nu} = g_{\mu\nu}\, e^{\mu} \otimes e^{\nu} \\ &= e_i \otimes e^i = \eta_{ij}\, e^i \otimes e^j \end{aligned} \tag{4.6}$$

($e^i =: h^i{}_{\mu}\, e^{\mu}$, $h^i{}_{\mu}$ the usual vierbein-fields and $g_{\mu\nu} = h^i{}_{\mu}\, h_{i\nu}$). The Cartan equations for that basis now read

$$de^i + \kappa^i{}_j \wedge e^j = 0 \tag{4.7a}$$

$$d\kappa^i{}_j + \kappa^i{}_k \wedge \kappa^k{}_j = R^i{}_j \tag{4.7b}$$

where $\kappa^{ij} := \eta^{jk} \kappa^i{}_k = \kappa^{[ij]}$ because of metricity. The Lagrangian (4.2/5) now becomes

$$L_E = \frac{1}{2} R_{ij} \wedge {}^{*}e^{ij} \tag{4.8}$$

and thus the Einstein current ${}^{*}G^k$ (${}^{*}G^k = G^{km}\, {}^{*}e_m$, $G^{km} = G^{(km)}$ = Einstein tensor with respect to the basis $\{e^i\}$) gets

$${}^{*}G^k := \frac{1}{2} R_{ij} \wedge {}^{*}e^{ijk} \equi -{}^{*}\Pi^k + {}^{*}t^k \tag{4.9}$$

where

$$*\Pi^k := \frac{1}{2} \kappa_{ij} \wedge *e^{ijk} \tag{4.10}$$

corresponds to the so called "super-field strength" and

$$*t^k := -\frac{1}{2} \kappa_{ij} \wedge (\kappa^k{}_m \wedge *e^{ijm} + \kappa^j{}_m \wedge *e^{imk}) \tag{4.11}$$

to one of the pseudo-currents, representing the canonical energy/momentum current of the gravitational field (see [5] for further details). Because of (4.8) - (4.11) it looks as if there were two kinds of 1-forms in the theory, e and κ. But solving (4.7a) with respect to $\kappa^i{}_j$ yields, after some calculations

$$\kappa^{ij} = \frac{1}{2} [i(e^j)de^i - i(e^i)de^j + e_k \wedge i(e^i)\, i(e^j)de^k] \tag{4.12}$$

which exhibits that the six κ^{ij} are <u>completely determined</u> by the four basis 1-forms e^i. But if this is the case, why not formulate Einstein's theory in terms of e and de only? Insert (4.12) into (4.7b) and (4.8) to find $L_E = L_E(e,de)$, in particular

$$\begin{aligned} L_E(e,de) = &-d(e_i \wedge *de^i) - \frac{1}{2} de^i \wedge e^j \wedge *(de_j \wedge e_i) + \\ &+ \frac{1}{4} de^i \wedge e_i \wedge *(de^j \wedge e_j) \end{aligned} \tag{4.13}$$

and the Einstein current $*G^k$ turns out to be the Euler-Lagrange expression for the fields e^i :

$$*G^k := \frac{\delta L_E}{\delta e_k} \equiv \frac{\delta \tilde{L}_E}{\delta e_k} := \frac{\partial \tilde{L}_E}{\partial e_k} + d\, \frac{\partial \tilde{L}_E}{\partial de_k} \equiv *t^k + d*\Pi^k \tag{4.14}$$

with $*\Pi^k$, $*t^k$ as in (4.9/10/11), but now in terms of e and de:

$$*\Pi^k := \frac{\partial \tilde{L}_E}{\partial de_k} = -e^i \wedge *(de_i \wedge e^k) + \frac{1}{2} e^k \wedge *(de^i \wedge e_i) \tag{4.15}$$

$$*t^k := \frac{\partial \tilde{L}_E}{\partial e_k} = i(e^k)\tilde{L}_E - i(e^k)de_i \wedge \frac{\partial \tilde{L}_E}{\partial de_i} \tag{4.16}$$

and where $L_{\tilde{E}}$ equals L_E but the exact term $d(e_i \wedge {}^{*}de^i)$ discarded. This reformulation (4.13) - (4.16) of Einstein's general relativity in terms of the 1-forms e^i (and their exterior derivatives de^i) does not only simplify practical calculations considerably (no connection-, no curvature components are to be calculated), but in addition reveals that GRT fits into the general scheme of our Lagrangian formalism, $L = L(e,de)$, by which various field theoretical aspects are reflected in a more distinct way, e.g. the canonical energy/momentum character of ${}^{*}t^k$ in (4.16). However, as we want to look at GRT as a gauge theory and the only variables we are left with are 1-forms e^i, it looks as if we had found the desired gauge potentials.

4.3 Where are the gauge potentials? Before investigating whether the orthonormal basis 1-forms e^i may be interpreted as gauge potentials, there is a comment on the Levi-Civita connection forms $\lambda^i{}_j$ in place, which are frequently viewed to be gauge potentials of the homogeneous Lorentzgroup. And indeed, under local Lorentzrotations of the basis, infinitesimally written as

$$\delta_{rot}\, e^i = \varepsilon^i{}_j(x)\, e^j\,, \qquad \varepsilon^{ij} = \eta^{jk}\,\varepsilon^i{}_k = \varepsilon^{[ij]} \tag{4.17}$$

the λ^{ij} transform as Lorentz gauge potentials

$$\delta_{rot}\, \lambda^{ij} = -\, d\varepsilon^{ij} - \lambda^i{}_k\, \varepsilon^{kj} - \lambda^j{}_k\, \varepsilon^{ik} =: -\, D\varepsilon^{ij} \tag{4.18}$$

(compare (3.15), using $A^a \to \lambda^{ij}$, $\varepsilon^a \to \varepsilon^{ij}$, and $f^a{}_{bc} \to f^{mn}{}_{ijk\ell} =$ $= \eta_{pq}\,[\delta^{mp}_{ij}\,\delta^{qn}_{k\ell} + \delta^{np}_{ij}\,\delta^{mq}_{k\ell}])$. But as the six λ^{ij} are completely determined by four basis 1-forms e^i, they are not independent in the sense of condition (a), Sec. 3.4. Therefore, they will not suffice to act as compensating fields and are thus ruled out as gauge potentials of the Lorentzgroup, at least in strict sense we are dealing with.

Now we are left again with the orthonormal 1-forms e^i, which are obviously not gauge potentials of the Lorentzgroup, but rather Lorentz-fields (SO(3,1)-fields) according to (4.17). So let us try another transformation group frequently used in GRT, the (local) coordinate transformation group, its action infinitesimally written as

$$\delta_{coord}\, x^{\mu} = \varepsilon^{\mu}(x) \quad , \qquad \mu = 0,1,2,3 \tag{4.19}$$

for any coordinate functions x^{μ}. Interpreting (4.19) as expressing the action of a gauged group, the anholonomic 1-forms e^{i} are obviously not suited to serve as "compensating fields" for that group, as, by construction, they do not transform under (4.19) at all:

$$\delta_{coord}\, e^{i} \equiv 0 \quad , \qquad i = 0,1,2,3 \quad . \tag{4.20}$$

Thus it is not astonishing that the basis 1-forms e^{i} do not fit into the general scheme (a) - (g), whereby they are ruled out again as gauge potentials corresponding to (4.19). This result is, at the first glance, rather disappointing since (4.19) looks very much like the local ("gauged") version of infinitesimal translations in Minkowskian spacetime ($\delta x^{\mu} = \varepsilon^{\mu} =$ const $\forall\ \mu$), and any interpretation of GRT as a gauge theory of the (4-dimensional) translation group T(4) is well suited to explain the preference of the mass/energy aspect in this theory. Thus some authors insist on having formulated a "translational gauge theory" with use of a certain kind of orthonormal basis as "gauge potentials" of that group (see e.g. [8] and further references therein).

However, as the underlying idea of gauging the translation group T(4) seems to be very reasonable, we try to follow that line, thus interpreting (4.19) as expressing the local action of T(4).

4.4 Gauging the translation group. To find out the gauge potentials, we perform the gauge procedure outlined in Sec. 3.2. Thus start with ordinary field theory over Minkowskian spacetime M^4, based on $L = L(x,dx,\phi,d\phi)$, where the $x \in E_0(M^4,\mathbb{R}^4)$ denote chart mapping, $dx \in E_1(M^4,\mathbb{R}^4)$ orthonormal coordinate basis and $\phi \in E_p$ some matter fields, where

$$\delta_{transl}\, \phi \equiv 0 \quad , \qquad \delta_{transl}\, x = \varepsilon = \text{const} \tag{4.21}$$

as usual (global translations, ε not to be confused with (2.9)). Note that $\delta x = \varepsilon$ follows our usual terminology, but as x is a chart mapping,

$\delta x = \varepsilon$ relates translations to the spacetime diffeomorphism group (the same goes for the gauged case).

Considering invariance of $L(x,dx,\phi,d\phi)$ under (4.21) only, we can forget about the matter fields ϕ for the moment when $\delta_{\text{transl}}\, L =: \delta L$ is carried out:

$$\delta L \equiv \delta x \wedge \frac{\partial L}{\partial x}\bigg|_{ex} + \delta dx \wedge \frac{\partial L}{\partial dx} \tag{4.22}$$

(recall that we treat x and dx as $\mathbb{R}^4$- valued 0-forms and 1-forms, resp., quite analogous to the ϕ's in Sec. 3). Because of $\delta dx = d\delta x = d\varepsilon = 0$, invariance of L under (4.21) amounts in the well known condition

$$\frac{\partial L}{\partial x}\bigg|_{ex} \equiv 0 \tag{4.23}$$

i.e. the coordinates must not appear explicitly in the Lagrangian to guarantee global translation-invariance.

Now, if we replace ε by $\varepsilon = \varepsilon(x)$ (<u>local</u>, or "gauged" translations), the symmetry $\delta L \equiv 0$ does not hold, except in the trivial case $\partial L/\partial x|_{ex} \equiv$ $\equiv \partial L/\partial dx \equiv 0$ because

$$\delta L \equiv \varepsilon \wedge \frac{\partial L}{\partial x}\bigg|_{ex} + d\varepsilon \wedge \frac{\partial L}{\partial dx} \quad . \tag{4.24}$$

According to the general gauge idea, we introduce 4 exterior fields $c^\mu \in E_1$, $\mu = 0,1,2,3$ (or combined in compact notation $c \in E_1(M^4,\mathbb{R}^4)$, $\mathbb{R}^4 =$ = Lie-algebra of $T(4)$), to compensate the inhomogeneous terms in (4.24), i.e. take $L' = L'(x,dx,\phi,d\phi,c)$ and assume

$$\delta L' \equiv \delta x \wedge \frac{\partial L'}{\partial x}\bigg|_{ex} + \delta dx \wedge \frac{\partial L'}{\partial dx} + \delta c \wedge \frac{\partial L'}{\partial c} \equiv 0 \quad . \tag{4.25}$$

In order to restate the original symmetry condition (4.23), we perform the "minimal procedure" analogously to (3.9)ff, that is

$$\varkappa_j := \frac{\partial L'}{\partial dx} \equiv \frac{\partial L'}{\partial c} \tag{4.26}$$

$$\delta c \equiv - \delta dx \equiv - d\varepsilon \quad . \tag{4.27}$$

As T(4) is abelian, we know from the general considerations in Sec. 3.2 that (4.26/27) is sufficient to perform the gauge procedure, see remark (1). Furthermore, (4.27) reveals that c actually corresponds to a <u>gauge</u> potential of a 4-dimensional abelian group, as is the case with T(4).

Now, (4.26) simply says that c has to appear in L' through the combination $e := dx + c$ only, whereas (4.27) reveals that this combination is invariant under gauged translations, i.e.

$$\delta e \equiv 0 \quad , \qquad e := dx + c =: Dx \tag{4.28}$$

where $Dx := dx + c \wedge x = dx + c^{\mu} \wedge \hat{X}_{\mu} x = dx + c$, as $\delta x = \varepsilon^{\mu}\hat{X}_{\mu} x = \varepsilon^{\nu}\partial_{\nu}x^{\mu} = \varepsilon$ according to (3.4) and (3.13), that is, D corresponds to a T(4)-covariant exterior derivative. Note that $D\phi = d\phi + c \wedge \phi = d\phi$ by $\delta\phi \equiv 0$.

Observe that by (4.27), c may be transformed to zero locally by local translations. If then, as we assumed, dx is an orthonormal basis at that particular point where $c = 0$ and therefore $e = dx$, we see that e stays on an <u>orthonormal basis</u> throughout, because of (4.28). This orthonormal anholonomic 1-form basis has to be used to construct any local translation invariant Lagrangian L', which is of the type

$$L'(dx,\phi,d\phi,c) = L(e{=}Dx,\phi,d\phi) \tag{4.29}$$

in order to meet L in the case $c = 0$ ("minimal coupling" of the gauge potentials c to the chart mapping x, as the latter are assumed to be the only fields the translation group acts on, see (4.21)).

<u>Remark</u>. According to (3.17) one might have expected some (weak) conservation laws corresponding to the symmetry $\delta L \equiv 0$. As we are dealing with translations, the presumably weakly conserved "charge"-currents $\partial L/\partial c$ are expected to correspond to the energy/momentum current ${}^*t$. And indeed, this is the case as ${}^*t := \partial L/\partial e \equiv \partial L/\partial dx \equiv \partial L/\partial c$. However, the corresponding weak conservation law $d{}^*t = 0$ (or rather a generalized version of it, as the coordinate basis dx has to be replaced by an anholonomic one) is obtained in a different way, using the identity $\ell(e_i)L \equiv di(e_i)L \;\forall\, i$, which yields

$$i(e_j)d\phi \wedge {}^{*}T + (-)^p\, i(e_j)\phi \wedge d{}^{*}T \equiv d{}^{*}t_j - i(e_j)de^k \wedge {}^{*}t_k \tag{4.30}$$

where ${}^{*}T$ corresponds to the Euler-Lagrange expression (2.13); see [7] for further details.

4.5 Lagrangians for the free T(4)-gauge potentials. Analogously to Sec. 3.3 we assume $L_o = L_o(e,c,dc)$, where e is already taken as the local translation invariant orthonormal basis of (4.28). Then

$$C := dc \equiv de \tag{4.31}$$

by (4.28), C being the T(4)-field strength, analogously to (3.22) but the $A \wedge A$-term discarded because T(4) is abelian. But then $\delta L_o \equiv 0$ simply gives $\partial L_o/\partial c|_{ex} \equiv 0$, that is, the T(4)-gauge potentials have to appear in L_o only through the local translational invariant e and de. There are several independent possibilities for L_o, e.g.

$$\begin{aligned} L_o^{(o)} &= \frac{1}{2}\, e^i \wedge {}^{*}e_i \\ L_o^{(1)} &= \frac{1}{2}\, de^i \wedge {}^{*}de_i \\ L_o^{(2)} &= \frac{1}{2}\, de^i \wedge e_i \wedge {}^{*}(de^j \wedge e_j) \\ L_o^{(3)} &= \frac{1}{2}\, de^i \wedge e^j \wedge {}^{*}(de_j \wedge e_i) \end{aligned} \tag{4.32}$$

where $L_o^{(o)}$ corresponds to a "cosmological term" and the other are correlated to the so called "Weitzenböck-invariants" in a unique way, see [9]. Now, compare (4.32) with the Einstein Lagrangian L_E (4.13) ($L_{\tilde{E}}$, respectively) and observe that L_E is made up of $L_o^{(2)}$ and $L_o^{(3)}$ only (up to the dynamically irrelevant exact term). Thus conclude that Einstein's Lagrangian may be viewed as one possible Lagrangian for the free T(4)-gauge potential c which appear in $L_{\tilde{E}}$ via the T(4)-invariant e and de - and GRT becomes a gauge theory of the 4-dimensional translation group T(4).

Remark. Of course, $L_{\tilde{E}}$ is not the only possible choice one can make. However, any choice has to lead to a gravity theory comparable with experiments. By this criterion, for example, $L_0^{(1)}$ is ruled out as the corresponding field equations do not admit a "Newtonian-limit" version of the theory. On the other hand $L_0^{(2)}$ vanishes identically in the case of spherical symmetry (where $de^i \sim e^i$) and is thus not suited as a Lagrangian for its own. $L_0^{(3)}$ finally, which is used by Hehl et al. ([8], [9]) leads to a gravitation theory undistinguishable from GRT up to the 5th order of post-Newtonian approximation, as shown e.g. in [10]. Einstein's Lagrangian, however, has one essential property - it is, in addition, invariant under local (spacetime dependent) Lorentz rotations of the basis 1-forms e^i, the corresponding (symmetric) Einstein current $*G^k$ thus has to be coupled to a symmetric matter energy/momentum current, as one expects within a translational gauge theory, where the spin aspect is left unbased.

4.6 A natural extension: U_4-theories. The last point in the preceding remark already suggests a natural extension of the T(4)-gauge theoretical version of GRT, namely, to include also the spin aspect by gauging the homogeneous Lorentzgroup. To this end, start again with a T(4)-gauge theory based on (ℓ^2 = essentially the gravitational constant, corresponding to the squared translational coupling constant)

$$L(e,de,\phi,d\phi) = L_m(e,\phi,d\phi) + \frac{1}{\ell^2} L_T(e,de) \qquad (4.33)$$

and perform the "minimal procedure" to gauge the Lorentzgroup (viewing e and ϕ as SO(3,1)-fields), that is to say, replace the ordinary exterior derivative d by the Lorentz-covariant derivative D, whereby six Lorentz-gauge potentials $\omega^{ij} = \omega^{[ij]}$ (and therefore a metric connection) are introduced as compensating fields. Furthermore, add a suitable Lagrangian for the free gauge fields (see the collection (3.26)), furnished with a suitable coupling constant κ according to (3.27). The result will be

$$L(e,de,\phi,d\phi,\omega,d\omega) = L_m(e,\phi,D\phi) + \frac{1}{\ell^2} L_T(e,De=:\Theta) + \frac{1}{\kappa^\zeta} L_R(e,\Omega) \qquad (4.34)$$

where $D\phi := d\phi + \omega \wedge \phi$, $\Theta := De = de + \omega \wedge e$ (torsion) and $\Omega := d\omega + \omega \wedge \omega$ (Riemann-Cartan curvature). Thus one arrives at a Lagrangian (4.34) corresponding to a field theory over a Riemann-Cartan spacetime (U_4-spacetime) including torsion. Theories of this type are frequently called U_4-theories.

Take, for example, L_T as the Einstein-Lagrangian $L_{\tilde{E}}$ (where de is replaced by torsion Θ), observe that L_R linear in Ω leads to nonpropagating torsion (see e.g. [9]) and by this reason, choose one of the Lagrangians squared in Ω, say $L_0^{(1)} = \frac{1}{2}\,\Omega^{ij} \wedge {*\Omega_{ij}}$ ($\zeta = 2$ in (4.34)), and you get a theory proposed by the author in [9]. However, this is just one of a great variety of possibilities. In general, one is well advised to consider a linear combination of all Lagrangians in (3.26) and (4.32) (de $\rightarrow \Theta$) and investigate which combination will survive under certain conditions as the "Newtonian-limit" criterion, the existence of a (perhaps extended) Birkhoff theorem, or, as done in [11], the criterion not to have tachyonic- and ghost-solutions in the linearized version.

Remark on teleparallelism. Any decoupling of the Lorentzgroup by $\kappa \rightarrow 0$ (and therefore $\Omega^{ij} \rightarrow 0 \;\forall\; i,j$) ought to lead back to the T(4)-gauge theory based on (4.33), that is, in the case of $L_T = L_{\tilde{E}}$, to GRT ("translation-limit"). However, the corresponding condition $\Omega^{ij} \equiv 0 \;\forall\; i,j$ restricts the underlying spacetime now more than the one we started with, i.e. spacetime of a translational gauge theory has to admit a teleparallel basis (the one with respect to which $\omega \equiv 0$ globally) and is then called Weitzenböck-(T^4-) spacetime, in contrast to the more general Riemannian spacetime V^4 of GRT. In principle, this point of view is acceptable, once the T(4)-invariant basis in (4.28) is interpreted as a teleparallel basis (modulo global Lorentz rotations) and the field equations of GRT are accompanied by constraints for the Riemannian curvature $R^i{}_j$ according to the teleparallelism conditions $\Omega^i{}_j \equiv 0$.

5. Conclusion

In carrying over the concept of a gauge theory (outlined in Sec. 3) to Einstein's general relativity theory (GRT), we are led to the possibility of interpreting GRT as a (pure) gauge theory of the 4-dimensional translation group T(4), where the corresponding gauge potentials c are incorporated in a T(4)-invariant way via orthonormal anholonomic basis 1-forms (Sec. 4). This seems to be very reasonable, as GRT stresses in particular the mass/energy aspect of physics. To include also the spin aspect on an equal footing, we just indicate in Sec. 4.6, how GRT may be extended by performing in addition the gauge procedure for the Lorentzgroup, and by which a Riemann-Cartan manifold as underlying spacetime is suggested.

For obvious reasons, the concept of a gauge theory was outlined at the same level as GRT - over spacetime. But as in modern gauge theory the notion of fiber bundles is extensively used, there are some remarks to be made, to link both approaches. For the sake of briefness, the terminology is taken for granted, otherwise see [1].

In the fiber bundle formulation of a gauge theory, the Lie-groups we are dealing with are contained as typical fibers of principal fiber bundles $(P,M^4,\pi;G)$ over spacetime M^4, where P denotes the bundle space and $\pi: P \to M^4$ a projection. In that picture a gauge potential is viewed as a certain Lie-algebra-valued 1-form over P, $\tilde{A} \in E_1(P,\mathcal{G})$, called the connection of the principal fiber bundle. The usual "matter-fields", as counterparts of the gauge potentials, are V-valued, horizontal and equivariant p-forms over P, $\tilde{\phi} \in E_p(P,V)$, where V is a representation space of G. As G is the typical fiber of $(P,M^4,\pi;G)$ and as the matter fields are equivariant, any action of G on V is uniquely linked with vertical automorphisms in P, called "gauge transformations". Now, the gauge potentials $A \in E_1(M^4,\mathcal{G})$, G-fields $\phi \in E_p(M^4,V)$ and gauge transformations we were dealing with in Sec. 3 are connected with the fiber bundle objects via (local) sections of P (i.e. $s: M^4 \supseteq U \to P$, $\pi \circ s = id_U$), by $A := s^*\tilde{A}$ and $\phi := s^*\tilde{\phi}$, where s^* denotes pull back of forms. Note that A and ϕ are therefore section-dependent. As local sections provide local immersions of M^4 into P, any vertical automorphism in P is linked with a change of sections (gauge transformations).

For example, in the case of the Lorentzgroup, the corresponding principal fiber bundle is known to be the bundle of orthonormal bases and any local section corresponds to a local choice of a basis. As the gauge potential $\omega = s^{*}\tilde{\omega}$ depends on sections, it depends on the basis chosen. And the same goes for the gauge potential c for the translation group $T(4) = (\mathbb{R}^4,+)$ with the argument reversed. As by (4.27), the T(4)-gauge potentials depend on the choice of coordinates, the corresponding principle fiber bundle will be a "coordinate bundle", where local sections correspond to a local choice of coordinates (i.e. P equals $U \times \mathbb{R}^4$ locally, because $G = T(4) = (\mathbb{R}^4,+)$, but the image of any section has to be diffeomorphic to an open neighbourhood in $\mathbb{R}^4$, in order to provide a "coordinate section"). Further details will be traced in [7].

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References

[1] W. Drechsler/M.E. Mayer: "Fiber Bundle Techniques in Gauge Theories", Lecture Notes in Physics 67 (1977)

A. Trautman: "Fiber Bundles, Gauge Fields, and Gravitation", in "General Relativity and Gravitation. One Hundred Years after the Birth of Albert Einstein". Ed. A. Held. Vol. 1, Chap. 9. Plenum Press, New York

W. Thirring: "Gauge Theories of Gravitation", Acta Phys. Austr., Suppl. XIX, 439 (1978)

[2] R. Utiyama, Phys. Rev. 101, 1597 (1956)

[3] S. Weinberg, Rev. Mod. Phys. 46, 255 (1974)

H. Fritzsch, Acta Phys. Austr., Suppl. XIX, 249 (1978)

P. van Nieuwenhuizen, in Marcel Grossmann Meeting on General Relativity, ed. by R. Ruffini, North-Holland (1977)

[4] Y. Choquet-Bruhat/C. DeWitt-Morette/M. Dillard-Bleick: "Analysis, Manifolds and Physics", North-Holland (1977)

[5] W. Thirring: "Classical Field Theory", Springer (1980)

[6] C.W. Misner/K.S. Thorne/J.A. Wheeler: "Gravitation", Freeman (1973)

[7] R.P. Wallner, Thesis, to appear 1981

[8] F.W. Hehl: "Four Lectures on Poincaré Gauge Field Theory", in "Proceedings of the 6th Course of the International School of Cosmology and Gravitation on Spin, Torsion, Rotation, and Supergravity", Erice (1979), eds. P.G. Bergmann and V. de Sabbata, Plenum Press, New York (1980)

J. Nitsch: "The Macroscopic Limit of the Poincaré Gauge Field Theory of Gravitation", same issue (1980)

[9] R.P. Wallner: "Notes on Recent U_4-Theories", to appear in GRG-Journal (1981)

[10] M. Schweizer/N. Straumann, Phys. Lett. 71A, 493 (1979)

[11] K. Hayashi/T. Shirafuji: "Gravity from Poincaré Gauge Theory of the Fundamental Particles", preprint (1979)

E. Sezgin/P. van Nieuwenhuizen, Phys. Rev. D21, 3269 (1980)