Department
of
PHYSICS

UNIVERSITY OF BERGEN
Bergen, Norway

Scientific/Technical Report No. 110

Gauge Theories

C. Jarlskog

Department of Physics, University of Bergen
Bergen, Norway

Lectures given at "Advanced Summer Institute 1978 on New Phenomena in Lepton Hadron Physics", University of Karlsruhe - Germany, September 4 - 16, 1978.

GAUGE THEORIES

Cecilia Jarlskog

Department of Physics, University of Bergen,

Bergen, Norway

TABLE OF CONTENTS

The metric and notations, in these lectures, are as in J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields, McGraw - Hill, 1965.

1. INTRODUCTORY REMARKS

In these lectures, an introduction to the unified gauge theories, of weak and electromagnetic interactions, is given. Strong interactions are treated only briefly; they constitute the theme of Professor Nachtmann's lectures.

The idea of unifying seemingly different forces of nature is not new. More than a century ago, James Clerk Maxwell (1831-1879) succeeded in unifying electricity and magnetism. Within the last decade most of the effort has gone into unifying weak forces with electromagnetism. Such a unification was suggested, by Schwinger, to be realized through a triplet of vector fields, whose universal coupling would generate both the weak and electromagnetic forces (*J. Schwinger, Ann. of Phys.* 2, 407 (1957)). Salam and Ward (Nuovo Cimento 11, 568 (1959)) suggested that the vector bosons should be introduced à la Yang and Mills. It is interesting to note that at that time the muon-neutrino had not yet revealed itself; the known leptons e, μ, ν_e, were supposed to form a triplet.

The Schwinger model was extended by Glashow (Nucl. Phys. 22, 579 (1961)) who, in addition to the triplet of vector bosons, introduced a new neutral vector boson. In the Glashow model, the two neutral vector bosons, denoted by Z^3 and Z^s, mix. The linear combinations $Z^3\cos\theta + Z^s\sin\theta$ and $-Z^3\sin\theta + Z^s\cos\theta$ were assumed to correspond to the physical particles (photon and Z^0, in present terminology). Nowadays, θ is usually referred to as the Weinberg angle. The model predicted neutral currents. Glashow prophesized: "For no choice of θ is the interaction of neutral current small compared with weak interactions involving charged currents".

The model was constructed for leptons; we know that it is not easy to detect the leptonic neutral current, even when it is not small.

In the Glashow model of 1961 the vector meson masses were put in by hand and therefore the model is not renormalizable. The next major step was taken by Weinberg (Phys. Rev. Lett. 19, 1264 (1967)) who employed a spontaneous symmetry breaking mechanism to generate masses. He introduced four Higgs particles, whereof three disappear by giving masses to the intermediate bosons and one (denoted by φ^0) remains. The model relates the masses of the neutral and charged intermediate bosons, $M_W = \cos\theta\, M_Z$. Weinberg guessed that the model may be renormalizable.

Salam and Ward had also been working on introducing masses via spontaneous symmetry breaking (see the talk by A. Salam in the proceedings of the 1968 Noble Symposium, Ed. N. Svartholm).

The extension to hadrons, for four quarks, was given by Weinberg (Phys. Rev. D5, 1412 (1972)) who utilized the so-called GIM-mechanism (Glashow, Iliopoulos and Maiani, Phys. Rev. D2, 185 (1970)).

In 1960's the unified models and Yang-Mills type theories were not particularly popular (however, I remember Veltman giving a series of lectures in a Copenhagen summer school in the late 60's).

The unified theories became fashionable only after t'Hooft (Amsterdam International Conference 1971) demonstrated that the spontaneously broken gauge theories are renormalizable. Two years after, in 1973, neutral currents were discovered at CERN. Since then the unified theories have become more and more the standard framework for describing nature.

Nowadays, it is believed that, the weak and electromagnetic interaction are described by a gauge theory based on the (non-abelian Lie) group $SU(2) \times U(1)$ and strong interactions follow a $SU(3)$ in colour. One goes further and advocates a unification of all nongravitational interactions, for example as in the $SU(5)$ model of Georgi and Glashow (Phys. Rev. Lett. 32, 438 (1974)).

Of course, we have not yet seen any intermediate vector bosons (excepting the photon) nor any Higgs particles, which are the back bone of the modern unified theories; we have not had enough energy to produce any W or Z^0, etc. The fact that the experiments are in an amazingly good agreement with the simplest gauge models makes us believe that we are on the right track; with enough energy we shall be able to admire the glorious intermediate bosons.

2. INGREDIENTS OF GAUGE THEORIES

Gauge theories assume that the Lagrangian formalism provides a correct language in describing nature which is supposed to consist of three classes of particles:

a) The fundamental constituents of matter (leptons and quarks).
b) Intermediate particles.
c) Higgs particles.

a) The fundamental constituents of matter are fermions with spin ½. There are two types of constituents: leptons and quarks (hadrons).

The lepton family contains the observed ones ν_e, e^-, ν_μ, μ^-, ν_τ and τ^- (and of cource their antiparticles). The evidence for ν_τ, although indirect, is substantial. Leptons are integrally charged.

The quark (hadron) family contains the "observed" quarks up (u), down (d), strange (s), charm (c) and beauty or bottom (b). The need for an additional quark, called truth or top (t) is urgent. We hope that Petra will discover the truth.

In the conventional approach, each quark exists in three varieties (colours). For example, there are three up quarks, etc.

b) In gauge theories, the existence of the constituents of matter, together with the Holy Principle of the local gauge invariance (discussed below), leads to the existence of intermediate vector (spin 1) bosons. Their job is to mediate interactions. The family of mediators contains Maxwell's photon (the only member directly seen so far) which mediates electromagnetism. We believe that there are further members, such as $W^\pm$, Z^0 which mediate weak interactions at low energies; gluons G_i, $i=1, \ldots, 8$ who mediate strong interactions, etc. Mediators are integrally or fractionally charged depending on the theory.

c) The Higgs particles are theoretically needed bêtes noires. Some day, perhaps we will learn to live without them, but so far their existence is indispensable. They are believed to be the origin of all masses except their own. We shall discuss their properties in section 10.

3. SYMMETRIES AND CONSERVATION LAWS

The Holy Principle in gauge theories is the Principle of Local Gauge Invariance, as we shall discuss shortly. Before we do so, let us remind ourselves, that the concept of invariance (or covariance), which is very essential in physics, is neatly formulated in Lagrangian language. For example, the principle that the laws of physics are independent of the particular Lorentz frame chosen requires the Lagrangian density to be a scalar quantity; the equations of motion must be covariant, etc.

[Exercise: State the physical principles leading to the laws of energy-momentum conservation, angular momentum conservation and parity conservation (if it had been true). How are these conservation laws guaranteed in the Lagrangian formalism?]

Not all conservation laws have to do with space and time. There are a rather large number of physically established conservation laws, which pertain to "internal" properties. Let us give a few examples:

a) Conservation of charge.
b) Conservation of baryon number B, and lepton numbers L_e, L_μ and L_τ.
c) It is commonly believed that colour is an exact symmetry, and only the colour singlet states are observable. There are however theories in which the colour leaks out.

Experimentally, there is, up to now, no evidence against the conservation laws a - c. Approximate conservation laws have also been observed in nature, for example

d) conservation of strange-, charm-, beauty-(ness) by strong and electromagnetic interactions. These are violated by weak interactions.
e) Isospin symmetry in strong interactions.
f) SU(3)-symmetry, again in strong interactions.

We leave it to our readers to ponder on the physical principles responsible for the symmetries above. We may ask whether the conservation laws a) and b) are exact? Are all physical states, which go through our bubble chambers, calorimeters, etc, colour singlets?

We shall return to some of these questions later on in these lectures. Let us conclude this section by simply noting that the experimentally established conservation laws are easily incorporated as symmetries into our Lagrangians, even if we do not understand their origin.

4. LOCAL GAUGE INVARIANCE AND QUANTUM ELECTRODYNAMICS (QED)

We shall now construct the simplest example of a gauge theory employing the principle of local gauge invariance. Suppose that we had only a single massless fermion f. Then the free Lagrangian, describing f is given by

$$\mathcal{L}_o = i\,\bar{\psi}_f(x)\gamma_\mu \frac{\partial}{\partial x_\mu}\psi_f(x), \tag{4-1}$$

Note that $i\frac{\partial}{\partial x_\mu}$ is the four momentum operator in quantum mechanics. We often refer to this term as the kinetic energy term, in analogy with $L = T - V$, where the mass term goes into V. $\mathcal{L}_o$ is invariant under the global gauge transformation

$$\psi_f(x) \rightarrow e^{i\Lambda}\psi_f(x), \tag{4-2}$$

where Λ is a constant, viz. $\mathcal{L}_o \rightarrow e^{-i\Lambda}\mathcal{L}_o e^{i\Lambda} = \mathcal{L}_o$. This transformation is called global, because Λ is a universal constant, independent of space and time.

The Holy Principle in gauge theories is that the above invariance should hold locally, i.e., for $\Lambda = \Lambda(\bar{x},t)$, where Λ is a real function of the space and time. In other words the phase of $\psi_f(x)$ is unobservable and may be chosen at will. If we accept this law, we must modify our Lagrangian, because

$$\psi_f(x) \rightarrow e^{i\Lambda(x)}\psi_f(x) \tag{4-3}$$

implies

$$\mathcal{L}_o \rightarrow \mathcal{L}_o - \bar{\psi}_f(x)\,\gamma_\mu\,\psi_f(x)\,\frac{\partial\Lambda(x)}{\partial x_\mu}$$

whereby the Lagrangian is not invariant and the phase matters. Since in the additional term $\frac{\partial\Lambda}{\partial x_\mu}$ is a vector, we introduce a vector field $A_\mu(x)$, and replace $\mathcal{L}_o$ with $\mathcal{L}_1$ given by

$$\mathcal{L}_1 = i\,\bar{\psi}_f(x)\,\gamma_\mu\left[\frac{\partial}{\partial x_\mu} - i\,c\,A^\mu(x)\right]\psi_f(x) \tag{4-4}$$

where c is an arbitrary constant. The local gauge transformation is now

$$\psi_f(x) \rightarrow e^{i\Lambda(x)}\psi_f(x),\ \bar{\psi}_f(x) \rightarrow e^{-i\Lambda(x)}\bar{\psi}_f(x),\ A^\mu(x) \rightarrow A^\mu(x) + \delta A^\mu. \tag{4-5}$$

Requiring $\mathcal{L}_1$ to be invariant gives

$$\delta A^\mu = \frac{1}{c}\frac{\partial\Lambda(x)}{\partial x_\mu}\ . \tag{4-5a}$$

This simple example illustrates the major feature of gauge theories: the requirement of local gauge invariance (in our case under phase transformation $U(1)$ implies the existence of the intermediate (gauge) bosons (here the A^{μ} field). A^{μ} may be interpreted as the photon field (whereby $c = e\, Q_f$; $e \equiv$ unit of charge and $Q_f \equiv$ the charge of the fermion f) if we are allowed to add a kinetic energy term, for the photon, in accordance with the gauge principle. The term

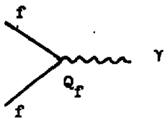
$$\mathcal{L}_o^{(\gamma)} = \frac{-1}{4} F_{\mu\nu}F^{\mu\nu} \; , \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{4-6}$$

has the desired properties. Notice that the photon mass must vanish identically, because a mass term $\frac{1}{2}\, m_{\gamma}^2\, A_{\mu}A^{\mu}$ violates the rule (4-5) and is therefore forbidden. A fermion mass term, on the other hand, may be added $- m_f\, \bar{\psi}_f(x)\, \psi_f(x)$, since it is invariant under (4-5). Collecting all term we have the Lagrangian of QED

$$\mathcal{L} = i\, \bar{\psi}_f(x)\, \gamma_{\mu} \left[\frac{\partial}{\partial x_{\mu}} - i\, e\, Q_f A^{\mu}\right] \psi_f(x) - \frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} -$$

$$- m_f \bar{\psi}_f(x)\psi_f(x) \; . \tag{4-7}$$

Putting f = electron, $Q_f = -1$, gives QED for electrons which has been the most successful theory we have ever had. The fine structure constant $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ is fortunately small; furthermore, the perturbative approach makes sence (the theory is renormalizable). The basic interaction in QED is represented by the diagram



The constants $|e|Q_f$ and m_f are not predicted by the theory but determined from experiment. QED is an example of an abelian (named after the Norwegian mathematician Niels Henrik Abel (1802-1829) gauge theory, i.e., the Λ's (if we had several of them) are just c-numbers and commute with each other. Note also that the gauge boson (γ) carries no charge and the coupling constant Q_f is not universal. The latter is seen by taking several basic fermions such as u, d, e, μ, ... and going through the arguments above, viz.,

$$\mathcal{L}_o + \mathcal{L}_1 = i \sum \bar{\psi}_j(x)\gamma_{\mu} \left[\frac{\partial}{\partial x_{\mu}} - ieQ_j\, A^{\mu}(x)\right] \psi_j(x) \tag{4-8}$$

$$\psi_j(x) \to e^{i\Lambda_j(x)} \psi_j(x) \text{ , } A^\mu(x) \to A^\mu(x) + \delta A^\mu(x) \qquad (4\text{-}9)$$

Substituting (4-9) in (4-8) together with the requirement of invariance yields

$$\Lambda_j(x) = Q_j \, \Lambda(x) \text{ , } \quad \delta A^\mu(x) = \frac{1}{e} \frac{\partial \Lambda(x)}{\partial x_\mu} \text{ ,} \qquad (4\text{-}10)$$

where $\Lambda(x)$ is arbitrary. The point is that the phases $\Lambda_j(x)$ of the fermion fields are no longer unobservable and arbitrary; we must choose them proportional to their empirical charges! Of course, there is nothing in the theory which tell us why $Q_u = 2/3$, $Q_d = -1/3$, $Q_e = -1$, etc. The gauge group U(1) has this ugly arbitrariness, i.e., it allows an arbitrary coupling constant for each fermion (or group of fermions, see below).

5. WEAK INTERACTIONS, QUANTUM FLAVOUR DYNAMICS (QFD)

In physics slang, each kind of particle is said to distinguish itself from other kinds by having a particular flavour. The observed leptonic flavours are ν_e, e, ν_μ, ..., and the hadronic flavours are supposed to be u, d, s, ...

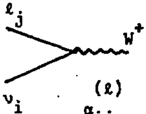
Electromagnetism is flavour blind (viz. 

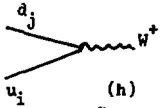
).

"Flavour dynamics" shows up in weak interactions, where the charged currents are flavour-changing. For example an electron-neutrino is turned into an electron, etc.

In this section we discuss the conventional theory of weak interaction, for four quark flavours and four leptons, i.e., the theory that we believed in until 1975, before the τ was discovered.

The charged-current weak interactions were described by the Cabibbo-theory supplemented by the Glashow-Iliopoulos-Maiani (GIM) mechanism

$\mathcal{L}^{c.c.} = g \sum_{i,j}$ 

+ 

+ h.c. =

$$g \sum_{i,j=1,2} \left\{ \alpha^{(\ell)}_{ij} \bar{\ell}_j \, \gamma^\lambda (1-\gamma_5) \nu_i + \alpha^{(h)}_{ij} \bar{d}_j \, \gamma^\lambda (1-\gamma_5) u_i \right\} W^+_\lambda + \text{h.c.} \qquad (5\text{-}1)$$

Here W^+_λ, ν_i, etc. refer to the field operators for W^+, neutrino, etc.; g is a constant $g^2/M_W^2 = G/\sqrt{2}$, M_W = mass of $W^\pm$, G = Fermi constant. Furthermore

$$\nu_1 = \nu_e \quad \nu_2 = \nu_\mu \cdot \quad u_1 = u \quad u_2 = c$$

$$\ell_1 = e \quad \ell_2 = \mu \quad d_1 = d \quad d_2 = s \ .$$

The α's are given by

$$\alpha_{ij}^{(\ell)} = \delta_{ij} \ , \ \alpha_{ij}^{(h)} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}_{ij} \quad , \ \theta_c = \text{Cabibbo angle.}$$

The elegant form (5-1) had gradually emerged from several decades of research on weak interactions. Each piece has a raison d'être and $\mathcal{L}^{cc}$, whenever applicable, gives a good description of data.

After the discovery of the τ-lepton and the quark flavour b, it is believed that we should modify (5-1) by introducing $\nu_3 = \nu_\tau$, $\ell_3 = \tau$, $u_3 = t$, $d_3 = b$ (where t is a to be discovered quark flavour) and let the sum over i and j run from 1 to 3. Furthermore $\alpha_{ij}^{(\ell)} = \delta_{ij}$ and $\alpha^{(h)}$ is a general three by three unitary matrix. We shall discuss the four and six quark models in detail later on.

After the discovery of neutral currents in 1973, we know for sure that the $\mathcal{L}^{cc}$ above cannot explain all weak interactions. As we shall see in the next section, the gauge theories provide a natural scene for neutral currents. What is much more, they allow the unification of weak and electromagnetic interactions which is, of course, marvelous. Moreover, it has been known for quite some time that the theory described by (5-1) is plagued by awful diseases, viz., some physical cross sections and transition probabilities are infinitely large and uncalculable. Gauge theories provide a framework in which the diseases are cured, one gets sensible expressions for cross sections, etc. Has Nature solved her problems by utilizing some local gauge theory? We hope so, and are encouraged by data which are in amazing agreement with the simple gauge model described in the next few sections.

6. CONSTRUCTION OF THE STANDARD SU(2) x U(1) MODEL

In this section we describe the construction of the standard model of weak and electromagnetic interactions, the so-called SU(2) x U(1) model. The model is "standard" in the sence that it seems to explain all data on neutral currents (a task which no other model can fulfill) provided we forget the confused situation in atomic physics experiments.

Let us, for simplicity,* assume that there were only two elementary fermions in nature, denoted by f and f' and that $Q_f = Q_{f'} + 1$, where Q is the electric charge. These assumptions are based on the observation that nature exhibits several such couples. For example (f,f') could be (ν_e, e), (ν_μ, μ), (u,d), etc. We now construct a gauge theory, proceeding along the main road laid out in section 4 but have to make appropriate detours and modifications. As in section 4 we introduce the kinetic term for the (massless) f and f'

$$\mathcal{L}_0(f,f') = i\left[\bar{f}\,\gamma\,\frac{\partial}{\partial x}\,f + \bar{f}'\gamma\,\frac{\partial}{\partial x}\,f'\right]\,,\quad f = \psi_f(x)\;,\quad f' = \psi_{f'}(x) \tag{6-1}$$

We remember (see eq. (5-1)) that the charged currents encountered in nature are of V-A form. In order to account for this fact, we decompose the fields into L = left and R = right components

$$f = \tfrac{1}{2}(1-\gamma_5)f + \tfrac{1}{2}(1+\gamma_5)f = f_L + f_R\,, \tag{6-2}$$

$$f_{L,R} = \tfrac{1}{2}(1 \pm \gamma_5)f\,,$$

and similarly for f'. Substituting into (6-1) yields

$$\mathcal{L}_0(f,f') = i\left[\bar{f}_L\gamma\,\frac{\partial}{\partial x}\,f_L + \bar{f}_R\gamma\,\frac{\partial}{\partial x}\,f_R + \bar{f}'_L\gamma\,\frac{\partial}{\partial x}\,f'_L + \bar{f}'_R\gamma\,\frac{\partial}{\partial x}\,f'_R\right]. \tag{6-3}$$

Again, we look at $\mathcal{L}^{cc}$, eq. (5-1), and observe that the left-handed fermions (f_L and f'_L) seem to be intimately related to each other; they go into each other by sending off a W. For example, in (5-1)

$$\bar{\ell}_j\,\gamma^\lambda(1-\gamma_5)\nu_i = \frac{1}{2}\,\bar{\ell}_j(1+\gamma_5)\gamma^\lambda(1-\gamma_5)\nu_i = 2\bar{\ell}_{jL}\gamma^\lambda\nu_{iL}\,.$$

Therefore it is natural to write

$$\mathcal{L}_0(f,f') = i\left\{(\bar{f}_L,\bar{f}'_L)\gamma\,\frac{\partial}{\partial x}\begin{pmatrix} f_L \\ f'_L\end{pmatrix} + \bar{f}_R\,\gamma\,\frac{\partial}{\partial x}\,f_R + \bar{f}'_R\,\gamma\,\frac{\partial}{\partial x}\,f'_R\right\}. \tag{6-4}$$

Now we shall impose the Local Gauge Principle. Here, we require that neither the "direction" of $\begin{pmatrix} f_L \\ f'_L\end{pmatrix}$, in a weak isospin space spanned by f_L and f'_L, nor the phases of the three fields

$$\psi_1 \equiv \begin{pmatrix} f_L \\ f'_L\end{pmatrix}\,,\quad \psi_2 \equiv f_R\,,\quad \psi_3 \equiv f'_R \tag{6-5}$$

---

* See section 12 for generalization to several f's and f''s.

should be measurable. The gauge group is, therefore, SU(2) x U(1) and $\mathcal{L}_0(f,f')$ must be modified to be invariant under

$$\psi_j(x) \to \left\{\exp\left(i\ \bar{a}(x)\cdot\frac{\bar{\sigma}}{2}\right)\exp\left(i\ \beta_j(x)\right)\right\}\psi_j(x)\ . \qquad (6\text{-}6)$$

$$\underset{SU(2)}{\uparrow} \qquad\qquad \underset{U(1)}{\uparrow}$$

Here $\bar{a}(x)$ are the three (real but otherwise arbitrary) parameters of the group SU(2); $\bar{\sigma}$ are the Pauli matrices,

$$\left[\frac{\sigma_j}{2}, \frac{\sigma_k}{2}\right] = i\ \epsilon_{jkm}\left[\frac{\sigma_m}{2}\right] .$$

$\frac{\bar{\sigma}}{2}$ represent the three generators of SU(2); $\bar{\sigma}\ \psi_2 = \bar{\sigma}\ \psi_3 = 0$, because $\psi_{2,3}$ are singlets. Finally $\beta_j(x)$, $j = 1,2,3$ are arbitrary real functions.

$\mathcal{L}_0$, as it stands, is not invariant under (6-6). From section 4, we know that we must introduce three vector bosons (one for each generator) for the SU(2) group and one additional vector boson for the U(1) group. Thus we replace

$$\mathcal{L}_0 = i\sum_j \bar{\psi}_j\ \gamma\ \frac{\partial}{\partial x}\ \psi_j$$

with

$$\mathcal{L}_1 = i\sum_{j=1}^{3} \bar{\psi}_j\ \gamma^\mu\left[\frac{\partial}{\partial x^\mu} - i\ g\ \frac{\bar{\sigma}}{2}\cdot\bar{W}_\mu - i\ g' y_j\ B_\mu\right]\psi_j\ . \qquad (6\text{-}7)$$

Here $\vec{W}^\mu = (W^{1,\mu}, W^{2,\mu}, W^{3,\mu})$; g and y_j are four constants, and the redundant constant g' has been introduced for convenience. Note that the W's couple only to the doublet ψ_1 and $\sigma\ \psi_{2,3} = 0$. By construction (6-7) gives the correct form for the charged currents (see eq. (5-1) up to mixings in the hadronic sector. Eq. (6-7) is the analog of (4-4). We now ask, what is the analog of (4-5a)? How should the fields W and B transform in order to ensure us invariance under SU(2) x U(1), eq. (6-6). For the U(1) part we copy the procedure in section 4. We infer, by comparing equs. (6-7) and (4-8), that we just have to replace

$$e \to g'\ ,\quad Q_j \to y_j\ ,\quad \Lambda_j(x) \to \beta_j(x)$$

whereby we have invariance provided $\beta_j(x) = y_j\beta(x)$ and the gauge transformation of the B field is according to

$$B^\mu \to B^\mu + \frac{1}{g'}\ \frac{\partial\beta(x)}{\partial x_\mu}\ . \qquad (6\text{-}8)$$

Here $\beta(x)$ is an arbitrary real function. It remains to add the kinetic term

$$\mathcal{L}_o(B) = -\frac{1}{4} B^{\mu\nu}B_{\mu\nu} \ , \ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad (6\text{-}9)$$

and the "U(1) problem" is solved.

We now focus our attention on the SU(2) part, which only affects ψ_1. The SU(2) group is a nonabelian Lie group*, i.e., the commutator of generators (here represented by σ's) is nonvanishing and equals a linear combination of the generators. To repeat, the question is how should W transform so that $\mathcal{L}_1$ is invariant under $\psi_1 \to \exp(i\ \vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2})\psi_1$? The answer to this question was given by Yang and Mills (Phys.Rev. 96,191(1954)). The reader may check that for an infinitesimal transformation (small $\vec{\alpha}(x)$) one gets

$$\vec{W}^\mu \to \vec{W}^\mu + \frac{1}{g}\frac{\partial\vec{\alpha}}{\partial x_\mu} - \vec{\alpha}\times\vec{W}^\mu \quad , \qquad (6\text{-}10)$$

i.e.,

$$W^j_\mu \to W^j_\mu + \frac{1}{g}\frac{\partial\alpha^j}{\partial x^\mu} - \varepsilon_{jkm}\,\alpha^k\, W^m_\mu \quad , \quad j = 1,2,3$$

Note, since $\vec{W}$ and $\vec{\alpha}$ are vectors, eq. (6-10) expresses the most general behaviour expected, viz., W gets translated as well as rotated.

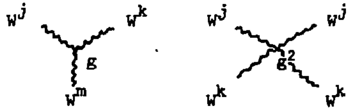
We now must supply W with a kinetic energy term which should remain invariant under (6-10). Again, here the analog of (6-9) does not work. Indeed, the requirement of invariance imposes self-interaction for W (the reader should check this)

$$\mathcal{L}_o(W) = -\frac{1}{4}\sum_{j=1}^{3} W^j_{\mu\nu}\, W^{\mu\nu,j} = -\frac{1}{4}\vec{W}_{\mu\nu}\cdot\vec{W}^{\mu\nu} \quad , \qquad (6\text{-}11)$$

where

$$W^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + g\,\varepsilon_{jkm}\,W^k_\mu\,W^m_\nu \qquad (6\text{-}12)$$

The self-interaction diagrams are



---

* Sophus Lie (1842-1899) was a mathematician from the most beautiful fjord area in Norway called Sognefjorden. I wish the interested reader will someday have the chance to visit that area.

Let us summarize: the interaction Lagrangian

$$\mathcal{L}_I = i \sum_j \bar{\psi}_j(x)\gamma^\mu \left[\frac{\partial}{\partial x^\mu} - i\, g\,\frac{\bar{\sigma}}{2}\cdot \bar{W}_\mu - i\, g' y_j B_\mu\right] \psi_j(x) \quad (6\text{-}13)$$

is invariant under the gauge transformation

$$\begin{aligned}
\psi_j(x) &\to \left\{\exp\left(i\,\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}\right)\exp\left(i\, y_j \beta(x)\right)\right\} (x) \\
B^\mu &\to B^\mu + \frac{1}{g'}\frac{\partial \beta(x)}{\partial x_\mu} \\
\bar{W}^\mu &\to \bar{W}^\mu + \frac{1}{g}\frac{\partial \bar{\alpha}}{\partial x_\mu} - \bar{\alpha}\times\bar{W}^\mu ,
\end{aligned} \quad (6\text{-}14)$$

$$\psi_1 = \begin{pmatrix} f_L \\ f'_L \end{pmatrix} , \quad \psi_2 = f_R,\ \psi_3 = f'_R \ .$$

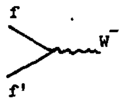
Furthermore, we learned how to add kinetic terms for the B and W fields. However, these particles must be massless, because a mass term $\frac{1}{2} M_B^2 B_\mu B^\mu$, etc. violates the gauge principle, eq. (6-14).

7. UNIFICATION OF WEAK AND ELECTROMAGNETIC INTERACTIONS IN THE STANDARD MODEL

In the last section we constructed an SU(2) x U(1) model which has the correct form for the charged currents (up to mixings). The charged current interactions are mediated via $W^\pm$. From (6-13)

$$\begin{aligned}
\mathcal{L}^{cc} &= g\,\bar{\psi}_1\,\gamma^\mu\,\frac{\sigma_1 W_\mu^1 + \sigma_2 W_\mu^2}{2}\,\psi_1 \\
&= \frac{g}{2}\left(\bar{f},\bar{f}'\right)\left(\frac{1+\gamma_5}{2}\right)\gamma^\mu \begin{pmatrix} 0 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & 0 \end{pmatrix} \frac{1-\gamma_5}{2}\begin{pmatrix} f \\ f' \end{pmatrix} = \\
&= \frac{g}{2\sqrt{2}}\left\{\bar{f}\,\gamma^\mu(1-\gamma_5) f'\, W_\mu^- + \text{h.c.}\right\} ,
\end{aligned} \quad (7\text{-}1)$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \pm i\, W_\mu^2\right)$$



$\mathcal{L}^{cc}$ above does not describe the real world, because W must be massless giving long-range forces, whereas, we know that weak interactions are short-ranged. Later on we shall see that we can have both massive W's and the Lagrangian described above provided we introduce the mass via the so-called Higgs mechanism. Let us assume that we had done so. Then, a comparison with the local V - A Lagrangian for μ-decay yields

$$\frac{g^2}{8M_W^2} = \frac{G}{\sqrt{2}} \tag{7-2}$$

where M_W is the mass of the charged W and G is the Fermi constant.

Now we turn to the neutral currents, which result from $\mathcal{L}_1$, eq. (6-13). These are mediated by B_μ and W_μ^3. Since these objects are massless and as yet unphysical, we may form two *orthogonal* linear combinations of them denoted by A_μ and Z_μ

$$\begin{pmatrix} B_\mu \\ \\ W_\mu^3 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{B\gamma} & a_{BZ} \\ \\ a_{W\gamma} & a_{WZ} \end{pmatrix}}_{\mathcal{R}} \cdot \begin{pmatrix} A_\mu \\ \\ Z_\mu \end{pmatrix} \tag{7-3}$$

where $\mathcal{R}$ is an orthogonal matrix with elements $a_{B\gamma}$, etc. We shall show later that via the Higgs mechanism, mentioned above, A_μ and Z_μ could be made to represent the physical particles, where A_μ (Z_μ) is a massless (heavy) particle. We wish to identify A_μ with the photon field, therefore, we must require that the terms in $\mathcal{L}_1$, eq. (6-13), containing A_μ must add up to give the electromagnetic Lagrangian, $\mathcal{L}^{em}$,

$$\mathcal{L}^{em} = i \sum_{j=1}^{3} \bar{\psi}_j \gamma^\mu(-ieQ_j)\psi_j A_\mu = e\left\{Q_f \bar{f}\gamma^\mu f + Q_{f'}\bar{f}'\gamma^\mu f'\right\} A_\mu \tag{7-4}$$

Here Q_j is the (matrix representation) of the electric charge, $\overline{\text{viz}}$.,

$$Q_1 = \begin{pmatrix} Q_f & 0 \\ \\ 0 & Q_{f'} \end{pmatrix} = \frac{\sigma_3}{2} + \left(Q_f - \frac{1}{2}\right), \quad Q_2 = Q_f, \quad Q_3 = Q_{f'}. \tag{7-5}$$

The three relations in (7-5) are summarised in the operator form

$$Q = \frac{\sigma_3}{2} + Y \,, \tag{7-6}$$

where

$$Q\psi_j = Q_j\psi_j, \quad Y\psi_j = Y_j\psi_j$$
$$Y_1 = Q_f - \frac{1}{2}\,, \quad Y_2 = Q_f\,, \quad Y_3 = Q_{f'}\,. \tag{7-6a}$$

Thus

$$\mathcal{L}^{em} = e \sum_j \bar{\psi}_j \gamma^\mu Q \psi_j A_\mu \equiv e\, j^{\mu,em} A_\mu \,. \tag{7-7}$$

Now we calculate the term in $\mathcal{L}$ which contains A^μ. Using eqs. (6-13) and (7-3), we find

$$\mathcal{L}^{em} = \sum_j \bar{\psi}_j \gamma^\mu \left[g\, a_{W\gamma} \frac{\sigma_3}{2} + g'\, a_{B\gamma}\, y_J\right] \psi_j A^\mu \,. \tag{7-8}$$

Comparing (7-8) with (7-7) yields

$$eQ_j = g\, a_{W\gamma} \frac{\sigma_3}{2} + g'\, a_{B\gamma}\, y_j \quad , \tag{7-9}$$

$$e\left(\frac{\sigma_3}{2} + Y_j\right) = g\, a_{W\gamma} \frac{\sigma_3}{2} + g'\, a_{B\gamma} y_j \tag{7-9a}$$

The relation (7-9) is valid provided

$$g\, a_{W\gamma} = e \quad \text{and} \quad e\, Y_j = g'\, a_{B\gamma}\, y_j \quad . \tag{7-10}$$

Remembering that the (redundant) constant g' may be choosen at will, we put

$$g\, a_{W\gamma} = e = g'\, a_{B\gamma} \tag{7-11}$$

$$y_j = Y_j \tag{7-12}$$

By orthogonality of the matrix , $(a_{W\gamma})^2 + (a_{B\gamma})^2 = 1$. Thus

$$e^2\left(\frac{1}{g^2} + \frac{1}{g'^2}\right) = 1 \,. \tag{7-13}$$

Putting $a_{W\gamma} = \sin\theta_W$ we have the famous relations

$$g = \frac{e}{\sin\theta_W} \quad , \quad g' = \frac{e}{\cos\theta_W} \quad , \tag{7-14}$$

where θ_W is usually called the Weinberg angle. Let us summarise: the SU(2) x U(1) model contains two neutral mediators W_μ^3 and B_μ, from which, by a rotation, we may form the fields A_μ and Z_μ. The model contains four "coupling constants", viz. g, $g'y_j$ and a parameter, θ_W. Identifying A_μ with the photon field fixes these constants, viz., $g = \frac{e}{\sin\theta_W}$, $g' = \frac{e}{\cos\theta_W}$, $y_1 = Q_f - \frac{1}{2}$, $y_2 = Q_f$ and $y_3 = Q_{f'}$. Thus the model introduces one new parameter, the angle θ_W, which is empirically determined from the neutral current data. Note that the U(1) group is rather unpleasant; only for a very specific (and incomprehensible) choice of the phases $\beta_j(x) = y_j\beta(x)$, with y_j as given above, do the particles f and f' obtain their observed charges. The hope is that the SU(2) x U(1) model is a substructure in a more elaborate scheme, where the origin of the eigenvalues y_j and the quantization of charge is explained. The SU(5) model of weak, electromagnetic and strong interactions (see section 14) provides a good example of how such an elaborate scheme might look like.

The SU(2) x U(1) model unifies weak and electromagnetic interactions in the sence that A_μ and Z_μ are intimately related to each other (eq. 7-3) and to $W_\mu^\pm$. The weak and electromagnetic couplings are of the same order, viz. $g = \frac{e}{\sin\theta_W}$. Thus the apparent huge differences in the strength of these interactions are due to the differences in the masses, $M_\gamma = 0$,

$$M_W = \left(\frac{\sqrt{2}\, g^2}{8G}\right)^{\frac{1}{2}} = \left(\frac{\pi\alpha}{\sqrt{2}\, G \sin^2\theta_W}\right)^{\frac{1}{2}} = \frac{38 \text{ GeV}}{|\sin\theta_W|} .$$

We shall see later that $M_Z = \frac{M_W}{\cos\theta_W}$, if the Higgs mechanism is done à la Weinberg (Phys.Rev.Letters 19,1264(1967). Thus W and Z are beyond the reach of the present accelerators.

8. THE WEAK NEUTRAL CURRENT COUPLINGS IN THE STANDARD MODEL

We go back to the interaction Lagrangian of the standard model and, using eq. (7-3), determine the term proportional to Z_μ

$$\mathcal{L}^{n.c.} = \sum_j \bar{\psi}_j \gamma^\mu \left[g\, a_{WZ} \frac{\sigma_3}{2} + g'\, a_{BZ}\, y_j\right] \psi_j\, Z_\mu \tag{8-1}$$

Using $g\, a_{W\gamma} = g'\, a_{B\gamma} = e$ and $Q = \frac{\sigma_3}{2} + y$, obtained in section 7 we find

$$\mathcal{L}^{n.c.} = \frac{e}{a_{W\gamma}\, a_{B\gamma}} \sum_j \bar{\psi}_j\, \gamma^\mu \Big[a_{B\gamma}\cdot a_{WZ}\, \frac{\sigma_3}{2} + a_{W\gamma}\cdot a_{BZ}\Big(Q - \frac{\sigma_3}{2}\Big)\Big]\psi_j Z_\mu$$

$$= \frac{e}{a_{W\gamma}\cdot a_{B\gamma}} \sum_j \bar{\psi}_j\, \gamma^\mu \Big[\det(\mathcal{R})\cdot \frac{\sigma_3}{2} + a_{W\gamma}\, a_{BZ}\, Q\Big]\psi_j Z_\mu \qquad (8\text{-}2)$$

Since $\mathcal{R}$ is an orthogonal matrix and we had introduced $\sin\theta_W = a_{W\gamma}$, $\cos\theta_W = a_{B\gamma}$, we must have $\det(\mathcal{R}) = 1$, and $a_{BZ} = -\sin\theta_W$. Therefore,

$$\mathcal{L}^{n.c.} = \frac{e}{\sin\theta_W \cos\theta_W} \Big\{\bar{\psi}_1 \gamma_\mu \frac{\sigma_3}{2}\psi_1 - \sin^2\theta_W\, J^{em}_\mu\Big\} Z^\mu \equiv J_\mu^{\,n.c.}\cdot Z^\mu \qquad (8\text{-}3)$$

where $\psi_1 = \begin{pmatrix} f \\ f' \end{pmatrix}_L$; J^{em}_μ is the electromagnetic current, defined in eq. (7-7), and $J^{n.c.}_\mu$ is the weak neutral current. We may rewrite J^{em}_μ as

$$J^{em}_\mu = Q_f \bar{f}\gamma_\mu f + Q_{f'} \bar{f}'\gamma_\mu f' = Q_f(\bar{f}_L\gamma_\mu f_L + \bar{f}_R\gamma_\mu f_R) + \ldots$$

Thus

$$J^{n.c.}_\mu = \frac{e}{\sin\theta_W\cos\theta_W} \Big\{a^f_L\, \bar{f}_L\gamma_\mu f_L + a^f_R\, \bar{f}_R\gamma_\mu f_R + a^{f'}_L\, \bar{f}'_L\gamma_\mu f'_L + a^{f'}_R\, \bar{f}'_R\gamma_\mu f'_R\Big\} \qquad (8\text{-}4)$$

where the left and right coupling $a_{L,R}$ are defined by

$$a_L = I_{3L} - Q\sin^2\theta_W$$
$$a_R = -Q\sin^2\theta_W \qquad (8\text{-}5)$$

Here Q is the charge of the fermion involved and I_{3L} is the third component of its *weak isospin* ($I_{3L} = \frac{1}{2}$ *for* f *and* $I_{3L} = -\frac{1}{2}$ *for* f'). Although (8-4) was derived for a hypothetical couple of fermions (f,f') with $Q_f = Q_{f'}+1$, the relation (8-4) is valid also when there are several such pairs, provided they are all treated on the same footing. [We shall return to this question later on (see section 12)]. Accepting this fact, we may immediately write down the neutral current couplings for any pair of elementary fermions. For example

(i) $f, f' = \nu_e, e \Rightarrow I_{3L}^{\nu_e} = \frac{1}{2} = -I_{3L}^{e}\ , \quad Q_e = -1\ , \quad Q_\nu = 0$

$$a_L^{\nu_e} = \frac{1}{2}\ ,\ a_R^{\nu_e} = 0\ ,\ a_L^{e} = -\frac{1}{2} + \sin^2\theta_W\ ,\ a_R^{e} = \sin^2\theta_W \tag{8-6}$$

(ii) $f, f' = u, d \Rightarrow Q_u = \frac{2}{3}\ ,\ Q_d = -\frac{1}{3}\ ,\ I_3^u = -I_3^d = \frac{1}{2}$

$$a_L^u = \frac{1}{2} - \frac{2}{3}\sin^2\theta_W\ ,\ a_R^u = -\frac{2}{3}\sin^2\theta_W\ ,\ a_L^d = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\ ,$$
$$a_R^d = \frac{1}{3}\sin^2\theta_W\ . \tag{8-7}$$

Furthermore all sequential neutrinos (negative leptons) have the same couplings as $\nu_e(e)$. Similarly, in the hadronic sector, all charged 2/3 quarks have the same couplings as the up quark, etc.

We may derive a local four-fermion interaction Lagrangian from $\mathcal{L}_{n.c.}$, eq. (8-3), provided the Z is very heavy

$$\mathcal{L}^{n.c.} = -8\frac{G}{\sqrt{2}}\left\{a_L^j\,\bar{f}_j\,\gamma_\mu\,L\,f_j + a_R^j\,\bar{f}_j\,\gamma_\mu\,R\,f_j\right\}$$
$$\times\left\{a_L^k\,\bar{f}_k\,\gamma_\mu\,L\,f_k + a_R^k\,\bar{f}_k\,\gamma_\mu\,R\,f_k\right\}\ , \tag{8-8}$$
$$L = \frac{1}{2}(1-\gamma_5)\ ,\quad R = \frac{1}{2}(1+\gamma_5)\ .$$

The relation (8-8) describes the neutral current interactions of any two fermions f_j and f_k , $f_j \neq f_k$. For example to get the neutrino-up quark interactions we put $f_j = \nu$, $f_k = u$, $a_L^j = a_L^\nu$, $a_L^k = a_L^u$, etc. Note that we have used the Weinberg relation $M_W = \cos\theta_W \cdot M_Z$. For $f_j = f_k$ the factor 2 in front of G must be removed.

9. SPONTANEOUS SYMMETRY BREAKING (SSB)

In the previous sections we discussed two examples of local gauge theories, namely QED and the SU(2) x U(1) model. Indeed the unobservability of the local phases of the fields led in a natural way to the existence of interactions, mediated via massless spin one *(gauge)* bosons. This masslessness of mediators, necessitated by the gauge principle, is just fine for QED but catastrophic for weak interactions. We know from experiments that the weak mediators ($W^{\pm}$, Z^0, etc.), if they exist at all, are certainly heavier than 10 GeV.

At this point one might take the attitude that gauge theories, in spite of their beauty, Ward-identities which ensure renormalizability, etc. are relevant for QED but not for weak interactions. Within the last few years, only a small group of brave physicists have dared to work along this line. Unconventional models for weak interactions have been constructed, but so far no viable alternative to gauge theories has been found. The mainstream of activity has gone in the direction of taking for granted the gauge principle in a world where there are no masses. In the real world, the gauge world, the gauge symmetry is, by assumption "spontaneously" broken whereby the masses are created.

Below, we give a short survey of the SSB as it is employed in particle physics.

The phenomenon of SSB is known to occur in several branches of physics. Usually systems exhibiting SSB have an infinite number of degrees of freedom. Since field theory has also an infinite number of degrees of freedom, it might well be that the theory of constituents of matter is a spontaneously broken one.

Let us consider a system, described by a classical Lagrangian $\mathcal{L}$ which is invariant under a continuous group of transformations G. We examine the ground states of the system. If the system possesses a unique ground (vacuum) state (which must be invariant under G) we have a situation in which the symmetry properties of $\mathcal{L}$ and the vacuum are the same. The relevant theory is a normal one. However, it might happen that the system has several ground states, which transform into each other under G. Then, if by some reason, one of the ground states is singled out as the physical ground state of the system (the others being unphysical) the symmetry is lost and the relevant theory is said to be spontaneously broken.

It is known that the spontaneous breaking of a global symmetry leads to existence of massless particles (excitations) called Goldstone bosons. Very fortunately, our gauge theories are based on invariance under a local group of transformations. When the local invariance is broken, instead of getting additional massless bosons, we get massive gauge bosons. We shall now describe the SSB mechanism in a little bit more detail.

9.1 The Spontaneous Breakdown of a Global Symmetry

A nice example of the SSB phenomenon is provided by the Goldstone model. Let us consider the Lagrangian

$$\mathcal{L} = + \frac{\partial\varphi^*}{\partial x_\mu}\frac{\partial\varphi}{\partial x^\mu} - m^2 \varphi^*\varphi - h(\varphi^*\varphi)^2 \equiv T - V ,$$

$$V = + m^2 \varphi^*\varphi + h(\varphi^*\varphi)^2 , \qquad (9\text{-}1)$$

$$\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) ,$$

where φ_i, $i = 1,2$ are real scalar classical fields and m^2,h are constants. We shall only consider the case $h > 0$, because otherwise $V(|\varphi|) \to -\infty$ as $|\varphi| \to \infty$, i.e., the Hamiltonian has *no minimum* and there is *no* ground state. Suppose that $m^2 \neq 0$, then there are two distinct possibilities. If $m^2 > 0$, the Lagrangian (9-1) would describe a self-interacting charged scalar field. For $m^2 < 0$, we have an example of a spontaneously broken theory. Perhaps nature utilizes both possibilities. We now look for the ground states, i.e., for those values of φ which *minimize* the potential

$$\frac{\partial V}{\partial|\varphi|} = 0 \qquad (9\text{-}2)$$

For $m^2 > 0$ we find $V \geq 0$, the minimum of V corresponds to $|\varphi| = \varphi_1 = \varphi_2 = 0$. Thus there is a unique vacuum. Note that the Lagrangian (9-1) is invariant under the global U(1) group

$$\varphi(x) \to e^{i\Lambda}\varphi(x) \quad , \quad \Lambda = \text{real constant}. \qquad (9\text{-}3)$$

The unique vacuum $\varphi_1 = \varphi_2 = 0$ is also *invariant* under U(1). Now we focus our attention on the case $m^2 < 0$. Put $\mu^2 = -m^2$. The potential $V = -\mu^2|\varphi|^2 + h|\varphi|^4$ is minimum for

$$2|\varphi|^2 = \varphi_1^2 + \varphi_2^2 = \frac{\mu^2}{h} \equiv v^2 \qquad (9\text{-}4)$$

Thus the system has an infinite number of ground states, which 'lie' on a circle in the $\varphi_1 - \varphi_2$ plane. Under the action of the group, eq. (9-3), the ground states transform into each other.

Suppose now that we choose one of the ground states, for example $\varphi_o \equiv (\varphi_1 = v, \varphi_2 = 0)$ as the physical ground state then the symmetry is spontaneously broken, because the state φ_0 is not invariant under the transformation (9-3). To see the structure of the theory, we expand the potential near the vacuum state, where the theory is expected to be stable. Put

$$\varphi_1' = \varphi_1 - v \quad , \quad \varphi_2' = \varphi_2$$

$$V(\varphi') = V_o + \sum_{i=1,2} \left(\frac{\partial V}{\partial \varphi_i}\right)_0 \varphi_i' + \frac{1}{2} \sum_{i,j=1,2} \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \right)_o \varphi_i' \varphi_j' + \ldots \tag{9-5}$$

where the index o indicates that the quantities are evaluated at the point φ_o . We have

$$\frac{\partial V}{\partial \varphi_i} = \left[-\mu^2 + h(\varphi_1^2 + \varphi_2^2) \right] \varphi_i \quad , \left(\frac{\partial V}{\partial \varphi_i}\right)_o = 0$$

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} = \left[-\mu^2 + h(\varphi_1^2 + \varphi_2^2)\right]\delta_{ij} + 2h\varphi_i\varphi_j, \quad \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_0 = 2h(\varphi_i\varphi_j)_o \tag{9-6}$$

Thus the mass matrix, defined by

$$m_{ij}^2 = \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_0 = 2h(\varphi_i\varphi_j)_o \tag{9-7}$$

$$\varphi_o = (\varphi_1 = v \, , \varphi_2 = 0)$$

has only one non-zero element, $m_{11}^2 = 2h\ v^2 = \frac{2h\mu^2}{h} = 2\mu^2$, while $m_{12}^2 = m_{21}^2 = m_{22}^2 = 0$. Therefore the 'particle' φ_2' is massless

$$V(\varphi') = \mu^2 \varphi_1'^2 + \text{interaction terms}$$

$$\mathcal{L}(\varphi') = \frac{1}{2} \sum_i \frac{\partial \varphi_i'}{\partial x_\mu} \frac{\partial \varphi_i'}{\partial x^\mu} - \mu^2 \varphi_1'^2 + \text{interaction terms} \tag{9-8}$$

The spontaneous breakdown of the U(1) symmetry has led to apparition of a massless (Goldstone) boson, represented by the field φ_2'. The field φ_1' has the mass $\sqrt{2}\,\mu$. The reader may wonder what happens if she would choose a different vacuum, say $\varphi_o' \equiv (\varphi_1 = \frac{v}{\sqrt{2}} \, , \ \varphi_2 = \frac{v}{\sqrt{2}})$? She could repeat the above analysis by introducing $\varphi_i' = \varphi_i - \frac{v}{\sqrt{2}}$. Then from $V(\varphi') = \frac{1}{2} m_{ij}^2 \varphi_i' \varphi_j' +$ interaction terms, $m_{ij}^2 = 2h(\varphi_i \varphi_j)_o$, she would find that the mass matrix is not diagonal

$$V(\varphi') = hv^2(\varphi_1'^2 + \varphi_2'^2 + 2\varphi_1'\varphi_2') + \ldots = hv^2(\varphi_1' + \varphi_2')^2 + $$
$$+ 0\cdot(\varphi_1' - \varphi_2')^2 + \ldots$$

Thus the linear combination $\frac{1}{\sqrt{2}}(\varphi_1' + \varphi_2')$ has acquired the mass $\sqrt{2}\mu$ while the combination $\frac{1}{\sqrt{2}}(\varphi_1' - \varphi_2')$ represents a massless (Goldstone) boson. This solution is obtainable from the earlier one by a rotation of 45 degrees in the $\varphi_1 - \varphi_2$ plane. Clearly, this solution and the former one are physically equivalent.

The arguments in this section can be easily generalized. Suppose we have a Lagrangian, $\mathcal{L}(\varphi_i, \frac{\partial\varphi_i}{\partial x}) = T - V(\varphi_i)$, $i = 1,\ldots,N$ Here φ_i are real scalar fields. Assume that $\mathcal{L}$ is invariant under a local group of transformations G. Let the elements of the group be represented by $\exp(i\,\bar{\alpha}\cdot\bar{T})$, where α^j are real constants and T^j, $j=1,\ldots,n$ are the matrices representing the generators

$$\varphi = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_N\end{pmatrix}, \quad \varphi \to \exp(i\,\bar{\alpha}\cdot\bar{T})\varphi\ , \quad \delta\varphi = i\,\bar{\alpha}\cdot\bar{T}\,\varphi\ , \text{ small } \bar{\alpha} \tag{9-9}$$

$$T^j = N \times N \text{ matrices}\ , \quad j = 1,\ldots, n$$

Look for the vacuum state by using $\frac{\partial V(\varphi)}{\partial\varphi_j} = 0$. Suppose that there are several vacuua. These will transform into each other under the group. Choose a particular vacuum

$\varphi_o = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_N\end{pmatrix}_o$, and disregard the rest. Under an infinitesimal action of the group the vacuum is changed by the amount $\delta\varphi_o = i\,\bar{\alpha}\cdot\bar{T}\,\varphi_o$. We know that the vacuum is then non-invariant under some of the operations in the group, namely those for which $T^k\varphi_o \neq 0$. Suppose there are n' such generators, $k=1,\ldots, n'$, $1 \leq n' \leq n$. Then the remaining generators leave the vacuum invariant, $T^k\varphi_o = 0$, $k = n' + 1,\ldots, n$. We shall now show that to each generator which breaks the symmetry corresponds a Goldstone boson. Introduce $\varphi' = \varphi - \varphi_o$, and expand as before

$$V = V_o + \sum_j \left(\frac{\partial V}{\partial \varphi_j}\right)_o \varphi'_j + \frac{1}{2}\sum \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_o \varphi'_i \varphi'_j + \text{interaction terms},$$

$$\left(\frac{\partial V}{\partial \varphi_j}\right)_o = 0 \quad , \quad m^2_{ij} = \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_o \quad . \qquad (9\text{-}10)$$

The potential is invariant under G implying

$$\delta V = \frac{\delta V}{\partial \varphi_j} \delta \varphi_j = 0 \quad ,$$

i.e.,

$$\frac{\partial V}{\partial \varphi_j} T^r_{jk} \varphi_k = 0 \qquad r = 1, \ldots, n \qquad (9\text{-}11)$$

Take the derivative $\frac{\partial}{\partial \varphi_\ell}$ of (9.11)

$$\frac{\partial^2 V}{\partial \varphi_\ell \partial \varphi_j} T^r_{jk} \varphi_k + \frac{\partial V}{\partial \varphi_j} T^r_{j\ell} = 0 \qquad (9\text{-}12)$$

At $\varphi = \varphi_o$, using (9-10), we find

$$m^2{}_{\ell j} T^r_{jk} (\varphi_k)_o = 0 \qquad r = 1, \ldots, n \qquad (9\text{-}13)$$

By our assumption $T^r \varphi_o \neq 0$ for $r = 1, \ldots, n'$. Thus $T^r \varphi_o$ span a n' - dimensional subspace in the N dimensional space spanned by the φ's, and therefore, $m^2{}_{\ell j}$ has n' zero eigenvalues, or in other words there are n' Goldstone bosons in the theory.

Let us apply this formalism to the special case considered above

$$\varphi \rightarrow (\varphi + i\Lambda\varphi)$$

$$\delta\varphi = i\Lambda\varphi$$

$$\delta\varphi_1 + i\delta\varphi_2 = i\Lambda(\varphi_1 + i\varphi_2)$$

$$\delta\varphi_1 = -\Lambda\varphi_2 \ , \quad \delta\varphi_2 = \Lambda\varphi_1$$

Thus the generator of the U(1) group in the $\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$ space has the form

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$m^2_{ij}\, T_{jk}(\varphi_k)_o = 0$$

$$\begin{pmatrix} m^2_{11} & m^2_{12} \\ m^2_{21} & m^2_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = 0 \; .$$

From here we obtain $m^2_{12} = m^2_{22} = 0$, i.e., the φ_2' is massless, in accordance with previous results.

When a local symmetry is spontaneously broken the situation becomes quite different, as we shall discuss below.

In quantized field theory the SSB is assumed to be realized by letting the field operators acquire vacuum expectetion values $\phi_i = \varphi_i' + v_i$, $\langle 0|\phi_i'|0\rangle = 0$, $\langle 0|\phi_i|0\rangle = v_i$. The c-numbers v_i are chosen such as to minimize the classical potential.

10. SPONTANEOUS BREAKING OF A LOCAL SYMMETRY

We start with the same Lagrangian as in the previous section (eq. (9-1)) and require invariance under the local phase transformation $\varphi(x) \to e^{i\Lambda(x)}\varphi(x)$, where Λ is real. From our previous considerations (section 4) we know that we must introduce a massless vector field B_μ and replace the derivative ∂_μ by the covariant derivative $\partial_\mu - igB_\mu$, where g is a constant

$$\mathcal{L} = \left(\frac{\partial\varphi^*}{\partial x^\mu} + igB_\mu\varphi^*\right)\left(\frac{\partial\varphi}{\partial x_\mu} - igB^\mu\varphi\right) - V(\varphi) - \frac{1}{4}\, B_{\mu\nu}B^{\mu\nu} \; ,$$

$$V(\varphi) = m^2\varphi^*\varphi + h(\varphi^*\varphi)^2 \qquad (10\text{-}1)$$

$$B_{\mu\nu} = \frac{\partial B_\nu}{\partial x^\mu} - \frac{\partial B_\mu}{\partial x^\nu} \; , \quad \varphi = \frac{1}{\sqrt{2}}\,(\varphi_1 + i\varphi_2)$$

$\mathcal{L}$ is invariant under the joint gauge transformations

$$\varphi \to \exp(i\Lambda(x))\,\varphi(x) \; , \quad B_\mu \to B_\mu + \frac{1}{g}\,\frac{\partial\Lambda(x)}{\partial x^\mu} \; . \qquad (10\text{-}2)$$

Again h is taken to be positive. For $m^2 > 0$, $\mathcal{L}$ is the QED Lagrangian for a self-interacting charged spin zero particle, if we identify B^μ with the photon field. The vacuum of the φ-sector corresponds to $\varphi_1 = \varphi_2 = 0$ and is invariant under (10-2).

As before we turn to the interesting case $m^2 = -\mu^2 < 0$, and assume that the ground state is defined by the minimum of the potential $V(\varphi)$, $\frac{\partial V(\varphi)}{\partial \varphi_j} = 0$, whereby we obtain the relation (9-4), $\varphi_1^2 + \varphi_2^2 = v^2$. Among these infinite number of vacuum we select one, e.g. $\varphi_1 = v$, $\varphi_2 = 0$. Now the symmetry is spontaneously broken. Expanding near the physical vacuum, $\varphi_1' = \varphi_1 - v$, $\varphi_2' = \varphi_2$, we find

$$\mathcal{L} = \frac{1}{2}\left|\frac{\partial(\varphi_1' + i\varphi_2')}{\partial x_\mu} - igB^\mu(\varphi_1' + v + i\varphi_2')\right|^2 - \mu^2\varphi_1'^2 + 0 \cdot \varphi_2'^2 -$$

$$- \frac{1}{4} B_{\mu\nu}B^{\mu\nu} + \text{interactions terms involving } \varphi_1' \text{ and } \varphi_2'. \tag{10-3}$$

The Lagrangian (10-3) is indeed remarkable. It seems that we have a massive scalar φ_1' (with mass $\sqrt{2}\mu$), a massless one φ_2' and a <u>massive</u> vector particle B. Note that there is a term $\frac{1}{2} g^2v^2B^\mu B_\mu$, telling us that the B field has acquired a mass equal to $|gv|$. However this cannot be true because to begin with we had four degrees of freedom, one for each scalar field and two for the massless B. Now we have five, because a massive vector field has three spin states, $S_z = 0, \pm1$. Thus there is a redundant degree of freedom in (10-3). Indeed the massless φ_2' is superfluous and can be "gauged away". To see this, we go back to our original Lagrangian, eq. (10-1) and perform a gauge transformation
$\varphi = e^{i\Lambda}\hat{\varphi}$ and $B = \hat{B} + \frac{1}{g}\frac{\partial\Lambda}{\partial x}$; The Lagrangian is of course invariant.

We may now choose Λ such that $\hat{\varphi}_2 = 0$, viz.,

$$\varphi = e^{i\Lambda}(\hat{\varphi}_1 + i\hat{\varphi}_2)/\sqrt{2} \quad , \quad \hat{\varphi}_2 = 0$$

$$\varphi = \frac{1}{\sqrt{2}}(1 + i\Lambda + ..)\hat{\varphi}_1 \ ,$$

The exponential $e^{i\Lambda}$ disappears and the scalar field becomes real.

The spontaneous symmetry breaking can be done as before, viz. $\hat{\varphi}_1^2 = v^2$, for vacuum. Put $\hat{\varphi}_1 = \varphi_1 + v$, we find

$$\mathcal{L} = \frac{1}{2}\left|\frac{\partial\varphi_1}{\partial x_\rho} - igB_\rho(\varphi_1 + v)\right|^2 - \mu^2\varphi_1^2 - \frac{1}{4} B_{\rho\sigma}B^{\rho\sigma} -$$

$$- \frac{\mu^2}{v}\varphi_1^3 - \frac{\mu^2}{4v^2}\varphi_1^4$$

$$= \frac{-1}{4} B_{\rho\sigma}B^{\rho\sigma} + \frac{1}{2} v^2g^2B_\rho B^\rho + (g^2v\varphi_1 + \frac{1}{2}g^2\varphi_1^2)B_\rho B^\rho +$$

$$+\frac{1}{2}\left[\frac{\partial\varphi_1}{\partial x_\rho}\frac{\partial\varphi_1}{\partial x^\rho}-2\mu^2\varphi_1^2\right]-\frac{\mu^2}{v}\varphi_1^3-\frac{\mu^2}{4v^2}\varphi_1^4 \quad . \qquad (10\text{-}4)$$

Note that gauging away $\hat{\varphi}_1$ instead of $\hat{\varphi}_2$ leads to an unphysical . From (10-4), $M_B = |gv|$, $M_{\varphi_1} = \sqrt{2}\mu$, where M refers to the mass. $\mathcal{L}$ describes a massive vector particle interacting in a very specific way with a real scalar field. In the quantized theory the prescription is as given as the end of the last section. The particle φ_1 is usually referred to as a Higgs particle and the mechanism described above is an example of the Higgs, Kibble, etc. mechanism.

It is widely believed that <u>all</u> masses are generated via the Higgs mechanism, the reason being two-fold. Firstly it is appealing to imagine that a beautiful massless world built upon the local gauge doctrine suddenly undergoes a spontaneous breaking whereby all masses are created. If the reader does not buy this argument the second one can be given, i.e., no alternative has been found so far. Ever since the heroic discovery, by t'Hooft, that the SSB does not spoil the renormalizability, such theories are much cherished.

For each specific gauge theory, a spontaneous breaking scheme has to be cleverly invented in order to achieve the purposes in mind. We shall give some examples later on.

It is fair to say that the Higgs sector of gauge theories usually does not match at all the beauty of the rest. The Higgs sector introduces new parameters, sometimes many of them. Let us hope that some day we will understand the origin of the masses.

11. SPONTANEOUS BREAKING SCHEME FOR THE STANDARD SU(2) x U(1) MODEL

The standard model, described in sections 6-8, has initially 4 gauge bosons $W^\pm$, W^0 and B. The breaking mechanism should be devised such that $W^\pm$ and (the linear combination of W^0 and B corresponding to) Z^0 become massive, whereas the photon remains massless. Thus we need to introduce at least 4 real scalar fields φ_i. These "Higgses" must transform as multiplets under $SU(2) \times U(1)$, so as to couple to the W's and B. We must make sure that the Higgs potential $V(\varphi)$ allows degenerate vacuua; then we would choose one of them as the physical vacuum and let $\varphi_i = \varphi_i' + v_i$, where $v_i = \langle 0|\varphi_i|0\rangle$. Since we believe in the conservation of the electric charge, we can only allow neutral scalars acquire vacuum expectation values. Thus $Q^Y v = 0$, where Q^Y is the charge operator and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$. The group SU(2) x U(1) has four generators

T^k, k = 1,2,3, and y, where $Q = T^3 + y$. Clearly v should be invariant only under Q, then 3 "symmetries" are broken and the corresponding Goldstone bosons may be gauged away to yield three massive bosons. The four Higgses are introduced elegantly, à la Weinberg, in a doublet

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^o \end{pmatrix}, \quad \varphi^+ = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \; \varphi^o = \frac{1}{\sqrt{2}}(\varphi_3 + i\varphi_4) \; , \qquad \text{(11-1)}$$

where φ_i are real. The Higgs Lagrangian now reads

$$\mathcal{L} = \left|\left(\frac{\partial}{\partial x} - ig\,\frac{\bar{\sigma}}{2}\cdot \bar{W} - ig'yB\right)\varphi\right|^2 - V(\varphi) \qquad \text{(11-2)}$$

where the requirement of local gauge invariance under SU(2) x U(1) has given the specific couplings to W and B. We may write $V(\varphi) = -\mu^2\varphi^*\varphi + h(\varphi^*\varphi)^2$, whereby SSB takes place. The arguments in the previous section go through *mutatis mutandis*

$$v = \text{vacuum} = \begin{pmatrix} 0 \\ v^o \end{pmatrix} \quad , \quad \varphi = \varphi' + v$$

$$\left|\left(g\,\frac{\bar{\sigma}}{2}\cdot \bar{W} + g'yB\right) v\right|^2 = \frac{1}{2}\, m^2_{ij}\, W^i\, W^j \quad , \quad W^o = B \qquad \text{(11-3)}$$

Here m^2_{ij} is the mass matrix. We check easily that the photon is massless, $T_3 \equiv \sigma_{3/2}$

$$A_\mu A^\mu \left|\left(\underbrace{g\, a_{W_3\gamma}}_{e}\; T_3 + \underbrace{g' a_{B\gamma}}_{e} y\right) v\right|^2 = A_\mu A^\mu \left| eQ^\gamma v\right|^2 = 0 \; . \qquad \text{(11-4)}$$

The mass of the Z is obtained from

$$\left|\left(g\; a_{W_3 Z}\; T_3 + g' a_{BZ}\; y\right) v\right|^2 Z_\mu Z^\mu = \frac{1}{2}\, M_Z^2\; Z_\mu\; Z^\mu \; . \qquad \text{(11-5)}$$

Using $y = Q - T_3$, $Qv = 0$, we find

$$M_Z^2 = \frac{v^{o2}\, e^2}{2(a^2_{W_3\gamma} a^2_{B\gamma})} \qquad \text{(11-6)}$$

Finally, the mass of the charged W's

$$\left| g\,\frac{\sigma_1 W^1+\sigma_2 W^2}{2}\, v \right|^2 = \frac{g^2 v^{o2}}{4}\,(W_1+iW_2)(W_1-iW_2) = \frac{g^2 v^{o2}}{2}\, W^+ W \; ,$$

$$M_W^2 = \frac{g^2 v^{o2}}{2} = \frac{e^2 v^{o2}}{2(a_{W_3\gamma})^2} \; . \qquad \text{(11-7)}$$

From (11-6) and (11-7) we obtain the Weinberg relation

$$M_W^2 = a_{B\gamma}^2 M_Z^2 = \cos^2\theta \; M_Z^2 \qquad \text{(11-8)}$$

Using the empirical relation $M_W = \frac{38\text{ GeV}}{|\sin\theta|}$, we find that for the present preferred value $\sin^2\theta = \frac{1}{4}$, the mediators are awfully heavy $M_W \sim 76$ GeV, $M_Z \sim 88$ GeV.

The analysis of this section can be generalized. Clearly the photon remains massless as long as the electric charge is conserved, viz.

$$\left| \sum g_k \; T_k \; W_k \; v \right|^2 = \frac{1}{2}\, W_i \; W_j \; m_{ij}^2$$

$$W_k = a_{W_k\gamma} A + \ldots$$

$$A^\mu A_\mu \left| \sum_j g_j \; a_{W_j\gamma} T_j \; v \right|^2 = \frac{1}{2}\, m_\gamma^2 \; A^\mu A_\mu$$

Here g, T and W are the coupling constants, generators and gauge bosons respectively. The sum over j involves only those terms where $a_{W_j\gamma} \neq 0$. For these we have the condition

$$g_j \; a_{W_j\gamma} = e \; , \quad \text{and} \quad Q^\gamma = \sum_j T_j \; . \quad \text{Thus } m_\gamma^2 \sim |Q^\gamma v|^2 = 0$$

because v is electrically neutral.

The Higgs doublet (11-1) is sufficient for giving masses to all fundamental fermions. Of course fermion mass terms $m\bar{\psi}\psi$ do not spoil renormalizability and could be added by hand. Nevertheless, it is more attractive if all masses should have the same origin, say Higgses. Suppose we want to give masses to up and down quarks. We have the multiplets $\begin{pmatrix} u \\ d \end{pmatrix}_L$, u_R, d_R, $\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^o \end{pmatrix}$. Note that $\varphi^c = \begin{pmatrix} \varphi^{o*} \\ -\varphi^- \end{pmatrix}$ has the same transformation properties as $\begin{pmatrix} \varphi^+ \\ \varphi^o \end{pmatrix}$. The gauge invariant Higgs-fermion interaction reads

$$\mathcal{L} = c_d(\bar{u},\bar{d})_L \begin{pmatrix}\varphi^+\\ \varphi^o\end{pmatrix} d_R + c_u(\bar{u},\bar{d})_L \begin{pmatrix}\varphi^{*o}\\ \varphi^-\end{pmatrix} u_R + \text{h.c.} \qquad (11\text{-}9)$$

Upon spontaneous breakdown $\varphi^o \rightarrow \varphi^o + v^o$, and after gauging away $\varphi^{\pm}$ we find

$$\mathcal{L}_{mass} = (c_d v^o)\ \bar{d}_L d_R + (c_u v^o)\ \bar{u}_L u_R + \text{h.c.}$$

If the c's are real (this can always be arranged as we shall discuss later)

$$c_d v^o(\bar{d}_L d_R + \bar{d}_R d_L) = c_d v^o \bar{d}d \equiv m_d \bar{d}d\ .$$

Thus

$$\mathcal{L} = m_d\bar{d}d + m_u\bar{u}u + \frac{m_d}{v^o}\,\bar{d}d\varphi^o + \frac{m_u}{v^o}\,\bar{u}u\varphi^o \qquad (11\text{-}10)$$

A very specific prediction of the spontaneously broken SU(2) x U(1) model is the existence of the scalar boson φ^o, which couples to each elementary fermion with a strength proportional to the mass of that fermion. Unfortunately, the vacuum expectation value v^o is huge (~200 GeV), see eqs. (7-2) and (11-7). Therefore only heavy elementary fermions do care about Higgses. The massive vector bosons couple strongly to Higgses, see eq. (10-4).

12. MORE QUARKS AND LEPTONS, MASSES AND THE CABIBBO ANGLES

In this section we start by reminding ourselves that Nature seems to be repeating herself and so far we have failed to understand her reasons for doing so. In fact by now there are three distinct families of leptons and quarks, viz;

the fundamental family	ν_e, e, u_1, u_2, u_3, d_1, d_2, d_3
the second (superfluous?) family	ν_μ, μ, c_1, c_2, c_3, s_1, s_2, s_3
the third (super superfluous?) family	ν_τ, τ, t_1, t_2, t_3, b_1, b_2, b_3.

(12-1)

The families are also sometimes referred to as generations, however, there does not seem to exist any mother-daughter relationship *among* them. In (12-1), the indices 1, 2, 3 refer to the colour degrees of freedom. As discussed before, the evidence for ν_τ, *although* substantial, is still indirect and the truth (t) remains to be discovered.

In *gauge* theories, one normally follows Nature's own pattern: a repeated world including n $(\text{super})^n$ fluous families is simply described by repeating the theory constructed for just one family, for example the fundamental family. (This state of affairs is, of course, not satisfactory.) Here we shall assume that there are n families,

$$\nu_j, \quad \ell_j, \quad u_j, \quad d_j \qquad j = 1, .., n$$

$$Q = 0, \quad -1, \quad 2/3, \quad -1/3, \tag{12-2}$$

where we have suppressed the colour indices. In (12-2), for example $u_1 = u$, $u_2 = c$, $u_3 = t$, etc. Furthermore, we shall assume that the standard $SU(2) \times U(1)$ model is correct, the left multiplets are doublets and the right-handed ones singlets. In constructing the theory, we follow the steps in section 6, including of course all the elementary fermions in nature, (12-2). The kinetic energy term reads

$$\mathcal{L}_o = i \sum_{j=1}^{n} \left[\bar{\nu}_j \gamma \frac{\partial}{\partial x} \nu_j + \bar{\ell}_j \gamma \frac{\partial}{\partial x} \ell_j + \bar{u}_j \gamma \frac{\partial}{\partial x} u_j + \bar{d}_j \gamma \frac{\partial}{\partial x} d_j \right], \tag{12-3}$$

where the sum over the colour has not been explicitly exhibited. This relation is the analog of (6-1). The decomposition into left and right components may be done exactly as in (6-3). Since we have more than 2 elementary fermions, we do not know, a priori, how to choose the doublets, viz., there are infinite number of possibilities and the physics depends on the choice made. Of course, because of the charge assignments, we cannot put leptons and quarks in the same doublets (otherwise we would get W's with fractional charges).

We now restrict ourselves to the hadronic sector. The generalization to leptonic sector is trivial (see below). Due to our lack of knowledge, the most general choice for multiplets is to take (arbitrary) linear combinations

$$\psi_{jL} = \begin{pmatrix} u'_j \\ d'_j \end{pmatrix}_L , \quad u'_{jR} , \quad d'_{jR} \qquad j = 1,.., n \tag{12-4}$$

where

$$u'_{jL} = A^{(u,L)}_{jk} u_{kL} \qquad u'_{jR} = A^{(u,R)}_{jk} u_{kR} \tag{12-5}$$

$$d'_{jL} = A^{(d,L)}_{jk} d_{kL} \qquad d'_{jR} = A^{(d,R)}_{jk} d_{kR} , \; j = 1,..,n.$$

Here the A's are n x n matrices. They have to be unitary in order to leave $\mathcal{L}_o$ invariant (we shall return to the origin of these matrices soon). Suppressing the indices, etc.

$$\sum_j \bar{u}_j u_j = \sum_j \bar{u}'_j u'_j \quad , \quad \sum_j \bar{d}_j d_j = \sum_j \bar{d}'_j d'_j \quad , \tag{12-6}$$

$$A^+ = A^{-1}, \text{ for each } A,$$

$$\mathcal{L}_o = i \sum_j \left\{ (\bar{u}'_j , \bar{d}'_j)_L \gamma \frac{\partial}{\partial x} \begin{pmatrix} u'_j \\ d'_j \end{pmatrix}_L + \bar{u}'_{jR} \gamma \frac{\partial}{\partial x} u'_{jR} + \bar{d}'_{jR} \gamma \frac{\partial}{\partial x} d'_{jR} \right\}. \tag{12-7}$$

Now we "gauge" the kinetic energy term, whereby the mediators $\vec{W}$ and B are born (just as in section 6). Thus (see eq. 6-7)

$$\gamma \frac{\partial}{\partial x} \to \gamma^\mu D_\mu = \gamma^\mu \left[\frac{\partial}{\partial x^\mu} - i g \frac{\bar{\sigma}}{2} \cdot \bar{W}_\mu - i g' y B_\mu \right]$$

is *sandwiched* between the fields in (12-4) just as in (6-7); remembering $\bar{\sigma} u'_{jR} = \bar{\sigma} d'_{jR} = 0$, $y = Q - \frac{\sigma_3}{2}$, Q is the charge matrix, $\bar{\sigma}$ = Pauli matrices, viz.

$$\mathcal{L} = i \sum_j \left\{ (\bar{u}'_j , \bar{d}'_j)_L \gamma_\lambda D^\lambda \begin{pmatrix} u'_j \\ d'_j \end{pmatrix}_L + \bar{u}'_{jR} \gamma_\lambda D^\lambda u'_{jR} + \bar{d}'_{jR} \gamma_\lambda D^\lambda d'_{jR} \right\} ,$$

$$D = \frac{\partial}{\partial x} - i g \frac{\bar{\sigma}}{2} \cdot \bar{W} - i g' y B \tag{12-8}$$

So far all the (matter as well as gauge) fields are massless. According to our assumptions, in the real world, the SU(2) x U(1) symmetry is spontaneously broken, whereby the masses are generated. The breaking mechanism is assumed to be à la Weinberg, i.e., due to a doublet of Higgses. Thus we may repeat the arguments in section 11. The quark - Higgs interaction (see eq. 11-9) before SSB reads

$$\mathcal{L}(\text{quark-Higgs}) = \sum_{j,k} \left\{ c_{jk} (\bar{u}'_j , \bar{d}'_j)_L \begin{pmatrix} \varphi^{o*} \\ -\varphi^- \end{pmatrix} u'_{kR} + \tilde{c}_{jk} (\bar{u}'_j , \bar{d}'_j)_L \begin{pmatrix} \varphi^+ \\ \varphi^o \end{pmatrix} d'_{kR} \right\} + $$
$$+ \text{h.c.} \tag{12-9}$$

Here the c's and $\tilde{c}$'s are arbitrary constants. The SSB is achieved by letting $\varphi^o \to \varphi^o + v$ and dropping terms involving $\varphi^\pm$. Then

$$\mathcal{L}_{mass} = v \sum_{j,k} \left\{ \bar{u}'_{jL}\, c_{jk}\, u'_{kR} + \bar{d}'_{jk}\, \tilde{c}_{jk}\, d'_{kR} + h.c. \right\}$$
$$= \bar{u}'_L M' u'_R + \bar{d}'_L \tilde{M}' d'_R + \bar{u}'_R M'^{+} u'_L + \bar{d}'_R \tilde{M}'^{+} d'_L \qquad (12\text{-}10)$$

where

$$u'_{L,R} = \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_n \end{pmatrix}_{L,R} , \quad d'_{L,R} = \begin{pmatrix} d'_1 \\ d'_2 \\ \vdots \\ d'_n \end{pmatrix}_{L,R} , \; M'_{jk} = v\, c_{jk} , \; \tilde{M}'_{jk} = v\, \tilde{c}_{jk}$$

Now we see clearly the significance of the matrices A in eq. (12-5). They originate from diagonalizing the mass matrix and getting rid of the pseudoscalar terms in (12-10). For Q = 2/3 sector, we write M' in "polar coordinates"

$$M' = m\, U , \qquad (12\text{-}11)$$

where m is hermitian and U is unitary. m is diagonalized by a transformation $m = S^{-1} D\, S$, where S is unitary, and D is diagonal and real,

$$\bar{u}'_L M' u'_R + h.c. = \bar{u}'_L S^{-1} D\, S\, U\, u'_R + h.c.$$

We put

$$u_R = S\, U\, u'_R , \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$
$$u_L = S\, u'_L , \qquad (12\text{-}12)$$

$$\bar{u}'_L M' u'_R + h.c. = \bar{u}_L D\, u_R + \bar{u}_R D\, u_L = \bar{u}\, D\, u .$$

Thus the elements of D are the physical masses

$$D = \begin{pmatrix} m_u & & & \\ & m_c & & \\ & & m_t & \\ & & & \ddots \end{pmatrix} , \quad u = \begin{pmatrix} u_1 = u \\ u_2 = c \\ u_3 = t \\ \vdots \\ u_n \end{pmatrix} . \qquad (12\text{-}13)$$

Evidently, the charge -1/3 sector can be diagonalized in the same way. Comparing (12-12) with (12-5), yields

$$A^{(u,L)} = S^+ , \; A^{(u,R)} = U^+ S^+, \text{ etc.}$$

We now return to the interaction term in (12-8). As we have just seen the primed fields are not the physical ones and therefore we should express them in terms of the physical unprimed fields, by utilizing (12-5). It seems that we then get a very complicated expression. Fortunately, the generalized GIM - mechanism is operating here and there is no trace of the matrices A in the neutral current sector, viz.

$$\sum_j (\bar{u}'_j, \bar{d}'_j)_L \frac{\sigma_3}{2} \begin{pmatrix} u'_j \\ d'_j \end{pmatrix}_L = \frac{1}{2} \sum_j \left\{ \bar{u}'_{jL} u'_{jL} - \bar{d}'_{jL} d'_{jL} \right\} = \tag{12-14a}$$

$$= \frac{1}{2} \sum_j \left\{ \bar{u}_{jL} u_{jL} - \bar{d}_{jL} d_{jL} \right\} ,$$

$$\sum_j \bar{u}'_{jR} \, y \, u'_{jR} = \sum_j Q_u \, \bar{u}'_{jR} u'_{jR} = \frac{2}{3} \sum_j \bar{u}_{jR} u_{jR} \quad , \tag{12-14b}$$

$$\sum_j \bar{d}'_{jR} \, y \, d'_{jR} = - \frac{1}{3} \sum_j \bar{d}_{jR} d_{jR} \tag{12-14c}$$

The neutral currents are thus 'naturally' flavour conserving: there are no strangeness-changing or charm-changing, etc. neutral currents, independent of the specific form of the matrices A.

All we had to do is to treat all Q = 2/3 quarks on an equal footing (and similarly for Q = -1/3 quarks). The above arguments apply just as well for leptons, i.e., there are no flavour-changing neutral currents (such as $\bar{\mu} e$, $\bar{\nu}_e \nu_\mu$, etc. The absence of such currents in the standard model is one of its attractive features. Strangeness-changing neutral currents are known to be highly suppressed (for example from $K_L^0 \to \mu^+ \mu^-$). Similarly there are good limits on charm-changing neutral currents from $D^0 - \bar{D}^0$ transition. In summary the neutral current sector of the standard model is as given in (8-4) and (8-5), $I_{3L} = \frac{1}{2}$ for neutrinos and charge 2/3 quarks, $I_{3L} = -\frac{1}{2}$ for ℓ^- and all charge -1/3 quarks.

We now focus our attention on the charged current sector in eq. 12-8,

$$\mathcal{L}^{c.c.} = \frac{g}{2\sqrt{2}} \bar{u}'_j \, \gamma^\lambda (1 - \gamma_5) d'_j \, W^-_\lambda + h.c.$$

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \bar{u}_j \gamma^\lambda (1 - \gamma_5) \left\{ \left(A^{(u,L)}\right)^+ A^{(d,L)} \right\}_{jk} d_k W_\lambda^- , \quad (12\text{-}15)$$

where we have used (12-5). Note that { } is a unitary n x n matrix. Because of our lack of information, all we know about U = { } is that it is an arbitrary unitary matrix. A general n x n unitary matrix has n^2 real parameters; of these 2n - 1 may be removed by redefining the fields u and d, viz.

$$\bar{u}\, U\, d = \bar{u}\, \Phi^{-1}(\theta_1,\ldots,\theta_n)\Phi(\theta_1,\ldots,\theta_n) U\, \Phi^{-1}(\varphi_1,\ldots,\varphi_n)\Phi(\varphi_1,\ldots,\varphi_n) d,$$

$$\Phi(\theta_1,\ldots,\theta_n) = \begin{pmatrix} e^{i\theta_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & e^{i\theta_n} \end{pmatrix} \quad (12\text{-}16)$$

The reader may now write $U_{jk} = |U_{jk}| e^{i\zeta_{jk}}$ and convince herself that, by judicial choice of $(\theta) = (\theta_1,\ldots,\theta_n)$ and $(\varphi) = (\varphi_1,\ldots,\varphi_n)$, 2n - 1 of the parameters in $\Phi(\theta) U\, \Phi^{-1}(\varphi)$ can be removed. Of cource, in our $\mathcal{L}$, we must now redefine $\Phi(\varphi)d = (d)_{new}$, $\Phi(\theta)u = (u)_{new}$, but this does not change anything anywhere else. Thus the transformed matrix $\Phi(\theta) U\, \Phi^{-1}(\varphi)$, which we simply refer to as U, has $n^2 - (2n-1) = (n-1)^2$ real parameters.

13. THE FOUR QUARK AND SIX QUARK MODELS; CP - VIOLATION

Up to 1975 we had only four quark flavours (u, d, c, s) and four leptons (ν_e, e, ν_μ, μ). The charged current interactions were described by

$$\mathcal{L}^{cc} = \frac{g}{2\sqrt{2}} \bar{u}_j \gamma^\lambda (1 - \gamma_5) U_{jk}\, d_k W_\lambda^- + \text{h.c.} \quad , \quad (13\text{-}1)$$

where U is a 2 x 2 general unitary matrix. According to our discussions in the previous section U has just one parameter, which we take to be a rotation angle, viz.,

$$u = \begin{pmatrix} u \\ c \end{pmatrix}, \quad d = \begin{pmatrix} d \\ s \end{pmatrix}, \quad U = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}, \quad (13\text{-}2)$$

where θ_c is the Cabibbo angle. We may rewrite $\mathcal{L}^{cc}$ as

$$\mathcal{L}^{cc} = \frac{g}{2\sqrt{2}} \left\{\bar{u}_L \gamma^\lambda d_{cL} + \bar{c}_L \gamma^\lambda s_{cL}\right\} W^-_\lambda + \text{h.c.} ,$$

$$d_{cL} = \cos\theta_c \, d_L + \sin\theta_c \, s_L \qquad (13\text{-}3)$$

$$s_{cL} = -\sin\theta_c \, d_L + \cos\theta_c \, s_L .$$

Thus, the doublets chosen in nature are

$$\begin{pmatrix} u \\ d_c \end{pmatrix}_L , \quad \begin{pmatrix} c \\ s_c \end{pmatrix}_L , \qquad (13\text{-}4)$$

i.e., the most general ones one could have. Of course the single parameter θ could also be put in the u - c sector. If we start with the doublets in (13-4) and the right-handed singlets (u_R, c_R, d_R, s_R) and construct the standard SU(2) x U(1) model we could forget about all rotation matrices because all their effects are included in the single parameter θ_c which appears in the doublets (13-4).

The relations (13-4) summarize the celebrated GIM mechanism. In the standard model the GIM *mechanism* (which tells us that there are no flavour-changing neutral currents) is perfectly "natural". We "understand" why there should be a Cabibbo angle, but cannot compute its magnitude.

Why is there not a second Cabibbo-like angle in the leptonic sector? Why do we not get

$$\mathcal{L}^{cc} = \frac{g}{2\sqrt{2}} (\bar{\nu}_e, \bar{\nu}_\mu)\gamma^\lambda \begin{pmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} e \\ \mu \end{pmatrix} W^-_\lambda + \text{h.c.}, \qquad (13\text{-}5)$$

where θ' is a *new angle*? The answer is that if ν_e and ν_μ are massless such an angle is unobservable. Because, then we could identify

$$\begin{pmatrix} (\nu_e)_{\text{physical}} \\ (\nu_\mu)_{\text{physical}} \end{pmatrix} = \begin{pmatrix} \cos\theta'\nu_e - \sin\theta'\nu_\mu \\ +\sin\theta'\nu_e + \cos\theta'\nu_\mu \end{pmatrix} \qquad (13\text{-}6)$$

whereby θ' is eliminated. By definition ν_e is the neutrino which is the partner of the electron, etc. For quarks we cannot do so because the d and s quarks are distinguishable (they have different "masses").

In conclusion in the standard model, with four quarks and four leptons, the GIM mechanism is natural; the lepton numbers L_e and L_μ are also automatically conserved provided the neutrinos are massless.

Since 1975 the third family of leptons and quarks have entered into the arena. We know that the GIM mechanism (with just one Cabibbo angle), which was so natural and attractive as well as in very good agreement with data must be at most only approximately valid.

We now examine the generalization of the standard model to the case of six quark flavours (u, c, t, d, s, b) and six leptons. Applying the results in section 12, we find that the neutral current sector is flavour conserving and given by (8-4) and (8-5), where $I_{3L} = \frac{1}{2}$ for all Q = 2/3 quarks and neutrinos and $I_{3L} = -\frac{1}{2}$ for all Q = -1/3 quarks and negative leptons. The charged current sector is the only place where new parameters (generalized Cabibbos) can enter, viz.,

$$\mathcal{L}^{cc} = \frac{G}{2\sqrt{2}}\,(\bar{u},\ \bar{c},\ \bar{t})\gamma^\lambda(1-\gamma_5)U\begin{pmatrix} d \\ s \\ b \end{pmatrix} W_\lambda^- + \text{h.c.} \quad , \qquad (13\text{-}7)$$

where we have used (12-5). Here U is a 3 x 3 unitary matrix and (according to our discussions in section 12) may be taken to have $(3-1)^2 = 4$ parameters. These parameters are normally taken as three rotation angles and a phase à la Kobayashi and Maskawa. It is easily seen that the Kobayashi - Maskawa choice corresponds to taking

$$U = \mathcal{R}_1(\theta_2)\cdot\mathcal{R}_3(\theta_1)\phi(0,0,\delta)\mathcal{R}_1(\theta_3) \quad , \qquad (13\text{-}8)$$

where $\mathcal{R}_j$ is a rotation matrix about the axis j

$$\mathcal{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \quad , \quad \mathcal{R}_3(\theta) = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

$$c = \cos\theta \ , \quad s = \sin\theta \ , \quad \phi(0,0,\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \qquad (13\text{-}9)$$

Multiplying the matrix we have

$$U = \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & -c_1s_2c_3-c_2s_3e^{i\delta} & -c_1s_2s_3+c_2c_3e^{i\delta} \end{pmatrix} \quad (13\text{-}10)$$

this is the famous Kobayashi - Maskawa generalization of the Cabibbo - GIM - matrix which described the world as long as we had only four quarks. Comparing (13-10) and (13-2) we find that $\theta_1 = \theta_c$ (the Cabibbo angle) but now we have two more arbitrary angles and a phase δ. None of these is calculable. The phase could possibly be the origin of CP - violation. Note that this phase is removable if any of the angles θ_1, θ_2 or θ_3 vanishes. For example, if $\theta_3 = 0$ we have $_1(\theta_3) = 1$ and then putting $(d)_{new} = \Phi(0,0,\delta)d$, $d = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$, removes the phase. Similarly, using $\mathcal{R}_3(\theta_1)\Phi(0,0,\delta) = \Phi(0,0,\delta)\mathcal{R}_3(\theta_1)$, we see that for $\theta_2 = 0$ again the phase can be absorbed into $u = \begin{pmatrix} u \\ c \\ t \end{pmatrix}$. Finally if $\theta_1 = 0$, we get $U = \mathcal{R}_1(\theta_2)\Phi(0,0,\delta)\mathcal{R}_1(\theta_3)$. The reader can easily convince herself that U is of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & V \end{pmatrix} ,$$

where V is a 2 x 2 unitary matrix. Removing of phases in such a 2 x 2 matrix is rather trivial.

We now discuss very briefly the phenomenological consequences of the six quark model. How big are the angles θ_j, $j = 1,2,3$?

i) <u>Cabibbo universality</u>. Before knowing that there are more than four quarks we used to identify the strength of $u \to d$ transition with "$\cos\theta_c$" and $u \to s$ by "$\sin\theta_c$", see eq. (13-3). Furthermore the analysis of baryon decays, etc. gave always $\sin^2\theta_c + \cos^2\theta_c = 1$ to a good approximation. Nowadays we know better. From (13-10) we have that

transition	relative coupling	where to measure
$u \leftrightarrow d$	c_1	β-decay
$u \leftrightarrow s$	s_1c_3	hyperon decays

$c \leftrightarrow d$	$-s_1c_2$	$\nu N \to \mu^-\mu^+X$
$c \leftrightarrow s$	$c_1c_2c_3 - s_2s_3e^{i\delta}$, etc.	"

Since the Cabibbo universality worked well with just four quarks, i.e., "$\sin^2\theta_c$" + "$\cos^2\theta_c$" = $s_1^2c_3^2 + c_1^2 \approx 1$ we may conclude that θ_3 cannot be so large. Furthermore, since the four quark model predicted that charmed particles should predominantly decay into strange particles and this prediction is experimentally verified s_2 must also be a small angle. Remembering that the Cabibbo angle is also small we find that the Kobayashi - Maskawa matrix, eq. (13-10) is of the form

$$U \approx \begin{pmatrix} 1 & s_1 & 0 \\ -s_1 & 1 & s_3+s_2\,e^{i\delta} \\ 0 & -s_2-s_3\,e^{i\delta} & 1 \end{pmatrix} \qquad (13\text{-}11)$$

Now we consider the present knowledge on the angle in (13-10). The muon lifetime together with the ratio of the supperallowed β-transitions give $\cos\theta_c = c_1$. We find

$$\cos^2\theta_c = (0.972 \pm 0.004)(1 + \Delta_\mu - \Delta_\beta) \; , \qquad (13\text{-}12)$$

where $\Delta_\mu - \Delta_\beta$ is due to radiative corrections. Theoretically $\Delta_\mu - \Delta_\beta = -0.021$ for $\sin^2\theta_W = \frac{1}{4}$. Thus

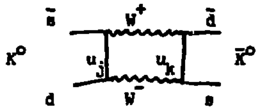
$$\begin{aligned} \sin^2\theta_c = \sin^2\theta_1 &= 1 - \cos^2\theta_c = \\ &= 0.028 \pm 0.004 \quad \text{for} \quad \Delta_\mu - \Delta_\beta = 0 \\ &= 0.047 \pm 0.004 \quad \text{"} \quad \Delta_\mu - \Delta_\beta = -0.021 \end{aligned} \qquad (13\text{-}13)$$

However, from the semileptonic decays of the baryon octet we obtain

$$\text{"}\sin^2\theta_c\text{"} = \sin^2\theta_1 \cos^2\theta_3 = 0.052 \pm 0.002 \; , \qquad (13\text{-}14)$$

where "$\sin^2\theta_c$" emphasizes that this quantity was improperly identified as $\sin^2\theta_c$. Putting together (13-13) and (13-14) we find $\theta_1 \equiv \theta_c \approx 13^\circ$ and $\theta_3 \lesssim 8^\circ$.

The t quark is presumably heavier than the b quark, otherwise it would have revealed itself in some of the experiments which have established the existence of the Υ-particle. Thus no direct information is so far available on the elements of the third row in (13-10) and (13-11). Information on the angle θ_2 may be obtained by, for example, comparing $b \to c\,\ell\,\bar{\nu}$ and $b \to u\,\ell\,\bar{\nu}$, $\ell = e, \mu, \tau$, which have respectively the couplings $s_3 + s_2\, e^{i\delta}$ and $s_1 s_3$. There is also some theoretical information, on θ_2, coming from the $\Delta s = 2$ weak processes responsible for the mass difference between K_L^0 and K_S^0, which is expected to be due to



(13-15)

$$\Delta M \sim \sum_{j,k} \int \frac{d^4k\; \bar{v}_L\, \gamma^\mu\, (k+m_j)\, \gamma^\nu\, u_L \bar{u}_L \gamma_\nu (k+m_k)\, \gamma_\mu\, v_L}{(k^2-M_W^2)^2 (k^2-m_j^2)(k^2-m_k^2)}\; F_{jk}$$

where u and v stand for the appropriate Dirac spinors; m denotes the mass. Note that the momenta of the quarks (inside the K^0 and $\bar{K}^0$) are small and thus have been neglected compared to the momentum flowing in the loop; furthermore the m's in the numerator drop out. F_{jk} is the product of the four coupling constants as the vertices in (13-15)

$$F_{jk} = a(u_j,s) \cdot a(d,u_j)\; a(u_k,s)\; a(d,u_k) \;, \qquad (13\text{-}16)$$
$$u_1 = u, \quad u_2 = c, \quad u_3 = t,$$

The a's are the elements of the unitary matrix U discussed in section 12, viz. $U_{j2} = a(u_j,s)$, $a(d,u_j) = a^*(u_j,d) = U^\dagger_{1j}$, etc. For the *four* and six quark models these quantities were given in (13-2) and (13-10). Note that the leading term in (13-15), which is obtained by putting $m_j = m_k = 0$ vanishes because

$$a(u_j,s) \cdot a(d,u_j) = (U^+U)_{12} = 0 \;.$$

For the four quark model, estimation of ΔM, gave an impressive prediction for m_c. Now that we have more than four quarks, the old prediction should not get spoiled. Estimation of (13-15) for the six quark model gives $\theta_2 < 27^\circ$ (see Ellis et al., Nucl Phys. B109, 213(76)) were the phenomenology of the model is discussed).

14. GRAND UNIFICATION

So far, in these lectures, we have discussed the conventional theory for weak and electromagnetic interaction. Nowadays, strong interactions are believed to be described by quantum chromodynamics (QCD), a nonabelien gauge theory based on the group SU(3); viz. three *colours* and eight mediators (gluons). This theory is not broken and the gluons are massless. In this section, we discuss the Georgi - Glashow model (Phys. Rev. Letters 32, 438(1974)) where the conventional SU(2) x U(1) model and chromodynamics are (grand) unified. Grand unification means that γ, $W^{\pm}$, Z^0, and the eight gluons should be the mediators (gauge bosons) of one and the same theory, where the latter is based on a simple or semisimple gauge groups, i.e., it contains only a single coupling constant. The SU(5) model by Georgi, and Glashow is the most economical scheme for unification of SU(2) x U(1) and (SU(3)) colour.

The SU(5) model does not explain the existence of the families 2 and 3 (see 12-1), so what we have to do is to write the theory for a hypothetical family

$$\nu_j', \ \ell_j', \ (u_j')_{1,2,3} \ , \ (d_j')_{1,2,3} \ , \qquad (14\text{-}1)$$

where j denotes the family number: $Q(\nu') = 0$, $Q(\ell') = -1$, $Q(u') = 2/3$, $Q(d') = -1/3$; the indices 1,2,3 denote the colour. The primes on the fields emphasize that the quantities in (14-1) do not necessarily represent the physical particles; to get the physical fields, we have to diagonalize the mass matrix (see section 12). Once we have constructed the theory for the family j, we just take a sum over all families, whereby all families are democratically represented. In the following, we simplify the notation by suppressing the primes, etc. and replacing (14-1) by ν, e, $u_{1,2,3}$, $d_{1,2,3}$; we shall discuss mixings later on.

The SU(5) theory may be constructed (analogously to theories discussed before) by starting with the kinetic term

$$\mathcal{L}_0 = i \sum_f (\bar{f}_L \gamma \frac{\partial}{\partial x} f_L + \bar{f}_R \ \gamma \frac{\partial}{\partial x} f_R) \ , \quad f = \nu, \ e, \ u, \ d \qquad (14\text{-}2)$$

How should we put f in SU(5) multiplets? The left-handed ν_e and e may be put into a fundamental 5-plet of SU(5)

$$\psi_R = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ e^+ \\ -\bar{\nu} \end{pmatrix}_R \qquad \begin{matrix} a_4 = e^+ = \psi_e^c \\ -a_5 = \bar{\nu} = \psi_\nu^c \end{matrix} \qquad (14\text{-}3)$$

together with the flavour a (to be determined) in three colours. Here $\psi^c = c\,\bar{\psi}$, where c denotes the charge conjugation matrix.

The reason for choice (14-3) is clear, viz., $(e_R^+, -\bar{\nu}_R) \sim (\nu_L, e_L)$ are intimately connected and form a doublet under SU(2). In (14-3) these leptons are "unified" with a triplet of hadrons $a_{1,2,3}$. Requiring invariance under $\psi_R \to \exp(i\,\bar{\alpha}(x)\cdot T)\psi_R$ (where the T/2 are 5 x 5 matrices representing the 24 generators of SU(5)) gives 24 gauge bosons V^j, j = 1,.., 24. Thus, we get a term

$$\mathcal{L} = i\,\bar{\psi}_R \gamma\left(\frac{\partial}{\partial x} - i\,g_0\,\bar{T}\cdot\bar{V}\right)\psi_R + \ldots \qquad (14\text{-}4)$$

The matrices T may be chosen analogously as is done in $SU(2)$ and $SU(3)$. For example

$$T^j = \left(\begin{array}{ccc|cc} & \lambda^j & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right), \quad j = 1,\ldots 8 \quad , \quad \lambda^j = \text{Gell-Mann matrices } (SU(3))$$

$$T^9 = \left(\begin{array}{ccc|cc} & 0 & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad T^{10} = \left(\begin{array}{ccc|cc} & 0 & & -i & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ \hline i & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right), \ldots\ldots \qquad (14\text{-}5)$$

We continue by putting 1 and $\pm i$ in the same pattern. It remains

$$T^{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{pmatrix}, \quad T^{24} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \qquad (14\text{-}5a)$$

The properties of T's are $Tr[T^j] = 0$, $T^{j+} = T^j$ so that the group elements $\exp(i\,\bar{\alpha}(x)\cdot\bar{T})$ are unitary and have determinant equal to unity. Furthermore $[T^j, T^k] = 2ic^{jkm}T^m$ defines the $SU(5)$ Lie algebra with c^{jkm} = structure constants. Note that $Tr[T^jT^k] = 2\delta_{jk}$. Since our theory is supposed to include all interactions (except gravity), one of the gauge bosons, i.e., one linear combination of the V's in (14-4) should be the photon field. The generator coupling to photon is the charge operator, $Q = \Sigma c_j\, T_j$, where c_j are constants. Thus

$$Tr\ Q = 0 \qquad (14\text{-}6)$$

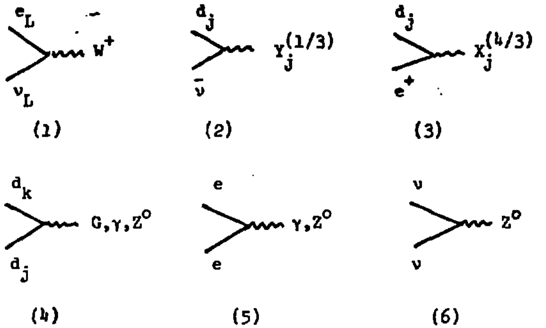
For the multiplet (14-3)

$$Tr\ Q = 3Q_a + Q_\nu + Q_{e^+} = 3Q_a + 1 = 0, \qquad (14\text{-}7)$$
$$Q_a = \frac{-1}{3}$$

where we have used the fact that a_1, a_2 and a_3, except for colour, are identical objects. Relation (14-7) implies that a should be identified with the d quark

$$\psi_R = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\bar{\nu} \end{pmatrix}_R \qquad (14\text{-}8)$$

We see that (14-8) includes a doublet and three singlets under the weak isospin group. The nature of mediators in SU(5) model is easily established from (14-8), where any member can transform into any other by sending off a mediator. Thus we have $\bar{\nu}_R \leftrightarrow e_R^{\pm}$, i.e., $\nu_L \leftrightarrow e_L^-$ via $W^{\pm}$ mediators; $d_j \leftrightarrow d_k$ via gluons, etc. However, there are also lepton-quark transitions via new exotic mediators denoted by $X_j^{(\pm 4/3)}$, $Y_j^{(\pm \frac{1}{3})}$, j = 1,2,3



Thus neither the lepton number nor the baryon number is conserved. The twelve exotics X and Y are sometimes referred to as leptoquarks, although they are neither leptons nor quarks. They must be heavy because otherwise the stable matter (protons and neutrons) would evaporate into leptons, as we shall discuss below.

We have altogether 15 helicity states to account for if the neutrino is massless (ν_L, e_L, e_R, u_{jL}, u_{jR}, d_{jL}, d_{jR}). For a massive neutrino there is also a ν_R, i.e., altogether 16 states. In (14-8), 5 of these states are accounted for, leaving us with 10 (11) unaccounted ones if the neutrino is massless (massive). The remaining ten states can be easily placed into a 10 dimensional representation of SU(5), as we discuss shortly. For massive neutrinos we would need to introduce an additional singlet ν_R. This is ugly and, therefore, we conclude that SU(5) has at least two beautiful features, (i) the fractionally charged quarks are natural and (ii) the neutrino is massless.

From the five objects b_j, $j = 1,\ldots, 5$ which transform as 5-plet of SU(5) we can form a ten-plet by taking the antisymmetric product

$$b_{jk} = \frac{1}{\sqrt{2}}\,(b_j\, b_k - b_k\, b_j)\ , \quad , j,k = 1,\ldots, 5 \qquad (14\text{-}9)$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{pmatrix} .$$

Evidently $b_{jk} = 0$ if $j = k$ and $b_{jk} = -b_{kj}$. For $j,k = 1,2,3$ we have the antisymmetric product of two colour triplets. From SU(3) relation $3 \times 3 = 6 + \bar{3}$, where the six is symmetric and $\bar{3}$ antisymmetric, we see that b_{jk}, $j,k = 1,2,3$ represent a colour antitriplet. Furthermore these three objects are singlets under weak - e m SU(2) and have $Q = -\frac{2}{3}$. Thus b_{jk} may be identified with $\varepsilon_{jkm}\,\bar{u}_{mL}$. Similarly $b_{j4}(b_{j5})$, $j = 1,2,3$ represents a colour triplet which has charge $\frac{-1}{3} + 1 = \frac{2}{3}\left(\frac{-1}{3} + 0 = \frac{-1}{3}\right)$ and the third component of weak isospin $\frac{1}{2}(-\frac{1}{2})$. These can be identified with u_{jL} and d_{jL}. The only remaining state is b_{45} which is singlet under colour and weak isospin and has $Q = +1$, i.e., e^+_L. Thus

$$\psi_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \bar{u}_3 & -\bar{u}_2 & u_1 & d_1 \\ -\bar{u}_3 & 0 & \bar{u}_1 & u_2 & d_2 \\ \bar{u}_2 & -\bar{u}_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L \equiv b_{jk} \quad (14\text{-}10)$$

Here $\bar{u}$ means $\psi^c_u = c\tilde{\bar{\psi}}_u$, i.e., the charge conjugated spinor.

The kinetic energy term, (14-2) can be expresses in terms of (14-8) and (14-10)

$$\mathcal{L}_o = i\left\{\bar{a}_j\ \gamma\ \frac{\partial}{\partial x}\ a_j + \bar{b}_{k\ell}\ \gamma\ \frac{\partial}{\partial x}\ b_{k\ell}\right\}$$

$$= i\ \mathrm{Tr}\left[\bar{\psi}_R\ \gamma\ \frac{\partial}{\partial x}\ \psi_R + \bar{\psi}_L\ \gamma\ \frac{\partial}{\partial x}\ \psi_L\right] + \mathcal{L}_1 \qquad (14\text{-}11)$$

$$\mathcal{L}_1 = i\ \mathrm{Tr}\left\{\bar{\psi}_R\ \gamma\left[\frac{\partial}{\partial x} - i\ g_o\ \bar{T}\cdot\bar{V}\right]\psi_R + \bar{\psi}_L\ \gamma\left[\frac{\partial}{\partial x} - i\ g_o\ \bar{T}'\cdot\bar{V}\right]\psi_L\right\}.$$

Here $2g_o$ is the grand unified coupling constant, $\bar{T}(\bar{T}')$ are the 24 matrices representing generators, for the 5 (10) dimensional representation. Furthermore

$$\bar{\psi}_R\ \bar{T}\cdot\bar{V}\ \psi_R = \bar{a}_j \sum_{\alpha=1}^{24} T^\alpha_{jk}\ V^\alpha\ a_k = \mathrm{Tr}(\bar{a}\ \bar{T}\cdot\bar{V}\ a)\ ,$$

$$\mathrm{Tr}(\bar{\psi}_L\ \bar{T}'\cdot\bar{V}\ \psi_L) = \bar{b}_{jk} \sum_{\alpha=1}^{24} (\acute{T}^\alpha\ V^\alpha)_{jk,j'k'}\ b_{j'k'} \tag{14-12}$$

Although the matrices representing generators look a bit frightening, for the left-handed 10-plet, they are quite simple because the b_{jk} decomposes into objects with simple transformation properties under colour and weak isospin. This ensures that, although some fermions are put into 5-plet and others into 10-plet, there is universality of strong, weak and electromagnetic interactions. For example the right-handed u's form a colour triplet - SU(2) singlet and are thus treated exactly as the right-handed d's, except for weak hypercharge; (u_{jL},d_{jL}) form doublets under weak isospin and are treated as (ν,e), etc.

Let us look a bit into the 5-plet sector. There we have 4 diagonal generators T_3, T_8, T_{15} and T_{24}, see (14-5). T_3 and T_8 couple to gluons G_3 and G_8. T^{15} is proportional to the (right-handed electric charge operator, $Q_R = -\sqrt{\frac{2}{3}}\,T^{15}$, i.e., V^{15}_μ is the photon field A_μ and $g_o = \sqrt{\frac{2}{3}}\,e$. If we are going to achieve unification, V^{24}_μ better be the Z_μ and $Q^Z_R = r\ T^{24}$, where r is a constant $\mathrm{Tr}[T^jT^k] = 2\ \delta_{jk}$, whereby $\mathrm{Tr}[Q^\gamma_R Q^Z_R] = 0$. This relation is normally not satisfied in the standard model [$Q^Z \sim (I_{3L} - Q\sin^2\theta_W)$, eq. (8-3)] as is seen by substituting I_{3L} and Q for the objects in (14-8) and remembering that $(e^+, -\bar{\nu})_R \to (\nu, e^-)_L$. We require

$$\mathrm{Tr}\left[Q^\gamma_R Q^Z_R\right] \sim \mathrm{Tr}(I_{3L}Q - \sin^2\theta_W Q^2) = 0\quad ,\quad Q = I_{3L} + Y$$

So $\sin^2\theta_W = \dfrac{\mathrm{Tr}(I^2_{3L})}{\mathrm{Tr}(Q^2)}$, where the sum goes over the members in (14-8),

i.e., $\sin^2\theta_W = \dfrac{2\cdot 1/4}{1 + 3\cdot\frac{1}{9}} = \dfrac{3}{8}$. Thus the unification hypothesis fixes the value of the angle θ_W to be about 38^o, at the unification energy, which is going to be a very large energy. [At our energies θ_W is expected to be smaller.]

Now we consider the left-handed multiplet, (14-10). The generators $\bar{T}'$, in (14-11) are easily obtained by taking infinitesimal transformations (see 14-9)

$$b_j \to [(1 + i\,\bar{\alpha}\cdot\bar{T})b]_j$$

$$b_{jk} \to [(1 + i\,\bar{\alpha}\cdot\bar{T})b]_j[(1 + i\,\bar{\alpha}\cdot\bar{T})b]_k - j \leftrightarrow k$$

Comparison with (14-12) yields

$$T'_{jk,j'k'} = \delta_{jj'}\,T_{kk'} + \delta_{kk'}\,T_{jj'} \quad , \tag{14-13}$$

$$\mathrm{Tr}(\bar{\psi}_L\,\bar{T}'\cdot\bar{V}\,\psi_L) = 2\,\bar{b}_{jk}\,\bar{T}_{kk'}\cdot\bar{V}\,b_{jk'} = 2\,\bar{b}_{jk}\,\bar{T}_{jj'}\,\bar{V}\,b_{j'k} \quad .$$

Again V^{15}_μ (V^{24}_μ) are respectively the photon (Z^o) fields. The reader can easily write down the various pieces of the interaction, e.g. the coupling to the Z^o of the right-handed electron is derived from

$$g_o\,\bar{\psi}_L\,\gamma_\mu\,T'^{15}\,\psi_L\,Z^\mu \to \left\{\sqrt{\frac{2}{3}}\;e\left\{\bar{b}_{54}\,T^{15}_{44}\,b_{54} + \bar{b}_{45}\,T^{15}_{55}\,b_{45}\right\}\right\}$$

$$= -\frac{3}{\sqrt{10}}\sqrt{\frac{2}{3}}\;e\;\bar{\psi}^c_{eL}\,\gamma^\mu\,\psi^c_{eL}\,Z_\mu = \sqrt{\frac{3}{5}}\,e\,\bar{\psi}_{eR}\,\gamma^\mu\,\psi_{eR}\,Z^\mu \;.$$

This agrees with (8-3), where we find

$$\frac{e\,Z^\mu}{\sin\theta_W\,\cos\theta_W}\,(-\sin^2\theta_W)Q_e\,\bar{\psi}_{eR}\,\gamma^\mu\,\psi_{eR} = e\sqrt{\frac{3}{5}}\,\bar{\psi}_{eR}\,\gamma^\mu\,\psi_{eR} \;,$$

Here we have used $\sin^2\theta_W = 3/8$.

15. MIXING ANGLES IN THE GEORGI-GLASHOW MODEL

In the previous section, we determined the structure of the couplings between the elementary fermions and the mediators. However, all our results are valid for the unphysical fermions (primes were suppressed) and massless intermediate bosons. We may now repeat the arguments in section 12 and rewrite the theory for the physical fermions, viz. $u'_L = A^{(u,L)}\,u_L$, $d'_L = A^{(d,L)}d_L$, etc., where as before,

$$u = \begin{pmatrix} u \\ c \\ t \\ \vdots \end{pmatrix}, \quad d = \begin{pmatrix} d \\ s \\ b \\ \vdots \end{pmatrix}, \quad \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \vdots \end{pmatrix}, \quad e = \begin{pmatrix} e \\ \mu \\ \tau \\ \vdots \end{pmatrix}$$

The six matrices $A^{(f,L)}$, $A^{(f,R)}$, $f = e,u,d$ (the neutrino is assumed to be massless) are all unitary. Furthermore, these matrices are independent of colour. We can now easily convince ourselves that in the interaction term

$$\mathcal{L} = g_o \, \mathrm{Tr}\left\{\bar{\psi}_R \gamma^\mu \bar{T} \cdot \bar{V}_\mu \, \psi_R + \bar{\psi}_L \gamma^\mu \cdot \bar{T}' \cdot \bar{V}_\mu \, \psi_L\right\} \qquad (15\text{-}1)$$

the A's cancel out in the neutral current sectors (couplings to the photon, Z^o and the gluons G). The reason is that in the interactions terms (because of the γ^μ) the L and R cannot mix and, therefore, in neutral current interactions we get terms of the type $\bar{u}'_L\{\ldots\}u'_L = \bar{u}_L(A^{(u,L)})^+\{\ldots\}A^{(u,L)}u_L = \bar{u}_L\{\ldots\}u_L$, etc. However, in the charged currents (couplings to $W^\pm$, $X_m^{\pm(4/3)}$ and $Y_m^{\pm(\frac{1}{3})}$, $m = 1,2,3$) the A's show up, that is, there we find all kinds of Cabibbo angles. More explicitly, the charged currents from $\bar{\psi}_R(\bar{T} \cdot \bar{V})\psi_R$ are of the form

$$\overline{e'^c_R} \, \nu'^c_R \, W \sim \bar{\nu}'_L \, e'_L \, W \, , \; \bar{d}'_R \, \nu'^c_R \, Y \, , \; \bar{d}'_R \, e'^c_R \, X \quad . \qquad (15\text{-}2)$$

From $\bar{\psi}_L(\bar{T}' \cdot \bar{V})\psi_L$, we get the charged currents of the form

$$\bar{u}'_L \, d'_L \, W \, , \; \overline{u'^c_L} \, d'_L \, Y \, , \; \overline{u'^c_L} \, u'_L \, X \, , \; \bar{u}'_L \, e'^c_L \, Y \, , \; \bar{d}'_L \, e'^c_L \, X \, . \qquad (15\text{-}3)$$

In (15-2) and (15-3) we have left out the γ-matrices, colour indices, etc. We may now express the charged currents in terms of the physical states. For example

$$\bar{u}'_L \, d'_L \, W = \bar{u}_L \underbrace{\left[A^{(u,L)}\right]^+ A^{(d,L)}}_{U^{(u,d)}} d_L \, W = \bar{u}_L \, U^{(u,d)} d_L \, W, \qquad (15\text{-}4)$$

where $U^{(u,d)}$ is a general $n \times n$ unitary matrix, n = number of families. For example, for three families U is the Kobayashi-Maskawa matrix, discussed in section 13. In the leptonic charged currents the corresponding U may be absorbed into the definition of the neutrino fields, exactly in the same fachion as in section 13.

$$\bar{\nu}'_L \, e'_L \, W = \bar{\nu}'_L \, A^{(e,L)} e_L \, W \equiv \bar{\nu}_L \, e_L \, W \quad . \qquad (15\text{-}5)$$

Here

$$\nu_L \equiv \left[A^{(e,L)}\right]^+ \nu'_L \quad ,$$

$$\nu_L \, e_L \, W = \left\{\bar{\nu}_{eL} \, e_L + \bar{\nu}_{\mu L} \, \mu_L + \bar{\nu}_{\tau L} \, \tau_L + \ldots\right\} \quad . \tag{15-6}$$

The remaining interactions in (15-2) and (15-3) may be treated analogously. Note that there are additional angles and phases in couplings to X and Y which are, in principle, completely different from the generalized Cabibbo angles and Kabayashi-Maskawa phases which are introduced to parameterize $U^{(u,d)}$. For example

$$\bar{d}'_R \, \nu'^c_R \, Y \sim \bar{\nu}'_L \, d'^c_L \, Y = \bar{\nu}_L \underbrace{\left(A^{(e,L)}\right)^+ \left(\overset{*}{A}{}^{(d,R)}\right)}_{U^{(\nu,d^c)}} d^c_L \, Y \ . \tag{15-7}$$

Again the γ-matrices, etc, have been suppressed. The specific form of these new mixing matrices will depend on how one chooses to break the symmetry and generate the fermion masses. The mediators X and Y are expected to be very very heavy (10^{15}-10^{16} GeV). Thus it is going to be very tough to study the mixing angles and phases (such as CP-violation) in the X and Y charged currents.

The matrices A, as discussed in section 13, originate from diagonalization of the fermion mass matrix. We believe that the masses have to do with the spontaneous symmetry breaking à la Higgs. Below we describe, briefly, how the Higgs mechanism is done for the SU(5) model.

16. THE HIGGSES OF THE SU(5) MODEL

SU(5) has 24 mediators ($X_j^{\pm(4/3)}$, $Y_j^{\pm(1/3)}$, $j = 1,2,3$, $W^\pm$, Z, γ, G_i, i = 1,..,8). Clearly X and Y should be very heavy so as to suppress the violation of bayron number and lepton numbers. $W^\pm$ and Z^0 should turn out to be as à la Weinberg, (eq. 11-8), in order to be in accordance with experiments; the gluons and the photon should remain massless. Therefore the symmetry breaking should occur at two levels, viz.

(i) SU(5) → SU(3) x SU(2) x U(1)

	colour	standard model	
massless:	G_i	$W^\pm$, Z, γ	(16-1)

ii) $SU(2) \times U(1) \rightarrow [U(1)]$.
standard charge

At the first stage one or more Higgses (which we have not introduced yet) should acquire a stupendous vacuum expectation value whereby X and Y get tremendous masses. Compared to these monsters W and Z are light as feather, i.e., their masses are due to a new class of Higgses which have acquired a much smaller vacuum expectation value (of the order of 100 GeV). After the second stage we have nine massless mediators (gluons and photon) and all symmetries except for colour and charge are broken.

The first stage of SSB is achieved by introducing 24 Higgses (in the adjoint representation, $5 \times \bar{5} = 1 + 24$) φ^j, $j=1,2,\ldots,24$. The charge and colour quantum numbers of these spin zero objects are just the same as for the gauge bosons (see after eq. (14-8)). Thus we have 8 gluonic Higgses,

$$\varphi_G \equiv \varphi^j \ , \ j = 1,\ldots,8 \quad \text{colour octet, electrically neutral.}$$

There are 12 Higgses with charges $\pm 4/3$, $\pm 1/3$. These form 2 colour triplets and two antitriplets

$$\varphi_X,\ \varphi_Y \equiv \varphi^j,\ j = 9\text{-}14,\ 16\text{-}21.$$

Similarly φ_W are two colour singlet Higgses with charges ± 1, viz. φ^j, $j=22,23$. Finally $\varphi_\gamma \equiv \varphi^{15}$, $\varphi_Z \equiv \varphi^{24}$ are colour singlets and electrically neutral. Clearly only φ_γ and φ_Z are permitted to acquire non-zero vacuum expectation values, because otherwise the conservation of electric charge or/and colour would be violated. We now copy the formalism in section 10 (adding more indices as needed). The kinetic term for the Higgses reads

$$\mathcal{L}_{KE} = + \frac{1}{2} \sum_{j=1}^{24} \frac{\partial \varphi^j}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x^\mu} = + \frac{1}{2} \sum_{j,k} \frac{\partial \varphi^k}{\partial x} \frac{\partial \varphi^j}{\partial x} \frac{1}{2} \mathrm{Tr}(T^j T^k)$$
$$= + \mathrm{Tr} \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} \quad , \tag{16-2}$$

where

$$\Phi = \frac{1}{2} \sum_{j=1}^{24} \varphi^j T^j \quad . \tag{16-3}$$

The T are as given by (14-5) and (14-5a). Now we let φ_γ and φ_Z break the symmetry spontaneously

$$\Phi \to \Phi' + \langle\Phi\rangle \; , \qquad\qquad (16\text{-}4)$$
$$\langle\Phi\rangle = \alpha\, T^{15} + \beta\, T^{24}$$

where α and β are as yet arbitrary constants. Of course, to do so we must add a potential to (16-2) and check that SSB can occur. We leave that task as an exercise to our reader or refer her to the literature quoted at the end of this article. As in section 10, $\mathcal{L}$ in (16-2) is not invariant under the local SU(5); we must replace

$$\frac{\partial\varphi^j}{\partial x^\mu} \to \frac{\partial\varphi^j}{\partial x^\mu} - i\, g_o \sum_{s,r=1}^{24} \mathbf{T}^r_{js}\, V^r_\mu\, \varphi^s \; . \qquad (16\text{-}5)$$

Here $\mathbf{T}^r$ are 24 matrices (24 by 24). They represent the generators in the 24 dimensional (adjoint) representation. It is easy to construct these matrices by taking infinitesimal transformations and remembering that φ^j transform as

$$\varphi^j \sim \bar{a}\, T^j\, a \; ,$$

where a stands for a 5 (same quantum numbers as in (14-8) and T^j are given in (14-5), viz.

$$\bar{a}\, T^j\, a \to \bar{a}(1 - i\,\bar{\alpha}\cdot\bar{T})T^j(1 + i\,\bar{\alpha}\cdot\bar{T})a \to \bar{a}(T^j - i[\bar{\alpha}\cdot\bar{T}, T^j]\, a \; . \qquad (16\text{-}6)$$

We have to compare eq. (16-6) with

$$\varphi^j \to \exp(i\,\bar{\mathbf{T}}\cdot\bar{\alpha})_{jk}\,\varphi^k \approx (1 + i\,\bar{\mathbf{T}}\cdot\bar{\alpha})_{jk}\,\varphi^k = (\delta_{jk} + i\,\bar{\mathbf{T}}_{jk}\cdot\bar{\alpha})\varphi^k \; . \qquad (16\text{-}7)$$

Thus we find

$$\mathbf{T}^r_{jk} \sim \mathrm{Tr}\Big([T^r, T^j]\, T^k\Big) \sim i\, c^{rjk} \; , \qquad (16\text{-}8)$$

where c^{rjk} are the SU(5) structure constants. This result is familiar from SU(3). [We apologize for being sloppy with factors of 2; T/2 are matrices representing the generators.]

Thus the vector boson mass matrix is obtained from

$$\frac{\partial\Phi}{\partial x} \to \frac{\partial\Phi}{\partial x} + c[\bar{T}\cdot\bar{V}, \Phi] \; , \qquad (16\text{-}9)$$

$$\mathrm{Tr}([\bar{T}\cdot\bar{V}, \langle\Phi\rangle][\bar{T}\cdot\bar{V}, \langle\Phi\rangle]) \sim (m^2)_{jk}\, V^j\, V^k \; . \qquad (16\text{-}10)$$

Here c is a constant and the vacuum expectation value $\langle\Phi\rangle$ is given in (16-4). We rewrite $\langle\Phi\rangle$ in the form

$$\langle\Phi\rangle = \begin{pmatrix} \begin{pmatrix} E & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \\ \begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & F \end{pmatrix} \end{pmatrix} \qquad (16\text{-}11)$$

where

$$F = a\, I_2 + b\, \sigma_3 \ , \qquad E = -\frac{2a}{3} I_3 \qquad (16\text{-}12)$$

Here a and b are arbitrary constants; σ is the Pauli matrix and I_n is the unit n by n matrix. Note that $Tr(\langle\Phi\rangle) = 0$. From eqs. (16-10), (16-11) and the definitions of T^j, we see that the vacuum is invariant under the colour SU(3), viz. the unit matrix E commutes with the colour generators T^j, $j=1,\ldots,8$. Thus the gluons are massless. Similarly the photon and Z^0 are massless;

$$[T^j, \langle\Phi\rangle] = 0 \ , \quad j = 15, 24 \ , \quad m_\gamma = m_Z = 0.$$

However for $b \neq 0$, in (16-12) the vacuum is noninvariant under the weak isospin and $W^\pm$ acquire masses. We avoid this by putting $b = 0$, without any good justification. Then the vacuum is only noninvariant under 12 operations in the group (generators corresponding to X's and Y's). The relevant generators are of the form (see (14-5))

$$T^j = \begin{pmatrix} (0) & (G^j) \\ (G^{j+}) & (0) \end{pmatrix} \qquad (16\text{-}13)$$

Eqs. (16-10) - (16-12) yield

$$M_X^2 = M_Y^2 \sim g_0^2\, a^2 \qquad (16\text{-}14)$$

where a is supposed to be tremendously large. Let us summarize: we introduced 24 Higgses and 12 gauge bosons (X_j, $\bar{X}_j$, Y_j and $\bar{Y}_j$) have acquired masses. Therefore half of the Higgses can be gauged away. In order to see who remains and who goes away, we go back to relation (9-13). Higgses corresponding to generator which break the invariance of vacuum must turn into Goldstone bosons, who get eaten up. The remaining Higgses are those corresponding to generators which leave the vacuum invariant. From (9-13) and (16-8) we find

$$T^{r}_{jk}\langle\varphi\rangle^{k} \sim \mathrm{Tr}\left([T^{r}, \langle\varphi\rangle]T^{j}\right) , \qquad (16\text{-}15)$$

$$[T^{r}, \langle\varphi\rangle] = 0 , \; T = T_{G}, T_{\gamma}, T_{Z}, T_{W} .$$

Thus, the Higgses in the theory, up to now, are the φ_G (eight of them), φ_W (two of them), φ_γ and φ_Z.

The second step in the SSB pattern (16-1) is achieved by introducing a quintet of Higgses θ^j, $j=1,..,5$, in the fundamental representation. So θ^j, $j=1,2,3$ have charge $-1/3$ and form a colour triplet; θ^4 (θ^5) is colour singlet and has $Q = 1$ ($Q = 0$). Of course these objects have also antiparticles, so altogether we have 10 degrees of freedom. Only θ^5 may be given a *nonzero* vacuum expectation value

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \end{pmatrix} , \quad \langle\theta\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \end{pmatrix} \qquad (16\text{-}16)$$

$$\frac{\partial\theta^{+}}{\partial x}\frac{\partial\theta}{\partial x} + \left|\left(\frac{\partial}{\partial x} - i\, g_{0}\, \bar{T}\cdot\bar{V}\right)\theta\right|^{2}$$

$$(m^2)_{jk}\, V^{j}\, V^{k} \sim g_{0}^{2}\, \langle\tilde{\theta}\rangle\, T^{j}\, T^{k}\, \langle\theta\rangle\, V^{j}\, V^{k} \; . \qquad (16\text{-}17)$$

Here v is a constant and the tilde stands for transposition. Clearly these *new* Higgses leave the photon and the gluons massless,

$$T^{j}\langle\theta\rangle = 0 , \quad j=1,\ldots,8, \text{ and } 15.$$

Furthermore the mass of the X is not altered, $T^{X}\langle\theta\rangle = 0$, however the Y are affected. We do not worry about this point because $|v| \ll |a|$. The $\langle\theta\rangle$ breaks the invariance under the weak isospin whereby W and Z become massive. In fact (θ^4,θ^5) are analogs of (φ^+, φ^0) of Weinberg and we obtain his relation, eq. (11-8), viz.

$$M_W^2 = K(0,\overset{*}{v})\,\sigma_1^2\begin{pmatrix}0\\v\end{pmatrix} = K|v|^2 ,$$

$$M_Z^2 = K\,\langle\tilde{\theta}\rangle\,(T^{24})^2\langle\theta\rangle = \frac{8}{5}K|v|^2 ,$$

$$M_W^2/M_Z^2 = \frac{5}{8} = \cos^2\theta_W . \qquad (16\text{-}18)$$

Again the angle is fixed at the unification energy. We have introduced 24 + 10 = 34 Higgses. Fifteen gauge bosons (all except the gluons and the photon) have become massive. Therefore there are 34 − 15 = 19 Higgses in the theory. The reader can easily convince herself that these consist of a neutral colour octet, three neutral colour singlets, two colour singlets with ±1 unit of charge, a colour triplet with charges −1/3 and the antiparticles of the latter. Although the Higgs sector of the SU(5) might seem outrageous, we must take it seriously because we lack an alternative for the generation of masses of the vector bosons.

The fermion masses are also believed to be due to the SSB mechanism. If the reader is familiar with the Young diagrams, she can easily see that

$$
\begin{aligned}
&\psi_R \sim 5 = \Box \quad , \quad \bar{\psi}_R \sim 5^* = \begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix} \\
&\psi_L \sim 10 = \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \quad , \quad \bar{\psi}_L \sim 10^* = \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix} \\
&\phi \sim 24 = \begin{smallmatrix}\Box&\Box\\\Box&\\\Box&\\\Box&\end{smallmatrix} \quad , \\
&\theta \sim 5 = \cdot\cdot \quad , \quad \theta^* \sim 5^* = \cdots
\end{aligned}
\tag{16-19}
$$

Furthermore the charge conjugated spinors transform as

$$
(\psi_R)^c \sim 5^* \quad , \quad (\psi_L)^c \sim 10^* \tag{16-20}
$$

Note also that $(\psi^c_L) = (\psi^c)_R$, etc. The Higgs-fermion invariant couplings connect left and right helicities. From (16-19) and (16-20) we find that

$$
\bar{\psi}_L \psi_R \theta, \text{ h.c. } , \quad \overline{(\psi_L)^c} \psi_L \theta, \text{ h.c.} \tag{16-21}
$$

with appropriate Clebsch-Gordan coefficients are the origin of masses. We have, in a short-hand notation, for example

$$
\begin{aligned}
\bar{\psi}_L \psi_R \langle\theta\rangle &\sim \sum_{j,k=1}^{5} \overline{b_{jk}}\, a_j \langle\theta^k\rangle \sim \sum_{j=1}^{5} \bar{b}_{5j}\, a_j \\
&\sim (\bar{d}_1 d_1 + \bar{d}_2 d_2 + \bar{d}_3 d_3 + \bar{e}\, e)
\end{aligned}
\tag{16-22}
$$

where we have left out the uninteresting factorn, and the primes denoting that the d and the e are not necessarily physical particles.

Similarly

$$\overline{\psi_L^c}\,\psi_L\,\theta \sim \sum_{1}^{4} \epsilon_{jkmn}\,\overline{b_{jk}^c}\,b_{mn} \sim (\bar{u}_1\,u_1 + \bar{u}_2\,u_2 + \bar{u}_3\,u_3) \quad (16\text{-}23)$$

17. THE PROTON/NEUTRON DECAY IN SU(5)

Let us recall that SU(5) has several nice features:

(i) It unifies the nongravitational forces.
(ii) The quantization of charge and fractional quark charges are naturally accounted for.
(iii) It has no room for a right-handed ν, unless the right-handed neutrino is, somewhat superficially, introduced as a singlet. So, we understand why the neutrino is massless.

Furthermore the leptons and quarks are unified, as they appear in the same multiplets. The reader might not like the 10 dimensional multiplets and the fact that the number of quark-lepton families is not explained. If she objects, we encourage her to produce a better theory.

What are the *predictions of* the model, in additional to the fact that there are 24 gauge bosons 19 Higgses, etc. as discussed before? First of all the model, by including the standard SU(2) x U(1) description of the weak and electromagnetic forces and SU(3) colour for strong interactions, reproduces the standard predictions of QCD, etc.

At our energies the weak, electromagnetic and strong interaction coupling constants are vastly different. The unification is expected to take place for energies well beyond all masses in the theory. The heaviest objects being the x and y gauge bosons, we have to go to energies $Q^2 \gg M_x^2, M_y^2$ in order to have unifications. As we move down to lower values of Q^2, the strong, weak and electromagnetic couplings seperate and follow different trajectories. The renormalization of the coupling constant is due to higher order corrections (loops involving gauge bosons) and involves factors such as $\log(M_x^2/Q^2)$. One may ask how big should $M_x(=M_y)$ be in order to reproduce the presently measured values of the coupling constants at our energies. The answer is (D.A. Ross, Nucl. Phys. B140, 1(1978))

$$M_x = M_y \sim 5 \times 10^{15}\ \text{GeV !}$$

Thus the unification happens for energies much larger than

10^{16} GeV, presumably not far from the Planck value of 10^{19} GeV, where the gravitational interactions have already taken over!

We cannot hope to measure the trajectory of coupling constants up to such energies, however from

$$g = e/\sin\theta_W \quad , \quad g' = e/\cos\theta_W$$

the renormalization of g and g' imply that value of θ_W is also changing as function of Q^2. The quantity $\sin^2\theta_W$ which was 3/8 at the unification energy falls off slowly as we go down in Q^2. At our energies $\sin^2\theta_W \sim 0.2$ for three families of quarks and leptons (A. Buras et al. Nucl. Phys. B135, 66(1978)).

The most spectacular predictions of the SU(5) model (as well as several other grand unified models) is that the protons and neutrons can evaporate into "instable" matter (leptons and pions). The reason for proton decay is the existence of the bosons X and Y which violate the conservation of the baryon and lepton numbers.

The proton decay has been treated by A. Buras, loc. cit and by the author and F.J. Yndurâin (to appear in Nucl. Phys. B). Here we follow the latter reference.

The dominant decay mechanism is that two quarks inside the nucleon annihilate, by exchanging either a X or Y boson. The third quark acts as spectator, viz.

(a)	$u + u \to e^+ + \bar{d}$	$P \to e^+ \pi^0$, etc.
	$\to \mu^+ + \bar{s}$	$P \to \mu^+ K^0$
(b)	$u + d \to e^+ + \bar{u}$	$P \to \pi^0 e^+$, $n \to \pi^- e^+$, ..
	$\to \mu^+ + \bar{c}$	forbidden by energy-momentum
(c)	$u + d \to \bar{\nu}_e + \bar{d}$	$P \to \pi^+ \bar{\nu}_e$, $n \to \pi^0 \bar{\nu}_e$, ..
	$\to \bar{\nu}_\mu + \bar{s}$	(small)
(d)	$d + d \to \bar{\nu}_e + \bar{u}$	$n \to \bar{\nu}_e \pi^0$, ..

The proton and neutron (baryon number violating) lifetimes are estimated to be

$$\tau_p \sim 10^{33} \text{ yrs} , \quad \tau_n \sim 2 \times 10^{33} \text{ yrs}.$$

The errors are large due to present uncertainties in the input parameters. The lifetimes could be shorter or longer. Experimentally, the present best limit on the proton lifetime, $\tau_p \geq 2 \times 10^{30}$ yrs, is based on looking for muons (F. Reines and M.F. Crouch, Phys. Rev. Letters 32, 493(1974)). However, if the Georgi-Glashow model is correct, the branching ratio to muons is small, because the dominant channel (b) cannot produce muons.

Other measurements give limits which are more than an order of magnitude shorter than the quoted best limit.

Truly, the proton lifetime is much longer than the lifetime of our theories and our universe (if we believe the standard 10^{10} yrs). Measuring it is a great challenge.

ACKNOWLEDGEMENTS

I wish to thank Anne Grete Frodesen for reading these notes and Janny Asphaug for her efficient typing of the manuscript.

REFERENCES

An incomplete list of references on gauge theories and phenomenology includes

1. E.S. Abers and B.W. Lee, Phys. Reports 9C, 1 (1973).
2. M. Böhm and H. Joos, DESY Report 78/27 (in German); this article includes an abundant list of references.
3. H. Harari, Phys. Reports 42C, 235 (1978).
4. J. Iliopoulos, CERN Yellow Report 77-18, pp. 36-78.
5. J.C. Taylor, Gauge theories of weak interactions, Cambridge University Press (1976).

The SU(5) model and its consequences may be found in, for example,

6. H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32, 438 (1974).
7. H. Georgi, H. Quinn and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).
8. A. Buras, J. Ellis, M. Gaillard and D. Nanapoulos, Nuclear Phys. B135, 66 (1978) and references therein.
9. C. Jarlskog, F.J. Yndurain, to be published in Nuclear Phys.
10. D.A. Ross, Nucl. Phys. B140, 1 (1978).