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SUPERSYMMETRIC DIRAC PARTICLES IN RIEMANN-CARTAN SPACE-TIME

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Abstract

A natural extension of the supersymmetric model of Di Vecchia and Ravndal yields a nontrivial coupling of classical spinning particles to torsion in a Riemann-Cartan geometry. The equations of motion implied by this model coincide with a consistent classical limit of the Heisenberg equations derived from the minimally coupled Dirac equation. Conversely, the latter equation is shown to arise from canonical quantization of the classical system. The Heisenberg equations are obtained exact in all powers of $\hbar$ and thus complete the partial results of previous WKB calculations. We touch also on such matters of principle as the mathematical realization of anticommuting variables, the physical interpretation of supersymmetry transformations, and the effective variability of rest mass.

1. Introduction

Physicists have been interested in the formulation of a classical mechanics of spinning particles ever since the spin of the electron was discovered. The traditional approach, which treats the electron as a classical relativistic top, dates back to Frenkel [1], and has recently been revived by Hanson and Regge [2] and, in a gravitational context, by Hojman [3], to mention only a few. It is fair to say, however, that all these attempts have been only of limited success if compared with the predictions of quantum theory. A second line of investigations was opened by Pauli [4] in 1932, when he derived a classical equation of the translational motion of the electron in an external electromagnetic field by a WKB approximation to the Dirac equation. Pauli's result, which contained no spin-orbit coupling, was challenged later by de Broglie [5]. It was only in 1964 that the WKB method was applied to the spin motion by Rubinow and Keller [6], who thus reproduced the classical equation of Bargmann, Michel and Telegdi (BMT) [7] with $g = 2$. A purely algebraic scheme of deriving classical equations of motion from Dirac's theory was indicated by Corben [8] and further elaborated by the author [9]. The ultimate version of that method will be presented in Sec. 6 of this paper. In our opinion the physical interpretation of the method, as well as of the Dirac equation itself, becomes clear only in a supersymmetric framework.

The merits of a description of classical particle spin dynamics in terms of "anticommuting c-numbers" had been recognized already in the fifties (see [10] for a list of early references). These numbers turned out to be the natural objects from which to construct the Feynman path integral for fermions [11], and they allowed a concise field-theoretical formulation of the fermionic dual models [12]. Unprecedented interest in the use of Grassmann variables was finally aroused by the discovery of supersymmetry. Curiously enough, local supersymmetry was employed even before global supersymmetry in models of spinning particles. In the papers of Brink et al. [13] this effort was inspired by the desire to build a one-dimensional model of the interaction between supermatter and supergravity. Berezin and Marinov [10], who were the first to treat massive particles in this manner, were led by the analogy with the "supergauge" group introduced in the dual model by Gervais and Sakita [12]. Essentially the same massive particle model was obtained also by Collins and Tucker [14] as a by-product of their treatment of the Neveu-Ramond-Schwarz string. Invariance under change of the evolution parameter and local supersymmetry are maintained in the massive model only at the price of a considerable complication of the formalism, especially with respect to quantization. A much simpler theory results, however, if the requirements of parametrization invariance and local supersymmetry are dropped (while maintaining global supersymmetry), as was shown by Di Vecchia and Ravndal [15].

A detailed exposition of the general features of the globally supersymmetric model will be given in Secs. 2 and 3 of this paper. While in the application of the model Di Vecchia and Ravndal had confined themselves to the derivation of previously established results, the complete classical equations of motion for Dirac particles in a Riemann-Cartan (U_4) space-time will be derived in Sec. 4. Only the leading terms, in an expansion in powers of the spin tensor, have been obtained so far by WKB methods [9,16,17]. Our result contrasts that of Barducci et al. [18] who had concluded from their model (which is not supersymmetric) that Dirac particles do not couple to torsion. Canonical quantization in general coordinates in a U_4 space-time is carried out in Sec. 5, and the minimally coupled Dirac equation and the exact Heisenberg equations of motion are obtained in Sec. 6. The latter equations reduce to the classical dynamics of the model in an obvious classical limit.

Apart from these concrete results we hope that this paper will also contribute to a better understanding of the foundations of supersymmetry. Simple though the model is, it excludes a replacement of the Grassmann algebra of variables by a non-associative algebra that was proposed recently [19]. Moreover we point out that the apparent unobservability of the supersymmetry of the model is due to quantization. A similar remark applies to the variability of the rest mass due to the interaction of the spin with external fields.

2. Free Motion and Generalities

According to [15] the supersymmetric Lagrangian for a free spinning particle of mass m is

$$L = \frac{m}{2} \eta_{ab}(\dot{x}^a\dot{x}^b - i\xi^a\dot{\xi}^b) \; . \tag{2.1}$$

The dot denotes differentiation with respect to a proper time parameter s. The x^a and ξ^b are even and odd Grassmann variables, respectively. The Grassmann algebra may be assumed to be four-dimensional . We shall not attach any direct physical meaning to the "spin variables" ξ, but only to the spin tensor constructed out of them according to (2.9).

The spin part of the Lagrangian is essentially determined by the requirement that the free Hamiltonian should be independent of the ξ^a, which hence are essentially canonically conjugate to themselves:

$$H = p_a\dot{x}^a + L\frac{\overleftarrow{\partial}}{\partial\xi^a}\dot{\xi}^a - L = \frac{p^2}{2m} \tag{2.2}$$

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = m\,\eta_{ab}\,\dot{x}^b \; .$$

Left differentiation has been indicated in (2.2). A factor i has to appear in (2.1), as with any definition of an involution in the Grassmann algebra the product $\xi^a\xi^b$ is "antihermitian" if the ξ^a are "hermitian".

The Lagrangian (2.1) is invariant under the supersymmetry transformations

$$\delta x^a = i\varepsilon\xi^a \tag{2.3a}$$

$$\delta\xi^a = \varepsilon\dot{x}^a \tag{2.3b}$$

with ε a (finite) anticommuting parameter. Recently a different interpretation of the anticommuting variables has been proposed [19], in which $\xi^a\xi^b$ is an ordinary imaginary number. As a consequence the algebra becomes non-associative, which makes differentiation at least more complicated. In the present model such a reinterpretation of the ξ's would destroy the supersymmetry, since then products which contain an anticommuting factor twice would no longer vanish automatically. As a consequence also the relationship of the model with the Dirac equation would be lost. Thus an alteration of the Grassmann character of the dynamical variables seems highly undesirable.

A consistent definition of the generalized Poisson bracket of two observables A, B is

$$[A,B\} := \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} + \frac{i}{m}\,\eta^{ab}\,A\,\frac{\overleftarrow{\partial}}{\partial\xi^a}\,\frac{\overrightarrow{\partial}}{\partial\xi^b}\,B\ . \tag{2.4}$$

The consistency will be made explicit in a more general context in Sec. 5. The bracket (2.4) defines a Z_2-graded algebra that carries over into quantum theory (classically, the grading is that of odd and even Grassmann numbers),

$$\deg\,[A,B\} = \deg A + \deg B = -\,(-1)^{\deg A\cdot\deg B}\,[B,A\} \tag{2.5}$$

(deg denotes the degree in the grading). Note that in the odd sector of phase space spanned by the ξ^a there is no natural distinction between "coordinates" and "momenta", although due to the even dimensionality of space-time a "pseudo-symplectic" structure can be made manifest by defining "coordinates" ζ^A and "momenta" β_A as follows:

$$\begin{aligned} \zeta^1 &= (m/2i)^{1/2}(\xi^o + \xi^1)\ , \qquad & \beta_1 &= (m/2i)^{1/2}(\xi^o - \xi^1) \\ \zeta^2 &= (m/2i)^{1/2}(\xi^2 + i\xi^3)\ , \qquad & \beta_2 &= (m/2i)^{1/2}(\xi^2 - i\xi^3)\ . \end{aligned} \tag{2.6}$$

Therefore all the ξ^a should be considered as phase space rather than configuration space variables from the outset.

As Lorentz transformations are generated by

$$J^{ab} = L^{ab} + S^{ab} \qquad (2.7)$$

$$L^{ab} = x^a p^b - x^b p^a \qquad (2.8)$$

$$S^{ab} = im\xi^a \xi^b \qquad (2.9)$$

the quantity S^{ab} has to be interpreted as the spin tensor, while L^{ab} and J^{ab} are the orbital and total angular momentum, respectively. In contrast to these objects the generator Q of the supersymmetry transformation of an observable O implied by (2.3) via

$$\delta O = [O, i\varepsilon Q\} \qquad (2.10)$$

cannot be defined indepently of the dynamics. However iteration of (2.3) implies quite generally that for any observable O

$$[[O,Q\},Q\} = -i\ [O,H\} \quad . \qquad (2.11)$$

By virtue of the graded Jacobi identity

$$[[A,B\},C\} = [A,\ B,C\}\} - (-1)^{\deg A \cdot \deg B}\ [B,[A,C\}\} \qquad (2.12)$$

this implies

$$[Q,Q\} = 2iH \qquad (2.13)$$

and

$$[Q,H\} = 0 \quad . \qquad (2.14)$$

Eq.(2.13) does not mean that the value of Q in the Grassmann algebra is fixed by that of H. In fact there is a continuum of possible values of Q even if H is fixed. As we shall see in Sec. 6 this situation is completely changed by quantization.

The Lagrangian (2.1) is invariant under separate Poincaré transformations of the x^a and ξ^a. The corresponding constants of motion are p_a, L^{ab}, ξ^a and S^{ab}. Thus orbital and spin angular momentum of a free particle are conserved separately. The "supercharge" is

$$Q = p_a \, \xi^a \tag{2.15}$$

and has vanishing bracket with p_a and J^{ab}, but not with ξ^a, L^{ab} and S^{ab}. Therefore a supersymmetry transformation constitutes a transformation of the particle without change of its momentum and total angular momentum, which forces the particle to change its spin angular momentum. Classically this leaves the possibility for the particle to "spin faster or slower". Quantum mechanically, however, the magnitude of the spin vector becomes fixed, and this reduces the supersymmetry to a part of ordinary Lorentz invariance.

3. Explicit Supersymmetry and Interaction with External Fields

The supersymmetry of the Lagrangian (2.1) can be made manifest by introducing an anticommuting evolution parameter Θ and a "super-coordinate" X^a which depends both on s and Θ and which comprises x^a and ξ^a via the familiar Taylor expansion about $\Theta = 0$:

$$X^a(s,\Theta) = x^a(s) + i\Theta\xi^a(s) \ . \tag{3.1}$$

The supersymmetry transformation (2.3) can now be represented by

$$\Theta \to \Theta + \varepsilon \ , \qquad s \to s - i\varepsilon\Theta \ . \tag{3.2}$$

The supercovariant derivative

$$D = \frac{\vec{\partial}}{\partial\Theta} + i \ \Theta \frac{\partial}{\partial s} \tag{3.3}$$

is form-invariant under (3.2) and a "square root" of $i\partial/\partial s$:

$$\frac{\partial X^a}{\partial s} = - iD^2X^a \ . \tag{3.4}$$

With the usual definition [11] of the "integral" over Grassmann variables,

$$\int d\Theta = 0 \quad , \qquad \int d\Theta\Theta = \ell \tag{3.5}$$

(ℓ is a real number with the dimension of length), the Lagrangian (2.1) can be written as

$$L = - \frac{m}{2\ell} \int d\Theta(D^2X^a)DX_a \tag{3.6}$$

which is of the desired form.

There are two possible directions in which this Lagrangian can be generalized. One way is to require invariance under arbitrary reparametrizations of the evolution parameter s and local supersymmetry. These requirements are equivalent to general coordinate invariance in the (s,Θ) superspace and can be fulfilled by introducing a bosonic ("einbein") and fermionic "gauge coordinate", and, in the massive case considered here, an anticommuting Lagrange multiplier ξ_5. The dynamics that emerges can be viewed as a one-dimensional model of supermatter interacting with supergravity (which in the massive case includes a "cosmological term"). Details can be found in the work of Brink et al. [13]. Interesting as this model is in its own right, it contains more degrees of freedom than are needed for the description of a spinning particle. Therefore we shall proceed in a different direction, namely to introduce "interactions" and "self-interactions" among the x and ξ coordinates that will produce couplings of the particle to external fields.

A fundamental role of supersymmetry is suggested by the fact that it uniquely determines the coupling to external fields so as to anticipate the Dirac equation. We illustrate this by the example of an external electromagnetic potential A^a. The only input that is needed for the construction of the corresponding Lagrangian is the demand that for a scalar particle ($\xi^a = 0$) the latter should reduce to

$$L_{sc} = \frac{m}{2}\dot{x}^2 + e\,A^a\,\dot{x}_a\ . \qquad (3.7)$$

All the rest is determined by supersymmetry and is most conveniently evaluated using the explicitly supersymmetric formalism. The result is

$$L = -\frac{m}{2\ell}\int d\Theta\ [D^2X^a + 2i\,\frac{e}{m}\,A^a(X)]\ DX_a \qquad (3.8)$$

$$= \frac{m}{2}\,(\dot{x}^2 + 2\,\frac{e}{m}\,A^a\,\dot{x}_a + i\dot{\xi}^a\xi_a + 2i\,\frac{e}{m}\,A_{a,b}\ \xi^b\xi^a)\ . \qquad (3.9)$$

Because of the anticommuting character of the parameter Θ the "superfunction" $A^a(X)$ is simply

$$A^a(X) = A^a(x) + i\Theta A^a{}_{,b}(x)\ \xi^b\ . \qquad (3.10)$$

Variation of L yields the following equations of motion:

$$m\ddot{x}^a = e\ F^a{}_b\dot{x}^b + \frac{e}{2m}\,\eta^{ad}\ F_{bc,d}\ S^{bc} \qquad (3.11)$$

$$\dot{\xi}^a = \frac{e}{m}\,F^a{}_b\ \xi^b\ . \qquad (3.12)$$

The spin-orbit coupling appearing in (3.11) is exactly that which has to be expected for a particle whose gyromagnetic ratio is $g = 2$. Eq. (3.12) is formally identical with the BMT equation [7] for $g = 2$. But since the identification of ξ^a with the polarization vector is problematic, the comparison with the BMT result is better carried out in terms of the spin tensor (2.9),

$$\dot{S}^{ab} = \frac{e}{m}\,(F^a{}_c\ S^{cb} + F^b{}_c\ S^{ac}) \qquad (3.13)$$

which indeed establishes the equivalence. However whereas the BMT equation was derived only under the assumption of homogeneity of the external field, eqs.(3.11) - (3.13) are not subject to this restriction. Note also that any anomalous magnetic moment would spoil the supersymmetry of the Lagrangian (3.9).

The Hamiltonian and the "supercharge" in the electromagnetic field are given by

$$H = \frac{1}{2m} (p^a - eA^a)(p_a - eA_a) - \frac{ie}{2} F_{ab} \xi^a \xi^b \tag{3.14}$$

$$Q = (p_a - eA_a) \xi^a \tag{3.15}$$

$$p^a = m\dot{x}^a + eA^a . \tag{3.16}$$

It will become evident in Sec. 6 that they imply the minimally coupled Dirac equation after quantization. As H is a constant of motion, the spin term in (3.14) introduces an effective variability of the rest mass of the particle. Therefore the parameter s can no longer be interpreted as the arc length along the particle worldline. This will be discussed further at the end of Sec. 6.

4. A Generalized Supersymmetric Action

Recently alternative theories of gravitation based on a Riemann-Cartan (U_4) geometry have attracted much interest. A space-time endowed with this geometry is not only the natural arena of supergravity, but arises also in classical gauge theories of gravitation that employ the Poincaré group (see e.g. [20]). It is well known that torsion will influence the motion of spinning test particles but not that of scalar ones. In the so-called teleparallelism theories of gravitation [16,21] this effect provides the only conceivable tool to detect deviations from Riemannian geometry experimentally. Therefore the deviation of classical equations of motion for spinning particles in a U_4 background is not merely of academic interest. For Dirac particles the leading terms of these equations in an expansion in powers of $\hbar$ have been obtained recently by WKB methods [9,16,17]. In this section we derive the complete version of these equations (which contains also terms quadratic in the spin tensor) from a supersymmetric Lagrangian variational principle.

In contrast to the electromagnetic and Riemannian gravitational case (which were both considered already in [15]) the supersymmetric action in a U_4 space-time cannot be determined from the knowledge of its scalar part alone, as scalar particles do not couple to torsion. However supersymmetry strongly restricts the possible generalizations, and the following one is essentially unique, if a third-rank tensor is to be involved, as is suggested by the presence of torsion:

$$L = -\frac{m}{2\ell} \int d\Theta \, [g_{\mu\nu}(X) D^2 X^\mu D X^\nu + \hat{S}_{\alpha\beta\gamma}(X) DX^\alpha DX^\beta DX^\gamma] \tag{4.1}$$

$$= \frac{m}{2} \, [g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + i g_{\mu\nu}(\dot{\xi}^\mu + \overset{*}{\Gamma}{}^\mu_{\lambda\rho} \, \dot{x}^\lambda \xi^\rho)\xi^\nu - \hat{S}_{\alpha\beta\gamma,\delta} \, \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta] \, . \tag{4.2}$$

Here $\hat{S}_{\alpha\beta\gamma}$ is a totally antisymmetric tensor, and

$$\overset{*}{\Gamma}{}^\lambda_{\mu\rho} = \{{}^{\lambda}_{\mu\rho}\} + 3\hat{S}_{\mu\rho}{}^{\lambda} \tag{4.3}$$

is a connection compatible with the metric $g_{\mu\nu}$, i.e.

$$\overset{*}{\nabla}_\mu \, g_{\alpha\beta} = 0 \quad , \tag{4.4}$$

$\overset{*}{\nabla}_\mu$ denoting the $\overset{*}{\Gamma}$-covariant derivative and { } the Christoffel symbol. In passing from (4.1) to (4.2) we have introduced the inverse $g^{\mu\nu}$ of the metric $g_{\mu\nu}$ (it enters through the Christoffel symbol in (4.3)). In order for this to exist we have to assume that the metric determinant g = = det $g_{\mu\nu}$ belongs to the multiplicative group of invertible elements of the Grassmann algebra (Grassmann numbers that are not in this group are nilpotent [22]). This assumption is very natural as in the special case $g_{\alpha\beta} = \eta_{\alpha\beta}$ it is certainly fulfilled. Note that due to the commutativity of the functions $g_{\mu\nu}(x)$ the evaluation of the matrix inverse proceeds exactly as in the case of real matrices.

All the indices appearing in (4.1) and (4.2) are to be interpreted as holonomic space-time indices, and using the fact that supersymmetry (2.3) requires the ξ^μ to transform as vectors it is easy to prove that L is a scalar under general coordinate transformations. The non-Riemannian contributions to L vanish if $\hat{S}_{\alpha\beta\gamma} = 0$. The equations of motion implied by (4.2) are the following:

$$\ddot{x}^{\mu} = - \{^{\mu}_{\alpha\beta}\}\dot{x}^{\alpha}\dot{x}^{\beta} + \frac{i}{2}\,\overset{*}{R}{}^{\mu}{}_{\nu\alpha\beta}\,\dot{x}^{\nu}\,\xi^{\alpha}\xi^{\beta} - \frac{1}{2}\,g^{\mu\nu}\,\overset{\{\}}{\nabla}_{\nu}\,\hat{S}_{[\alpha\beta\gamma,\delta]}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \quad (4.5)$$

$$\dot{\xi}^{\mu} = - \overset{*}{\Gamma}{}^{\mu}_{\alpha\beta}\,\dot{x}^{\alpha}\,\xi^{\gamma} - 2i\,g^{\mu\nu}\,\hat{S}_{[\alpha\beta\gamma,\nu]}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma} \quad . \quad (4.6)$$

We have denoted by $\overset{\{\}}{\nabla}_{\mu}$ the Christoffel covariant derivative and by $\overset{*}{R}_{\mu\nu\alpha\beta}$ the curvature tensor of the connection $\overset{*}{\Gamma}$. Due to the total antisymmetry of $\hat{S}_{\alpha\beta\gamma}$, { } may be replaced by $\overset{*}{\Gamma}$ wherever it appears in (4.5). Eq.(4.6) implies an especially simple evolution equation of the spin tensor (2.9):

$$\dot{S}^{\mu\nu} + \overset{*}{\Gamma}{}^{\mu}_{\alpha\beta}\,\dot{x}^{\alpha}\,S^{\beta\nu} + \overset{*}{\Gamma}{}^{\nu}_{\alpha\beta}\,\dot{x}^{\alpha}\,S^{\mu\beta} = 0 \quad (4.7)$$

i.e. parallel transport with respect to the connection $\overset{*}{\Gamma}$.

In the Riemannian case ($\hat{S}_{\alpha\beta\gamma} = 0$) equs.(4.5) and (4.7) accord with the well-known momentum and spin propagation equations of Mathisson [23] and Papapetrou [24] for a classical pole-dipole particle, if the simplest momentum-velocity relation, namely $P^{\mu} = m\,\dot{x}^{\mu}$, is assumed. For the U_4 case analogous classical propagation equations have been derived by Hehl [25], Trautman [26], and, under slightly more general assumptions, by Yasskin and Stoeger [27]. In this general case there is at least no obvious choice of a momentum-velocity relation for which the latter equations would agree with (4.5) and (4.7). What is particularly remarkable about these equations is that due to the total antisymmetry of $\hat{S}$ the connection $\overset{*}{\Gamma}$ is not the most general one that is metric-compatible (this property requires only antisymmetry of the torsion tensor in the last two indices). On the other hand in the Hehl-Trautman equations the connection is allowed to be of this general type. Thus one might suspect that the supersymmetric particle couples only to part of the full U_4 geometry. But of course the meaning of the "full" geometry is not clear as long as this geometry has not been given operational significance. If one takes the point of view that the particle model under consideration constitutes the very attempt to explore a new kind of geometry, one might very well be inclined to interpret this geometry as that of a U_4 with connection $\overset{*}{\Gamma}$ (and hence torsion $3\hat{S}$). At the level of the Lagrangian (4.2) this is certainly legitimate. However we are going to quantize this theory in the next section and thus to construct a field theory. The corresponding field will turn out to be the Dirac field minimally coupled to a U_4 geometry with connection Γ and torsion S related to $\overset{*}{\Gamma}$ and $\hat{S}$ in the following way:

$$\hat{S}_{\alpha\beta\gamma} = S_{[\alpha\beta\gamma]} \tag{4.8}$$

$$\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + 2S_{\gamma\cdot\beta}{}^{\alpha} \quad . \tag{4.9}$$

It is for this reason that we will eventually consider Γ to be the Cartan connection rather than $\overset{*}{\Gamma}$, although the latter appears to be more relevant in an operational sense.

5. Hamiltonian Formalism and Canonical Quantization

It follows from the covariant version of the bracket (2.4) that in a general Riemannian metric (and hence even in Minkowski space, if curvilinear coordinates are introduced) the ξ^μ and p_ν cannot together be canonical degrees of freedom, since

$$[\xi^\mu(x), \xi^\nu(x)\} = \frac{i}{m} g^{\mu\nu}(x) \tag{5.1}$$

and hence

$$[p_\lambda, [\xi^\mu, \xi^\nu\}\} = -\frac{i}{m} g^{\mu\nu}{}_{,\lambda} \neq 0 \ . \tag{5.2}$$

This in itself suggests to base a Hamiltonian dynamics on the anholonomic variables

$$\xi^a := e^a{}_\mu(x)\ \xi^\mu \tag{5.3}$$

with $e^a{}_\mu$ an orthonormal tetrad field,

$$\eta_{ab}\, e^{a}{}_{\mu}\, e^{b}{}_{\nu} = g_{\mu\nu} \; . \qquad (5.4)$$

But in order to substantiate the very definition of the bracket (2.4) we shall in the following supply a more explicit argument for the replacement (5.3), which is derived from Dirac's analysis of Hamiltonian systems with constraints [28]. Although this theory was originally developed for a real phase space, its generalization to Grassmann variables is straightforward.

In a first step, consider the ξ^μ'as configuration space variables and evaluate their conjugate momenta from the Lagrangian (4.2),

$$\pi_\mu = L \frac{\overleftarrow{\partial}}{\partial \dot{\xi}^\mu} = -\frac{i}{2} m\, g_{\mu\nu}(x)\, \xi^\nu \; . \qquad (5.5)$$

The fact that $\dot{\xi}^\mu$ is not expressible in terms of ξ and π is due to the existence of 4 primary constraints ϕ'_μ, which are conveniently taken from (5.5) to be defined as

$$\phi'_\mu(\xi,\pi) = \pi_\mu + \frac{i}{2} m\, g_{\mu\nu}\, \xi^\nu \approx 0 \qquad (5.6)$$

where "≈" denotes equality in the weak sense [28]. According to general theory the equations of motion can be obtained from a total Hamiltonian

$$H'(x^\mu, p'_\nu, \xi^\alpha, \pi_\beta, \lambda_\rho) = p'_\mu\, \dot{x}^\mu + \pi_\mu \dot{\xi}^\mu - L + \lambda_\mu \phi^\mu \qquad (5.7)$$

with

$$p'_\mu = \left.\frac{\partial L}{\partial \dot{x}^\mu}\right|_{x^\alpha,\, \xi^\beta} \qquad (5.8)$$

and anticommuting Lagrange multiplies λ_μ. The evolution equations are

$$\dot{x}^\mu = \frac{\partial H}{\partial p'_\mu} \; , \qquad \dot{p}'_\mu = -\frac{\partial H}{\partial x^\mu} \; , \qquad (5.9)$$

$$\dot{\xi}^\mu = \frac{\overrightarrow{\partial}}{\partial \pi_u} H \; , \qquad \dot{\pi}_\mu = \frac{\overrightarrow{\partial}}{\partial \xi^\mu} H \; . \qquad (5.10)$$

From these equations we infer the following definition of the "primitive" Poisson bracket:

$$[A,B\}_{P.} = \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p'_\mu} - \frac{\partial A}{\partial p'_\mu}\frac{\partial B}{\partial x^\mu} + A\frac{\overleftarrow{\partial}}{\partial \xi^\mu}\frac{\overrightarrow{\partial}}{\partial \pi_\mu}B + A\frac{\overleftarrow{\partial}}{\partial \pi_\mu}\frac{\overrightarrow{\partial}}{\partial \xi^\mu}B \,. \qquad (5.11)$$

Due to the plus sign connecting the last two terms (5.11) defines also a grading. In the U_4 case H' is given by

$$H' = \frac{p'^2}{2m} - \frac{i}{2}\overset{*}{\Gamma}{}^{\lambda}_{\rho\sigma}\, p_\lambda\, \xi^\rho\xi^\sigma - \frac{m}{8}g^{\mu\nu}\overset{*}{\Gamma}_{\mu\alpha\beta}\overset{*}{\Gamma}_{\nu\gamma\delta}\,\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta + \frac{m}{2}S_{\alpha\beta\gamma,\delta}\,\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta +$$

$$+ \lambda^\mu(\pi_\mu + \frac{i}{2}m\, g_{\mu\nu}\,\xi^\nu) \,, \qquad (5.12)$$

$$p'_\mu = m\, g_{\mu\nu}\,\dot{x}^\nu + i\,\frac{m}{2}\overset{*}{\Gamma}_{\mu\alpha\beta}\,\xi^\alpha\xi^\beta \,. \qquad (5.13)$$

Solving (5.9) and (5.10) yields

$$\dot{\xi}^\mu = -\lambda^\mu \qquad (5.14)$$

and the equations of motion (4.5) and (4.6).

For the purpose of quantization we would like to get rid of the Lagrange multipliers in the Hamiltonian. This is indeed possible by a "canonical reduction" to the true dynamical degrees of freedom. To this end the canonical structure is redefined by the so-called Dirac bracket:

$$[A,B\}_{D.} = [A,B\}_{P.} - [A,\Phi'_\mu\}_{P.}\, C'^{\mu\nu}[\Phi'_\nu,B\}_{P.} \qquad (5.15)$$

with

$$C'^{\alpha\beta}[\Phi'_\beta,\Phi'_\gamma\}_{P.} = \delta^\alpha{}_\gamma \,. \qquad (5.16)$$

(In general C' is the matrix inverse of the Poisson brackets of the so-called second-class constraints, which in our case coincide with the primary ones, however.) Specializing again, we have

$$C'^{\mu\nu} = \frac{1}{im}g^{\mu\nu} \qquad (5.17)$$

$$[\xi^\mu,\xi^\nu\}_{D.} = \frac{i}{m}\, g^{\mu\nu} \tag{5.18}$$

$$[p'_\alpha,p'_\beta\}_{D.} = -\frac{im}{4}\, g^{\mu\nu}\, g_{\mu\rho,\alpha}\, g_{\nu\sigma,\beta}\, \xi^\rho\xi^\sigma \tag{5.19}$$

$$[p'_\alpha,\xi^\mu\}_{D.} = -\frac{1}{2}\, g_{\mu\nu,\alpha}\, \xi^\nu . \tag{5.20}$$

Now we see from (5.19) and (5.20) that the reduced set of canonical variables cannot encompass both the ξ^μ and the p'_α. The dilemma is solved by replacing all of them, the ξ^μ by (5.3), and the p'_μ by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}\bigg|_{x^\nu,\xi^a} = m\, g_{\mu\nu}\, \dot{x}^\nu + i\,\frac{m}{2}\, \overset{*}{\Gamma}_{\mu ab}\, \xi^a\xi^b . \tag{5.21}$$

This replaces the constraints ϕ'_μ by

$$\phi_a = \pi_a + \frac{im}{2}\, \eta_{ab}\, \xi^b \tag{5.22}$$

and makes the Dirac bracket indeed coincide with definition (2.4)! Rewriting the Lagrangian (4.2) in terms of x^μ and ξ^a yields most readily the "reduced" Hamiltonian

$$H(x^\mu,p_\nu,\xi^a) = \frac{p^2}{2m} - \frac{i}{2}\, \overset{*}{\Gamma}{}^\lambda_{ab}\, p_\lambda\, \xi^a\xi^b - \frac{m}{8}\, g^{\mu\nu}\, \overset{*}{\Gamma}_{\mu ab}\, \overset{*}{\Gamma}_{\nu cd}\, \xi^a\xi^b\xi^c\xi^d + \frac{m}{2}\,(\overset{*}{\nabla}_d\, S_{abc})\, \xi^a\xi^b\xi^c\xi^d . \tag{5.23}$$

Of course $H = H'$, if p'_μ and ξ^α, π_β are expressed in terms of p_μ and ξ^a.

The Hamiltonian (5.23) generates the evolution of any observable O according to

$$\dot{O} = [O,H\} . \tag{5.24}$$

The supercharge Q generating the supersymmetry transformations (2.3) via (2.10) is given by

$$Q = p_\mu\, \xi^\mu - \frac{im}{2}\, \hat{\Gamma}_{\mu ab}\, \xi^\mu\xi^a\xi^b \tag{5.25}$$

where

$$\hat{\Gamma}^{b}_{\mu a} = \{^{b}_{\mu a}\} + \hat{S}_{\mu a}{}^{b}. \tag{5.26}$$

is another metric connection. It is instructive to express the conserved quantities Q and H in terms of x, $\dot{x}$ and ξ only:

$$Q = m\, g_{\mu\nu}\, \dot{x}^{\mu}\, \xi^{\nu} + im\, \hat{S}_{abc}\, \xi^{a}\xi^{b}\xi^{c} \tag{5.27}$$

$$H = \frac{m}{2}\, (\dot{x}^{2} + \hat{S}_{\alpha\beta\gamma,\delta}\, \xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta})\ . \tag{5.28}$$

We recognize in (5.28) an effective variable rest mass term as in the electromagnetic case (3.14), whereas the second term in (5.27) has no analog in electromagnetic and Riemannian backgrounds.

Having cast the dynamics into Hamiltonian form we can now perform canonical quantization in the same way as for the free system (conf. [15]). Thus we regard the x^{μ}, p_{ν} and ξ^{a} as self-adjoint generators of an abstract algebra with involution and postulate

$$\begin{aligned} &[A,B\} \to \frac{i}{\hbar}\, [A,B] \qquad \text{if deg } A\cdot\text{deg } B \text{ is even} \\ &[A,B\} \to \frac{i}{\hbar}\, \{A,B\} \qquad \text{if deg } A\cdot\text{deg } B \text{ is odd}\ , \end{aligned} \tag{5.29}$$

[,] and { , } denoting the algebraic commutator and anticommutator, respectively. In particular we have

$$[x^{\mu}, p_{\nu}] = -\, i\hbar\delta^{\mu}{}_{\nu} \tag{5.30}$$

$$\{\xi^{a}, \xi^{b}\} = \frac{\hbar}{m}\, \eta^{ab} \tag{5.31}$$

$$[x^{\mu}, \xi^{a}] = [p_{\nu}, \xi^{b}] = 0\ . \tag{5.32}$$

The representation of this algebra is essentially unique and is generated by the standard position and momentum operators $\hat{x}^{\mu}$ and $\hat{p}_{\nu}$ and by the matrices

$$\hat{\xi}^a = (\hbar/2m)^{1/2}\,\gamma^a \tag{5.33}$$

where γ^a are the Dirac matrices. These operators act on the space of Dirac spinors ψ with the indefinite scalar product

$$\langle\psi_1,\psi_2\rangle = \int d^4x\,(\det e^a{}_\mu)\,\bar{\psi}_1\psi_2\ , \tag{5.34}$$

$\bar{\psi}$ denoting the Dirac-adjoint of ψ. The indefiniteness of (5.34) is a consequence of the indefinite Minkowski metric appearing in (5.31). Although this scalar product is neglected in standard quantization approaches, one may attribute to it a fundamental role in the definition of physical states in quantum field theory in external fields [29].

Eqs.(5.30) - (5.32) define the quantum kinematics of a supersymmetric particle. Its quantum dynamics is implied by (5.24) and (5.29),

$$\dot{O} = \frac{i}{\hbar}\,[O,H]\ . \tag{5.35}$$

The version of relativistic quantum mechanics embodied by (5.30), (5.35) may seem unconventional in that the evolution parameter s is distinct from the physical time x^0, which is just an observable like the other coordinates. As a matter of fact, this approach dates back to Stueckelberg [30], and there exists an extensive literature on it (for a recent article with a fairly complete list of references see [31]). In our opinion the "proper time formalism" suggests itself in view of the difficulties inherent in the more common formulation of relativistic quantum theory (see e.g. [32]). Its role in quantum field theory is discussed in [33], where also further references can be found.

6. The Dirac Equation and Its Classical Limit

The spin 1/2 particle states ψ observed in nature are all eigenstates (or very nearly so) of the universal constants of motion H and Q,

$$H\psi = \frac{m}{2}\,\psi \qquad (6.1)$$

$$Q\psi = \pm\,(m\hbar/2)^{1/2}\,\psi \,. \qquad (6.2)$$

Because of

$$\{Q,Q\} = 2\hbar H \qquad (6.3)$$

(cf. (2.13) and (5.29)) the "mass condition" (6.1) is implied by the "superselection rule" (6.2). This is the reason why in the standard treatment more emphasis is laid upon the Dirac equation (the two possible choices of sign in (6.2) are physically equivalent), although it is (6.1) that governs the dynamical evolution of the particle. This has given rise to many misunderstandings. For instance in many textbooks on relativistic quantum mechanics one finds the statement that the spin and the orbital angular momentum of a free particle are not conserved separately. This statement is based on a "Hamiltonian" constructed out of the time-independent part of Q and appears to be completely out of place from the point of view of the formalism considered here. In fact there is no experimental evidence supporting that statement. In contrast to the classical situation (cf. the remarks following (2.14) and (2.15)) supersymmetry is made unobservable by the condition (6.2).

The Heisenberg equations of motion (5.35) were first proposed by Corben [8] on rather flimsy heuristic grounds. A more elaborated motivation was attempted in [9], but still rested heavily on the apparent success of (5.35) in the electromagnetic case, where it yields (3.11) and (3.13). Of course in the supersymmetric framework (5.35) is self-evident.

Before we derive the Heisenberg equations of motion in the U_4 case we must resolve a difficulty which arises in a general Riemannian metric (and hence even in Minkowski space, if curvilinear coordinates are used). It is connected with the fact that in the standard representation the momentum operator $\hat{p}_\mu = i\hbar\partial_\mu$ is not (even formally) self-adjoint with respect to the scalar product (5.34):

$$\bar{\hat{p}}_\mu = \hat{p}_\mu + i\hbar \left\{ {\lambda \atop \mu\lambda} \right\} . \tag{6.4}$$

(The bar denotes the Dirac adjoint.) One could live with this fact, but it turns out that it is related to a problem of factor ordering that introduces undesired order $\hbar$ and $\hbar^2$ terms into the Heisenberg equations. For simplicity we shall illustrate this by the example of the scalar particle in a Riemannian metric. In the standard representation one has

$$\hat{H}_{sc.} = \bar{\hat{p}}_\mu \, g^{\mu\nu} \, p_\nu = -\hbar^2(-g)^{-1/2}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu \tag{6.5}$$

$$\hat{\dot{x}}^\mu = \frac{1}{2m} \left(g^{\mu\nu} \, \hat{p}_\nu + \bar{\hat{p}}_\nu \, g^{\mu\nu}\right) . \tag{6.6}$$

Now if (6.6) is inverted in order to express $\hat{p}_\mu$ in terms of $\hat{\dot{x}}^\nu$, one obtains an $\hbar$ term from (6.4) which makes the Heisenberg equations very messy. Without attempting to write down these equations we can conclude this also from

$$\hat{H} = \frac{m}{2} \hat{\dot{x}}^\mu \, g_{\mu\nu} \, \hat{\dot{x}}^\nu + \frac{\hbar^2}{4m} \left\{ {\alpha \atop \mu\alpha} \right\} g^{\mu\nu} \left\{ {\beta \atop \mu\beta} \right\} \tag{6.7}$$

and from the apparent impossibility of getting rid of the second term at the right hand side of (6.7) by rearranging the factors in the first term.

The problem is solved by choosing a different representation based on wavefunctions

$$\tilde{\psi} = (-g)^{1/4} \, \psi . \tag{6.8}$$

Now

$$\langle\tilde{\psi}_1,\tilde{\psi}_2\rangle = \langle\psi_1,\psi_2\rangle = \int d^4x \, \bar{\tilde{\psi}}_1 \, \tilde{\psi}_2 \tag{6.9}$$

and $\hat{p}_\mu$ is self-adjoint. The modified Hamiltonian implied by (6.8) is

$$H_{sc.} = \frac{1}{2m} (-g)^{-1/4} \; p_\mu (-g)^{1/2} \; g^{\mu\nu} \; p_\nu (-g)^{-1/4} \tag{6.10}$$

and yields

$$\dot{x}^\mu = \frac{1}{2m} (g^{\mu\nu} \; p_\nu + p_\nu \; g^{\mu\nu}) \tag{6.11}$$

whence

$$H_{sc.} = \frac{m}{8} [(-g)^{-1/4} (g_{\mu\rho} \dot{x}^\rho + \dot{x}^\rho g_{\mu\rho}) (-g)^{1/2} \; g^{\mu\nu} (g_{\nu\sigma} \dot{x}^\sigma + \dot{x}^\sigma g_{\nu\sigma}) (-g)^{-1/4}]. \tag{6.12}$$

Similarly the Heisenberg equation for x implied by (6.10) is still rather lengthy, but it is free of $\hbar$ terms and coincides with the geodesic equation up to factor ordering. Obviously (6.10) is to be considered as the correct version of $H_{sc.}$ also in the abstract algebra of observables.

According to what we have just said the quantum version of the supercharge Q in a U_4 spacetime is given by

$$Q = \xi^\mu (-g)^{1/4} \; p_\mu (-g)^{-1/4} - \frac{1}{2} \hat{\Gamma}_{\mu ab} \; \xi^\mu \; S^{ab} \tag{6.13}$$

(cf. 5.24). In the standard representation this corresponds exactly to the Dirac differential operator minimally coupled [34] to a U_4 with torsion S related to $\hat{S}$ via (4.8). Note that Q is self-adjoint, although neither of the two terms it consists of has this property. The quantum Hamiltonian is

$$H = \frac{1}{2m} [(-g)^{-1/4} (p_\mu - \frac{1}{2} \overset{*}{\Gamma}_{\mu ab} \; S^{ab}) (-g)^{1/2} \; g^{\mu\nu} (p_\nu - \frac{1}{2} \overset{*}{\Gamma}_{\nu ab} \; S^{ab}) (-g)^{-1/4} + \\ + \frac{\hbar^2}{4} \overset{\{\}}{R} - \frac{1}{4} S_{[\alpha\beta\gamma,\delta]} \; S^{\alpha\beta} \; S^{\gamma\delta}] \; . \tag{6.14}$$

The Christoffel curvature scalar term stems from

$$\overset{\{\}}{R}_{\mu\nu\alpha\beta} \; S^{\mu\nu} \; S^{\alpha\beta} = - 2\hbar^2 \; \overset{\{\}}{R} \tag{6.15}$$

and has no counterpart in (5.23), as $R_{\mu\nu\alpha\beta}\,\xi^\mu\xi^\nu\xi^\alpha\xi^\beta$ vanishes classically due to a symmetry property of the Riemann tensor. We should remark that the algebraic square of Q corresponds exactly to the covariant iteration of the Dirac equation,

$$\hat{\nabla}_\mu\gamma^\mu\hat{\nabla}_\nu\gamma^\nu = g^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu - \hat{R}_{\mu\nu\alpha\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta} - \hat{S}_{\mu\nu\rho}\sigma^{\mu\nu}g^{\rho\sigma}\hat{\nabla}_\sigma \tag{6.16}$$

($\hat{\nabla}_\mu$ is the covariant derivative with respect to the connection $\hat{\Gamma}$). Hence (6.14) provides an alternative representation of (6.16).

Because of lack of space we write down the Heisenberg equations of motion for a Dirac particle in a U_4 space only modulo factor ordering in x and $\dot{x}$:

$$\ddot{x}^\mu = -\left\{{\mu \atop \alpha\beta}\right\}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2m}\overset{*}{R}{}^\mu{}_{\nu\alpha\beta}\dot{x}^\nu S^{\alpha\beta} - \frac{\hbar^2}{8m^2}\overset{\{\}}{R}{}_{,\mu} + \frac{1}{2m^2}g^{\mu\nu}\overset{\{\}}{\nabla}_\nu S_{[\alpha\beta\gamma,\delta]}S^{\alpha\beta}S^{\gamma\delta} \tag{6.17}$$

$$\dot{S}^{ab} = -\overset{*}{\Gamma}{}^a_{\mu c}\,\dot{x}^\mu\,S^{cb} - \overset{*}{\Gamma}{}^b_{\mu c}\,\dot{x}^\mu\,S^{ac}\ . \tag{6.18}$$

If the standard polarization vector of the Dirac representation

$$w^\mu = \gamma_5\gamma^\mu \tag{6.19}$$

is introduced, it obeys formally the same equations as ξ^μ in (4.6) (again modulo factor ordering; the same holds for the vector formed by the Dirac matrices γ^μ themselves). A comparison with the classical equations (4.5) - (4.7) shows that they formally coincide with (6.17) and (6.18) except for the $\hbar^2$ term in (6.17). Thus the complete classical equations of motion can be consistently obtained from the Heisenberg equations in a limit defined by replacing the Clifford by a Grassmann algebra and letting $\hbar \to 0$.

Prior to this work only partial information about the classical limit of the Dirac equation in a U_4 geometry had been obtained by methods which either started directly from the iterated Dirac equation [9,16] or employed its conserved current [17]

$$j_\mu = \frac{i}{2m} \bar{\psi} \overset{\leftrightarrow}{\overset{*}{\nabla}}_\mu \psi \qquad (6.20)$$

($\overset{*}{\nabla}_\mu$ is the $\overset{*}{\Gamma}$-covariant derivative). The latter coincides with the "convective part" of a generalized Gordon decomposition of the ordinary Dirac current. It can easily be inferred from (5.21) that in a formal classical limit (6.20) becomes the velocity distribution associated with the classical position probability $\bar{\psi}\psi = \rho$ of the particle. Therefore the conservation of (6.21) corresponds to the classical continuity equation

$$(\rho\, \dot{x}^\mu)_{,\mu} = 0 \qquad (6.21)$$

whereas the conservation of the conventional current $\bar{\psi}\gamma^\mu\psi$ reflects the continuity of spinning matter under the supersymmetry transformation (2.3), (3.1), (3.2)

$$(\rho\, \frac{dx^\mu}{d\theta})_{,\mu} = 0\ . \qquad (6.22)$$

We conclude this section with a discussion of the effective mass term appearing in the iterated Dirac equation due to (3.14) in the electromagnetic case and due to (6.14) in the U_4 case. In the former case the relevant term is $(e\hbar/4m)F_{ab}\sigma^{ab}$ with

$$\sigma^{ab} = \frac{i}{2}\, \gamma^{[a}\, \gamma^{b]}\ . \qquad (6.23)$$

Even if restricted to positive frequency states, this term has imaginary eigenvalues if F_{ab} is electric, and real ones if F_{ab} is magnetic. In the latter case, if the magnetic field B is strong enough, one of the eigenvalues may be smaller than $-m^2$ (corresponding to a state whose magnetic moment is aligned with the field). However, as is well known, a critical field strength at which electrons become tachyonic does not exist. The reason is that the zero point energy associated with the motion perpendicular to the electric field is proportional to B and renders the particle energy always non-negative. The imaginary contribution to m^2 in the electric case does not imply the existence of "antidamped" or "resonance" states, because there is no unusual asymptotic behavior of the wavefunctions if measured in the norm implied by (5.34) (cf. [29]). In the U_4 case (6.14), the $\overset{\{\}}{R}$ term is real and the last term is of the "electric" type. In the Riemannian massless case the $\overset{\{\}}{R}$ term assures conformal invariance as does an analogous (but non-minimal) completion of the scalar wave equation.

7. Conclusion

The purpose of this paper was twofold. Firstly we wanted to work out the classical and quantum implications of the supersymmetric particle model in more detail and in a more general setting than in [15]. Secondly we were interested in the information that can be obtained if the electron is employed as a probe of space-time structure.

As to the first aspect, supersymmetry has proved to be an extremely efficient guiding principle that not only correctly determines the interactions of a classical particle, but also clarifies the status of the Dirac equation in relativistic quantum mechanics. The correct limit $\hbar \to 0$ of this equation involves a contraction of the Heisenberg and Clifford algebras to a Grassmann algebra of "observables" and hence "pre-quantum theory" might be a better term for it than "classical limit".

It had already been known prior to this work that an electron effectively couples only to 4 of the 24 gravitational degrees of freedom that are present in a U_4 space-time in addition to the Riemannian metric. Likewise it had been known that in the leading order of a WKB expansion the motion of the particle is determined by the connection $\overset{*}{\Gamma}$ rather than Γ. This result has been confirmed, and completed by new terms, by the exact Heisenberg equations of motion of Sec. 6 as well as by the pseudoclassical equations derived in Sec. 4. The peculiar contrast between the representation-theoretical and the Grassmann-algebraic approach to spin-1/2 particles is exemplified once again by the fact that in a general Riemannian metric the specification of an orthonormal tetrad field is an indispensible prerequisite of the former approach whereas in the latter it is needed only for the canonical formulation.

The apparent success and the striking simplicity of the model considered clearly point towards a fundamental role of supersymmetry also at the field-theoretical level.

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