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On the Dirac Groups of Rank n

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Abstract:

The group theoretical properties of the Dirac groups of rank n are discussed, together with the properties and construction of their IR's. The cases n even and n odd show distinct features. Furthermore, for n odd, the cases $n=4k+1$ and $n=4k+3$ exhibit some different properties too.

1. Introduction

The present work is devoted to finite groups defined by

$$G(n)=\{\pm\gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_n^{p_n}\}\ ,\quad p_i=0,1\ ,\quad i=1,2,\dots,n\ , \tag{1.1}$$

with

$$\{\gamma_i,\gamma_j\}\equiv\gamma_i\gamma_j+\gamma_j\gamma_i=2\delta_{ij}I\ . \tag{1.2}$$

For a fixed integer n, the abstract group so defined is said to be of rank n, since it is generated by the n linearly independent elements $\gamma_1,\gamma_2,\dots,\gamma_n$ with I as the unit element. One may choose to call it the Dirac group of rank n, since the most relevant case for physics ($n=4$) is currently known as the Dirac group[1,2].

The groups $G(n)$, for n even, have been treated by Lomont[1] in his well-known book, particulary from the viewpoint of group representation theory. Their irreducible representations (IR) provide representations for creation and annihilation operators of fermionic systems with $n/2$ degrees of freedom.

In this article we treat the general case corresponding to an arbitrary integer n and show that the odd n

case possesses peculiar features not present in the n even case. In fact, for odd n, the cases $n=4k+1$ and $n=4k+3$ are shown to have some distinct group theoretical properties.

Our main emphasis was directed to the search and explicit construction of the IR's of $G(n)$, making full use of the theory of group representations due to the classical works of Frobenius, Schur and Burnside although several elementary properties of $G(n)$ were investigated as well.

To our mind, the treatment of the groups $G(n)$, besides some interest of his own, has a remarkable pedagogical value since it provides a non-trivial example which can be worked out completely and allows to illustrate many of the beautiful results of the theory of finite groups and their IR's[3,4] in a systematic way.

This work is organized as follows. In Section 2, the main group theoretical properties of $G(n)$ such as its order, subgroups of interest, conjugate classes and their number, the center of $G(n)$ and its related properties, are discussed. For later use, a closed expression, valid for arbitrary n, is also given for the number $\mathcal{E}(I)$ of group elements whose square is equal to the unit element I. Section 3 deals with the IR's of $G(n)$, their properties and explicit realization in terms of Kronecker products of Pauli spin matrices. The primitive characters of $G(n)$ is also explicitly constructed. In Section 4 we discuss the reality classes of the IR's, as well as the properties of the Clebsch-Gordan series for the decomposition of direct products of two IR's of $G(n)$. Finally, Section 5 is devoted to some remarks and conclusions. In particular, we briefly discuss the relation between the IR's of $G(n)$ with the IR's of the Clifford algebras $\mathbb{C}_n$ (Ref.5).

2. Some group-theoretical properties of $G(n)$

It is easily seen that $G(n)$, as defined by (1.1), is a group under multiplication. It suffices to note that associativeness is implicitly postulated in (1.1), that the inverse of $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}$ is $\gamma_{i_p}\cdots\gamma_{i_2}\gamma_{i_1}$ and that I is the unit element.

We briefly mention some particular cases. $G(0) = \{\pm I\}$ is isomorphic to the cyclic group C_2, $G(1)=\{\pm I, \pm\gamma_1\}$ is isomorphic to the Klein's four group D_2. The groups $G(n)$ of rank higher than that of $G(1)$ are no longer Abelian. $G(2) = \{\pm I, \pm\gamma_1, \pm\gamma_2, \pm\gamma_1\gamma_2\}$ is the Pauli spin group. The case $n=4$ corresponds to the Dirac group $G(4)$.

By a simple counting we conclude that the order of $G(n)$ is given by

$$g(n) = 2\sum_{k=0}^{n}\binom{n}{k} = 2^{n+1}. \tag{2.1}$$

We recall the well-known identity

$$(\pm\gamma_1\gamma_2\cdots\gamma_n)^2 = (-)^{\frac{1}{2}n(n-1)} \tag{2.2}$$

which can be easily proved by induction or directly by using (1.2). Another way of presenting (2.2) is by writting

$$(\pm\gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_n^{p_n})^2 = (-)^{\frac{1}{2}J(J-1)}, \tag{2.3}$$

where J stands for the number of exponents in the LHS of (2.3) which are equal to 1. Relations (2.2) and (2.3), which are basic, will be rather useful in the course of this work.

2.1 Subgroups

Among the subgroups of $G(n)$, the normal ones are of particular interest. As shown by Lomont[1], the minimal normal subgroups of $G(n)$ are $\{\pm I\}$ for all n even. It is not difficult to see that this is the only minimal normal subgroup also for $n=4k+3$. For $n=4k+1$, however, there are three minimal subgroups which are normal, namely, $\{\pm I\}$, $\{I, \gamma_1\gamma_2\ldots\gamma_n\}$, and $\{I, -\gamma_1\gamma_2\ldots\gamma_n\}$.

As far as the maximal subgroups are concerned, we note that since the order of $G(n)$ is 2^{n+1}, the maximal subgroups of $G(n)$ should be of order at most 2^n. Among them we have those isomorphic to $G(n-1)$, namely, the $G^{(i)}(n-1)$ subgroups for $i=1,2,\ldots,n$, obtained from $G(n)$ by omitting the element γ_i. A brief discussion on the minimal and maximal subgroups is given in Appendix 1.

2.2 Conjugate classes

According to the usual definition of the conjugate class of an element a of a group G, as the set of all elements of G conjugate to it, i.e., $C(a)=\{bab^{-1},$ for all $b\in G\}$ we show, in Appendix 2, that the number of conjugate classes of $G(n)$ is given by

$$C_n \begin{cases} 2^n+1 & \text{, for } n \text{ even ,} \\ 2^n+2 & \text{, for } n \text{ odd .} \end{cases} \tag{2.4}$$

The number of <u>ambivalent</u> classes (i.e., those satisfying $C(a) = C(a^{-1})$) will be discussed later on.

The set of <u>central</u> elements, i.e., those elements whose class contains only the element itself is known to be a normal subgroup and is called the <u>center</u> of the group. It is easily seen that the center of $G(n)$ is

$$Z(n) = \begin{cases} \{\pm I\}, & \text{for } n \text{ even} \\ \{\pm I, \pm \gamma_1 \gamma_2 \cdots \gamma_n\}, & \text{for } n \text{ odd} \end{cases} \tag{2.5}$$

This normal subgroup can be used to make a coset decomposition of $G(n)$, each coset having as many elements as the order of $Z(n)$, i.e., $g(Z,n) = 3-(-)^n$. The number of distinct cosets is then $g(n)/g(Z,n)$, the index of $Z(n)$ in $G(n)$.

Let us consider now the quocient group $G(n)/Z(n)$ whose elements are the cosets aZ, with $a \in Z(n)$ and satisfying the multiplication law

$$(a\, Z(n))(a' Z(n)) = a a' Z(n) = a' a\, Z(n). \tag{2.6}$$

The following isomorphisms hold:

$$G(n)/Z(n) \sim D_2^{(1,2)} \otimes D_2^{(3,4)} \otimes \cdots \otimes \begin{cases} D_2^{(n-1,n)} , & \text{for even} \\ D_2^{(n-2,n-1)} , & \text{for odd} \end{cases} \tag{2.7}$$

In (2.7) the $D_2^{(i,j)}$ groups are defined by

$$D_2^{(i,j)} = \{ I, \lambda_i, \lambda_j, \lambda_{ij} \} \ , \ i \neq j \tag{2.8}$$

with the λ's satisfying the multiplication law

$$\lambda_i^2 = \lambda_j^2 = I \quad , \quad \lambda_i \lambda_j = \lambda_j \lambda_i = \lambda_{ij} \ . \tag{2.9}$$

Clearly these $D_2^{(i,j)}$ are isomorphic to the Klein's four group D_2 . The proof of (2.7) can be found in Appendix 3.

2.3 Square roots of I

Foolowing Lomont, we now calculate the number $\zeta(I)$ of elements of $G(n)$ whose square is equal to the unit element I . Using (2.3) one obtains

$$\mathcal{E}(I) = 2\sum_{k=0}^{[n/4]} \binom{n}{4k} + 2\sum_{k=0}^{[(n-1)/4]} \binom{n}{4k+1}, \qquad (2.10)$$

where the symbol $[x]$ denotes, as usual, the largest integer less than or equal to x .

It is possible to show that (2.10) can be written in the form

$$\mathcal{E}(I) = 2^n + A_{n-1}, \qquad (2.11)$$

with

$$A_n = (1+i)^n + (1-i)^n. \qquad (2.12)$$

The A_n obey the 3-term recurrence relation

$$A_{n+1} = 2(A_n - A_{n-1}). \qquad (2.13)$$

This relation together with the initial values $A_0 = A_1 = 2$ allows one to obtain A_n for all n. The results obtained can be put in the form

$$A_n = (-)^{\left[\frac{n+3}{4}\right]} 2^{\left[\frac{n}{2}\right]+1} \epsilon(n) \qquad (2.14)$$

with

$$\epsilon(n) = \frac{1+(-)^{\left[\frac{n^2}{4}\right]}}{2}. \qquad (2.15)$$

The factor $\epsilon(n)$ in (2.13) accounts for the fact that A_n vanishes when $n = 2 \pmod 4$)

Finally, using (2.11) and (2.14) one gets

$$\mathcal{E}(I) = 2^n + \epsilon(n-1)(-)^{\left[\frac{n+1}{4}\right]} 2^{\left[\frac{n+1}{2}\right]} . \tag{2.16}$$

Notice that

$$\epsilon(n-1) = \begin{cases} 1 & \text{for } n \neq 4k+3 \\ 0 & \text{for } n = 4k+3 \end{cases} , \quad k = 0,1,2,\dots \tag{2.17}$$

Therefore $\epsilon(n-1) = 1$ for n even, in which case one gets

$$\mathcal{E}(I) = 2^n + (-)^{\left[\frac{n+1}{4}\right]} 2^{\left[\frac{n+1}{2}\right]} , \quad \text{for } n \text{ even} \tag{2.18}$$

a result entirely equivalent to that in Ref.(1), for even n.

3. Irreducible Representations

This section is concerned with the irreducible representations (IR) of the groups $G(n)$, their properties and realizations. The characters of those IR (the so called primitive characters) are also discussed.

3.1 Dimensions of the IR's

As it was seen in last section, the number of conjugate classes of $G(n)$ is different according to n is even or odd:

$$C_n = \begin{cases} 2^n + 1 & , \text{ for } n \text{ even} \\ 2^n + 2 & , \text{ for } n \text{ odd}. \end{cases} \tag{3.1}$$

This number is important since, as it is well known, for a finite group, it gives the number of inequivalent IR's of the group. Besides, calling d_i the dimension of the i^{th} IR, the dimensionality theorem gives

$$\sum_{i=1}^{C_n} d_i^2 = g(n). \tag{3.2}$$

Using this theorem one obtains the main results of this section, concerning the dimensionality of the IR's of $G(n)$.

Among the IR's of $G(n)$ there are those of dimension one, which are obtained by assigning a single numerical value to each element of the group. Since I is the unit element the value assigned to it must be +1. From $\gamma_i\gamma_j = (-I)\gamma_j\gamma_i$ for $i \neq j$, it follows that one must assign the value +1 to $-I$. The relation $\gamma_i\gamma_i = I$ implies that one may assign to the γ's only the values +1 and -1. There are, of course, 2^n distinct ways of making these numerical assignments. One of them is to ascribe the value +1 to every element of $G(n)$ what leads to the identical representation. The other ones are obtained by distributing the -1 values among the γ's in the following ways:

1) only 1 γ with value -1, giving $\binom{n}{1}$ IR's,
2) only 2 γ's with value -1, giving $\binom{n}{2}$ IR's,

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n) n γ's with value -1, giving $\binom{n}{n}=1$ IR's.

The total number u_n of one-dimensional IR's is then

$$u_n = 1 + \sum_{m=1}^{n} \binom{n}{m} = 2^n \, , \tag{3.3}$$

as antecipated above. All the one-dimensional IR's are, of course, unfaithful since by the above construction they are not one-to-one mappings.

By means of (2.1) and (3.3), the dimensionality theorem (3.2) tells us that besides the one-dimensional representations there is for n even one IR, which we will denote by (f), of dimension

$$d_f = 2^{n/2} , \quad n \text{ even} , \tag{3.4}$$

while for n odd there are two IR's, denoted by (f) and (f') both of dimension

$$d_f = d_{f'} = 2^{\frac{n-1}{2}} , \quad n \text{ odd} . \tag{3.5}$$

3.2 Primitive Characters

We shall assume that the IR's are unitary. This assumption is always possible for finite groups in view of the Schur-Auerbach theorem: any representation of a finite group is equivalent to a unitary representation.

A great deal of information can be deduced from the orthogonality relations for the primitive characters:

$$\sum_{i=1}^{C_n} \chi^{(p)^*}(i)\,\chi^{(q)}(i) = g(n)\,\delta_{p,q}\ , \tag{3.6}$$

$$\sum_{p=1}^{C_n} \chi^{(p)^*}(i)\,\chi^{(p)}(j) = \frac{g(n)}{g_i}\,\delta_{i,j}\ . \tag{3.7}$$

In the above relations, $\chi^{(p)}(i)$ denotes the primitive character of class $i^{\underline{th}}$ in the IR labelled by (p), while g_i stands for the number of elements of class $i^{\underline{th}}$.

Case n even

Let us apply (3.7) for $i=I$ and $j=-I$. Having in mind the results of the previous section and that $\chi^{(p)}(\pm I)=1$ for any one-dimensional IR, one obtains by using (3.2) that

$$\chi^{(f)}(-I) = -d_f = -2^{n/2} = -\chi^{(f)}(I), \tag{3.8}$$

what implies that

$$D^{(f)}(-I) = -\mathbb{1}\ , \tag{3.9}$$

where $D^{(f)}(-I)$ is the matrix representative of $-I$ in the IR (f) while $\mathbb{1}$ is the $d_f \times d_f$ unit matrix.

All classes, except those relative to the elements I and $-I$, have two elements,

$$C(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}) = \{\pm\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}\}\ , \text{ for } \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p} \neq \pm I. \tag{3.10}$$

Since both $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}$ and $-\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}$ belong to the same class, they have the same character. Then using (3.9) one concludes that

$$\chi^{(f)}(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})=0\,,\quad \text{for}\quad \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}\neq\pm I. \tag{3.11}$$

Case n odd

By the same arguments used in the even case one obtains

$$\chi^{(f)}(-I)=\chi^{(f')}(-I)=-d_f=-2^{\frac{n-1}{2}}=-\chi^{(f)}(I)=-\chi^{(f')}(I), \tag{3.12}$$

what implies (See Theorem 1 in pag.48 of Ref.1)

$$D^{(f)}(-I)=D^{(f')}(-I)=-\mathbb{1}. \tag{3.13}$$

Now (3.10) is true only for $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}\neq\pm I,\pm\gamma_1\gamma_2\cdots\gamma_n$ and therefore, in the same way, one obtains

$$\chi^{(f)}(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})=\chi^{(f')}(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})=0,\quad \text{for}\ \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}\neq\pm I,\pm\gamma_1\gamma_2\cdots\gamma_n. \tag{3.14}$$

Since now $\gamma_1\gamma_2\cdots\gamma_n$ as well as $-\gamma_1\gamma_2\cdots\gamma_n$, constitutes a class by itself, the orthogonality relation (3.6) for $p=q=(f)$ and $p=q=(f')$, and the previous results lead to

$$|\chi^{(q)}(\gamma_1\gamma_2\ldots\gamma_n)|=|\chi^{(q)}(-\gamma_1\gamma_2\cdots\gamma_n)|=2^{\frac{n-1}{2}},\ \text{for } q=(f) \text{ and } (f'). \quad (3.15)$$

Using again (3.6), now with p = identical representation, $q=(f)$ and (f'), one obtains

$$\chi^{(q)}(-\gamma_1\gamma_2\cdots\gamma_n)=-\chi^{(q)}(\gamma_1\gamma_2\ldots\gamma_n),\ \text{for } q=(f) \text{ and } (f'). \quad (3.16)$$

From the general theory of group characters one knows that

$$\chi(a^{-1})=\chi^{*}(a),\ \text{for all } a\in G. \quad (3.17)$$

For $n=4k+1$, Eq.(2.2) implies that $(\gamma_1\gamma_2\ldots\gamma_n)^{-1}=\gamma_1\gamma_2\ldots\gamma_n$. This result, together with (3.15), (3.16) and (3.17) allows us to make the assignments

$$\begin{aligned}\chi^{(f)}(\gamma_1\gamma_2\ldots\gamma_n)&=-\chi^{(f)}(-\gamma_1\gamma_2\cdots\gamma_n)=2^{\frac{n-1}{2}},\\ \chi^{(f')}(\gamma_1\gamma_2\ldots\gamma_n)&=-\chi^{(f')}(-\gamma_1\gamma_2\ldots\gamma_n)=-2^{\frac{n-1}{2}},\end{aligned}\quad \text{for } n=4k+1, \quad (3.18)$$

For $n=4k+3$, one has $(\gamma_1\gamma_2\dots\gamma_n)^{-1}=-\gamma_1\gamma_2\cdots\gamma_n$ and similarly one has

$$\begin{aligned}\chi^{(f)}(\gamma_1\gamma_2\dots\gamma_n) &= -\chi^{(f)}(-\gamma_1\gamma_2\dots\gamma_n)=i\,2^{\frac{n-1}{2}},\\ \chi^{(f')}(\gamma_1\gamma_2\dots\gamma_n) &= -\chi^{(f')}(-\gamma_1\gamma_2\dots\gamma_n)=-i\,2^{\frac{n-1}{2}}, \quad \text{for } n=4k+3.\end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that the corresponding matrix representatives are multiples of the unit matrix, i.e.,

$n=4k+1$:

$$\begin{aligned}D^{(f)}(\gamma_1\gamma_2\dots\gamma_n) &= -D^{(f)}(-\gamma_1\gamma_2\dots\gamma_n)=\mathbb{1}=D^{(f)}(I),\\ D^{(f')}(\gamma_1\gamma_2\dots\gamma_n) &= -D^{(f')}(-\gamma_1\gamma_2\dots\gamma_n)=-\mathbb{1}=D^{(f')}(-I),\end{aligned} \tag{3.20}$$

$n=4k+3$:

$$\begin{aligned}D^{(f)}(\gamma_1\gamma_2\dots\gamma_n) &= -D^{(f)}(-\gamma_1\gamma_2\dots\gamma_n)=i\,\mathbb{1},\\ D^{(f')}(\gamma_1\gamma_2\dots\gamma_n) &= -D^{(f')}(-\gamma_1\gamma_2\dots\gamma_n)=-i\,\mathbb{1}.\end{aligned} \tag{3.21}$$

Therefore (f) and (f') are unfaithful for $n=4k+1$ but faithful for $n=4k+3$. Notice that for $n=4k+1$, there are three minimal normal subgroups of order 2,

$$\{I,-I\},\ \{I,\ \gamma_1\gamma_2\dots\gamma_n\},\ \{I,-\gamma_1\gamma_2\dots\gamma_n\} \tag{3.22}$$

and therefore Burnside's theorem[1] does not apply, that is,in that case the theorem does not garantee the existence of a faithful representation, a result in agreement with our above result.

3.3 Realization of the IR's

In this subsection realizations of the IR's of $G(n)$ are discussed in terms of direct products of Pauli spin matrices. For n even, these realizations can be found in Ref.1. Since the result for n even is also relevant for the realization in the n odd case, we first briefly reproduce it here.

n even

The matrices q_j and p_j giving by

$$q_j = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 ,$$
$$p_j = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 , \quad j = 1, 2, \ldots, n/2, \tag{3.23}$$

containing $(j-1)$ factors of σ_3 and $(n/2-j)$ factors of the 2X2 unit matrix $\mathbb{1}_2$ obey the relations

$$q_i^2 = \mathbb{1}_2 \ , \quad p_i^2 = \mathbb{1}_2 \quad , i = 1, 2, \ldots, n/2$$

$$q_i q_j = -q_j q_i \ , \quad p_i p_j = -p_j p_i \ , \quad p_i q_j = -q_j p_i \quad , i \neq j = 1, 2, \ldots, n/2 . \tag{3.24}$$

Therefore, they provide a realization of the generators γ_i of $G(n$ even) by matrices of dimension $2^{n/2}$.

Relations (3.24) are derived making use of the well-known formula

$$(a_1 \otimes a_2 \otimes \ldots)(b_1 \otimes b_2 \otimes \ldots) = (a_1 b_1) \otimes (a_2 b_2) \otimes \ldots \qquad (3.25)$$

n odd

For $n = 4k+1$, one obtains, by using (3.20),

$$\begin{aligned} D^{(f)}(\gamma_{4k+1}) &= D^{(f)}(\gamma_1) D^{(f)}(\gamma_2) \cdots D^{(f)}(\gamma_{4k}) = D^{(f)}(\gamma_1 \gamma_2 \ldots \gamma_{4k}), \\ D^{(f')}(\gamma_{4k+1}) &= -D^{(f')}(\gamma_1) D^{(f')}(\gamma_2) \ldots D^{(f')}(\gamma_{4k}) = D^{(f')}(-\gamma_1 \gamma_2 \ldots \gamma_{4k}). \end{aligned} \qquad (3.26)$$

Therefore one obtains realizations for (f) and (f') by taking

$$\begin{aligned} & D^{(f)}(I) = D^{(f')}(I) = \mathbb{1}, \\ & D^{(f)}(-I) = D^{(f')}(-I) = -\mathbb{1}, \\ & D^{(f)}(\gamma_i) = D^{(f')}(\gamma_i) = \text{ matrices (3.23) of } G(4k) \\ & \qquad \text{for } i = 1, 2, \ldots, 4k, \end{aligned} \qquad (3.27)$$

and (3.26) for the matrix representative of γ_{4k+1}.

The last equal sign in Eqs.(3.26) tell us that the representations are unfaithful as they should be.

For $n=4k+3$, Eq.(3.21) give us

$$\begin{aligned} D^{(f)}(\gamma_{4k+3}) &= i D^{(f)}(\gamma_1) D^{(f)}(\gamma_2) \dots D^{(f)}(\gamma_{4k+2}), \\ D^{(f')}(\gamma_{4k+3}) &= -i D^{(f')}(\gamma_1) D^{(f')}(\gamma_2) \dots D^{(f')}(\gamma_{4k+2}), \end{aligned} \tag{3.28}$$

and realizations for (f) and (f') are obtained through

$$\begin{aligned} &D^{(f)}(I) = D^{(f')}(I) = \mathbb{1}, \\ &D^{(f)}(-I) = D^{(f')}(-I) = -\mathbb{1}, \\ &D^{(f)}(\gamma_i) = D^{(f')}(\gamma_i) = \text{ matrices of } \quad G(4k+2) \\ &\qquad\qquad \text{for } \quad i = 1, 2, \dots, 4k+2, \end{aligned} \tag{3.29}$$

together with (3.28) for the matrix representative of γ_{4k+3}. In this case the realizations are clearly faithful.

4. Other Properties of $G(n)$

In this section the reality classes of the IR's of $G(n)$ are discussed, together with the number of ambivalent classes. We also present results on the Clebsch-Gordan series for the reduction of the direct product of two IR's.

4.1 The number of ambivalent classes

We have already defined ambivalent classes in section 2. Their number r_a is given by the theorem[1]

$$r_a = \frac{1}{g(n)} \sum_{a \in G} \zeta^2(a) , \tag{4.1}$$

where $\zeta(a)$ denotes the number of elements in the group G whose square is equal to the element I.

The sum is easy to calculate in the case of the $G(n)$ groups because, due to (2.3) $\zeta(a)=0$ for $a \neq \pm I$. Therefore one has

$$r_a = \frac{1}{g(n)} \left[\zeta^2(I) + \zeta^2(-I) \right] \tag{4.2}$$

and

$$\zeta(I) + \zeta(-I) = g(n). \tag{4.3}$$

These results, together with (2.1) and (2.16) implies

$$r_a = 2^n + \epsilon(n-1)\,2^{2[\frac{n+1}{2}]-n}, \qquad (4.4)$$

and by using (2.15) and (2.4)

$$r_a = \begin{cases} C_n, & \text{for } n \neq 4k+3, \\ C_n - 2, & \text{for } n = 4k+3. \end{cases} \qquad (4.5)$$

In words, except for the case $n=4k+3$, all the classes are ambivalent. This is in agreement with our previous results according to which only the case $n=4k+3$ has non-ambivalent classes, viz., $\{\gamma_1\gamma_2\ldots\gamma_n\}$ and $\{-\gamma_1\gamma_2\cdots\gamma_n\}$.

4.2 Reality class

As it is well known, from the representation theory of finite groups, the IR's of a group given by matrices D are classified as follows: i) they are of first kind if D and D^* are equivalent and equivalent to a real representation; ii) they are of second kind if D and D^* are equivalent but <u>not</u> equivalent to a real representation; iii) they are of third kind if D and D^* are not equivalent IR's of the group.

The three possibilities are referred to as the relity class $R^{(p)}$ of IR and one has[1]

$$R^{(p)} = \frac{1}{g(n)} \sum_{a \in G} \chi^{(p)}(a^2) = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \tag{4.6}$$

according to whether the IR (p) is of $1^{\underline{st}}$, $2^{\underline{nd}}$ or $3^{\underline{rd}}$ class, respectively

Applying this result to $G(n)$ one observes that due to (2.3), $a^2 = \pm I$ and therefore

$$R^{(p)} = \frac{1}{g(n)} \left[\zeta_0(I)\chi(I) + \zeta_0(-I)\chi(-I) \right] \quad . \tag{4.7}$$

For the one dimensional IR's, $\chi(I) = \chi(-I)$ and (4.7) gives $R^{(p)} = 1$ as it should be.

For the IR's of dimension greater than 1, it was seen in sub-section 3.2 that

$$\chi(I) = -\chi(-I).$$

This result, together with (4.7) and (4.3) gives

$$R^{(p)} = \chi(I) \left[2\frac{\zeta_0(I)}{g(n)} - 1 \right], \tag{4.8}$$

for all IR's of dimension greater than 1.

For n even, $(p)=(f)$ and then (4.8) reduces to

$$R^{(f)} = (-)^{[\frac{n+1}{4}]} , \text{ for } n \text{ even,} \tag{4.9}$$

while for n odd one has

$$R^{(f)} = R^{(f')} = \begin{cases} (-)^{[\frac{n+1}{4}]} & , \text{ for } n = 4k+1 , \\ 0 & , \text{ for } n = 4k+3 , \end{cases} \tag{4.10}$$

in full agreement with the realizations given in sub-section (3.3).

4.2 Clebsch-Gordan series

It is a well-known result that, for a finite group, the coefficients $\alpha_k^{(i,j)}$ which give the multiplicity of the k^{th} IR in the reduction of the Kronecker product of the IR's i and j,

$$D^{(i)}(a) \otimes D^{(j)}(a) = \sum_{k=1}^{c_n} \oplus \, \alpha_k^{(i,j)} D^{(k)}(a) \tag{4.11}$$

are given by

$$\alpha_k^{(i,j)} = \frac{1}{g(n)} \sum_{a \in G} \chi^{(k)*}(a) \chi^{(i)}(a) \chi^{(j)}(a) . \tag{4.12}$$

n even

In the n even case, we have seen in Section 3 that one has one IR of dimension $d_f = 2^{n/2}$ denoted by (f) and 2^n one-dimensional IR's which will be denoted by (ℓ).

Using the results on the group characters given in sub-section 3.2, one obtains from (4.12)

$$\alpha_k^{(f,f)} = \frac{1}{2}\left[\chi^{(k)}(I) + \chi^{(k)}(-I)\right] = \begin{cases} 0, & \text{for } k = (f), \\ 1, & \text{for } k = (\ell). \end{cases} \tag{4.13}$$

This means that the Kronecker product of (f) by itself is reducible as the direct sum of all one-dimensional IR's.

Analogously,

$$\alpha_k^{(f,\ell)} = 2^{n/2-1}\left[\chi^{(k)}(I) - \chi^{(k)}(-I)\right] = \begin{cases} 0, & \text{for } k = (\ell'), \\ 1, & \text{for } k = (f), \end{cases} \tag{4.14}$$

as it should be since (ℓ) is one-dimensional.

For the one dimensional IR's one has

$$D^{(\ell_1)}(a) \otimes D^{(\ell_2)}(a) = \chi^{(\ell_1)}(a)\,\chi^{(\ell_2)}(a), \tag{4.15}$$

which may be only 1 or -1, being necessarily +1 for $a = \pm I$.

Since the 2^n one-dimensional IR's of $G(n)$ were obtained by assigning to $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ all n-ples $(k_1, k_2, \ldots, k_n)$ with $k_i = 1$ or -1, it is obvious that, given any two one-dimensional IR's (ℓ) and (ℓ'), there will always exist one one-dimensional IR (ℓ'') such that

$$\chi^{(\ell)}(a)\,\chi^{(\ell')}(a) = \chi^{(\ell'')}(a), \quad \forall a \in G(n). \tag{4.16}$$

n odd

Analogously to the even case one has

$$\alpha_k^{(d,d')} = \frac{1}{2^{n+1}}\left\{ 2^{n-1}\left[\chi^{(k)}(I)+\chi^{(k)}(-I)\right] + \chi^{(d)}(a)\,\chi^{(d')}(a)\left[\chi^{(k)}(a)+\chi^{(k)}(-a)\right]\right\}, \tag{4.17}$$

with $d, d' \neq (\ell)$ and $a = \gamma_1\gamma_2\cdots\gamma_n$. (4.17)

When $k = (f)$ or (f') the RHS of (4.17) vanishes, that is, as in the n even case, the Kronecker product of two (not one-dimensional) IR's reduces to a direct some of one-dimensional ones. Using the results of sub-section 3.2, one obtains, for $n = 4k+1$,

$$\alpha_\ell^{(f,f)} = \alpha_\ell^{(f',f')} = \begin{cases} 1, & \text{for } (\ell)=(\ell_+) \\ 0, & \text{for } (\ell)=(\ell_-) \end{cases} \; ; \quad \alpha_\ell^{(f,f')} = \begin{cases} 0, & \text{for } (\ell)=(\ell_+) \\ 1, & \text{for } (\ell)=(\ell_-) \end{cases}, \tag{4.18}$$

while, for $n = 4k+3$

$$\alpha_{\ell}^{(f,f)} = \alpha_{\ell}^{(f',f')} = \begin{cases} 0, \text{ for } (\ell)=(\ell_+) \\ 1, \text{ for } (\ell)=(\ell_-) \end{cases}, \quad \alpha_{\ell}^{(f,f')} = \begin{cases} 1, \text{ for } (\ell)=(\ell_+) \\ 0, \text{ for } (\ell)=(\ell_-) \end{cases} \tag{4.19}$$

where $(\ell_+)[(\ell_-)]$ means a one-dimensional IR with an even [odd] number of γ's with character $+1$.

As far as (i) and/or (j) in (4.11) is equal to (ℓ), one has the same results obtained in the n even case.

5. Comments and Conclusions

As shown in the previous sections, the structure of the Dirac groups of rank n is rather simple and can be worked out systematically, for arbitrary n, using the well-known results of the theory of finite groups. The n odd case, as compared with the even case, presents new features. In particular, the cases $n=4k+1$ and $n=4k+3$ were shown to exhibit distinct group theoretical properties.

The difining relation (1.2) corresponds, so to say, to the "Euclidianized" version of the Dirac group. We could as well have considered

$$\{\gamma_i, \gamma_j\} = 2 g_{ij} I \tag{5.1}$$

with diagonal (g_{ij}) but with a given signature implying that some γ's would have their squares equal to $-I$ instead of I. The groups so obtained, although having structures similar to those of $G(n)$, are <u>not</u> isomorphic to them, a fact that can be immediatly seen by noting that $\mathcal{L}(I)$ will be chan ged. To deal with those groups, it is enough to multiply by i the γ's whose squares are equal to $-I$.

The realizations of IR's of $G(n)$ presented in sub-section 3.3 are also realizations of the Clifford algebra generated by the γ's but in that case, as pointed out by

Boerner[5], for n odd, all the IR's are unfaithful.

Some of the results presented in this paper were derived more than once. We did it on purpose to check their consistency.

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Appendix 1. Minimal and Maximal subgroups of $G(n)$

The minimal proper subgroups are of the form

$$\{I, a\}$$

where $a \neq I$ and $a^2 = I$. Their number is obviously $\mathcal{L}(I)-1$. The condition of being normal implies that $[a,b]=0$ for all $b \in G(n)$. The trivial solution is $a=-I$, that is, $\{I,-I\}$. The other possibility for a is then to be a product of p distinct γ's, i.e., $a=\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}$. Since

$$\gamma_j(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})=\begin{cases}(-)^{p-1}(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})\gamma_j, & \text{for } \gamma_j\in\{\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_p}\},\\ (-)^{p}(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})\gamma_j, & \text{for } \gamma_j\notin\{\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_p}\},\end{cases} \tag{A1.1}$$

one sees that for $p<n$ it is impossible to have $[b,\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}]=0$ for all $b\in G(n)$, while for $p=n$ only for $n=4k+1$ one has $[b,\gamma_1\gamma_2\ldots\gamma_n]=0$, for all $b\in G(n)$.

One then concludes that for $n=4k,4k+2,4k+3$ the only minimal normal subgroup is $[I,-I]$, while for $n=4k+1$ the minimal normal subgroups are $\{I,-I\},\{I,\gamma_1\gamma_2\cdots\gamma_n\},\{I,-\gamma_1\gamma_2\cdots\gamma_n\}$.

Since the order of $G(n)$ is 2^{n+1} , Lagrange's theorem tells us that the maximum order a proper subgroup of can have is 2^n . One way of obtaining subgroups of order 2^n

is by considering the subgroups $G^{(i)}(n-1)$, for $i=1,2,\ldots,n$, obtained from $G(n)$ by omitting the element γ_i . Of course, those subgroups are isomorphic to $G(n-1)$.

Consider the elements of $G(n)$, $a=\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}$ and $b=\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_q}$ which are the product of p and q distinct γ's. The product ab will be then

$$ab=\pm\gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_r}\,, \tag{A1.2}$$

formed by r distinct factors of γ's with $r=p+q-2m$ where m is the number of γ's common to a and b . Hence the product of two elements formed each one by a product of an even number of γ's is also an element of this type. Therefore $H(n-1)=\{\pm I,\ a=\gamma_{i_1}\cdots\gamma_{i_p}$ with $i_1\neq i_2\neq\cdots\neq i_p$ and p even$\}$ constitutes a subgroup of $G(n)$. The order of this subgroup is

$$g(H(n-1))=2\sum_{i=0}^{[n/2]}\binom{n}{2i}=2^n=g(n-1). \tag{A1.3}$$

Clearly $H(n-1)$ is not isomorphic to any $G^{(i)}(n-1)$ since

$$\zeta(I)=\begin{cases}2(S_0^{(n-1)}+S_1^{(n-1)}), & \text{for } G^{(i)}(n-1)\\ 2\,S_0^{(n)}, & \text{for } H(n-1)\end{cases} \qquad \text{A1.4)}$$

and $S_0^{(n)}\neq(S_0^{(n-1)}+S_1^{(n-1)})$. In these formulas

$$S_i^{(p)} = \sum_{k=0}^{\left[\frac{p-i}{4}\right]} \binom{p}{4k+i} . \qquad \text{(A1.5)}$$

Since $G^{(i)}_{(n-1)}$ and $H(n-1)$ contains $-I$, it follows from (A1.1) that they are normal.

By omitting one γ_i one obtains a subgroup $H^{(i)}_{(n-1)}$ of $H(n-1)$.

One has then two kinds of chains of maximal normal subgroups

$$G(n) \supset G(n-1) \supset \ldots \supset G(0) \supset I , \qquad \text{(A1.6)}$$

$$G(n) \supset G(n-1) \supset \ldots \supset G(k) \supset H(k-1) \supset H(k-2) \supset \ldots H(2) \supset G(1) \supset G(0) \supset I,$$

$$n-1 \leq k \leq 2,$$

Since the index of both $G(n-1)$ and $H(n-1)$ in $G(n)$ is 2 , it follows that in both chains (A1.6) one has the same sequence of factor groups, viz. $C_2, C_2, \cdots, C_2$, in agreement with the Jordan-Hölder theorem.

Appendix 2. Conjugate classes

Let a be the product of p distinct γ's. Then, by (A1.1) one has

$$\gamma_j a \gamma_j^{-1} \equiv \gamma_j(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p})\gamma_j = \begin{cases} (-)^{p-1} a, & \text{for } j\in\{\gamma_{i_1},\gamma_{i_2},\dots,\gamma_{i_p}\}, \\ (-)^{p} a, & \text{for } j\notin\{\gamma_{i_1},\gamma_{i_2},\dots,\gamma_{i_p}\}. \end{cases} \quad (A2.1)$$

Therefore, for $p<n$ one has

$$C(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}) = \{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}, -\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_p}\}, \quad p=1,2,\dots,n-1, \quad (A2.2)$$

while, for $p=n$,

$$C(\gamma_1\gamma_2\cdots\gamma_n) = \begin{cases} \{\gamma_1\gamma_2\cdots\gamma_n\}, & \text{for } n \text{ odd}, \\ \{\gamma_1\gamma_2\cdots\gamma_n, -\gamma_1\gamma_2\cdots\gamma_n\}, & \text{for } n \text{ even}. \end{cases} \quad (A2.3)$$

Since I and $-I$ constitute a class by themselves, the total number of classes in then

$$C_n = 2 + \sum_{i=1}^{n-1}\binom{n}{p} + \begin{cases} 2 = 2^n+2, & \text{for } n \text{ odd} \\ 1 = 2^n+1, & \text{for } n \text{ even} \end{cases} \quad (A2.4)$$

Besides $\{I\}$ and $\{I\}$ all the classes (A2.2) are ambivalent. For n even the class (A2.3) is also ambivalent, so all classes are ambivalent in that case. Using (2.3) one sees that the same is true for $n=4k+1$ and false for $n=4k+3$, in the latter case the class (A2.3) being not ambivalent.

Appendix 3. Factor groups $G(n)/Z(n)$

The elements of $G(n)/Z(n)$ are the cosets aZ for $a \in G(n)$ with the composition law

$$(aZ)(bZ) = abZ \quad , \ a,b \in G(n). \qquad \text{(A3.1)}$$

Since $ab = baI$ or $ba(-I)$ and both I and $-I$ belong to $Z(n)$ one has that the composition law of $G(n)/Z(n)$ is commutative, i.e.,

$$(aZ)(bZ) = abZ = baZ \ , \ a,b \in G(n). \qquad \text{(A3.2)}$$

Using (1.1) and (1.2), the elements of $G(n)/Z(n)$ can be written as

$$A = \gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_n^{p_n}Z \ , \ B = \gamma_1^{q_1}\gamma_2^{q_2}\cdots\gamma_n^{q_n}Z \ , \ etc., \qquad \text{(A3.3)}$$

and the composition law (3.2) reads then

$$AB = BA = \gamma_1^{\overline{p_1+q_1}}\gamma_2^{\overline{p_2+q_2}}\cdots\gamma_n^{\overline{p_n+q_n}}Z, \qquad \text{(A3.4)}$$

where $\overline{p_i+q_i} = (p_i+q_i)(\mathrm{mod}\ 2)$.

Consider now the sets of 4 elements

$$\mathcal{D}_2^{(i,j)} \equiv \{I, \lambda_i, \lambda_j, \lambda_{ij}\}, \quad i \neq j = 1, 2, \ldots, n, \tag{A3.5}$$

with the multiplication law

$$\lambda_i \lambda_i = \lambda_j \lambda_j = I \, ; \; \lambda_i \lambda_j = \lambda_j \lambda_i = \lambda_{ij}, \quad i \neq j \, . \tag{A3.6}$$

They constitute groups isomorphic to the Klein's four group $\mathcal{D}_2$. For n even one can form a group by making direct product of those groups,

$$\mathcal{D} = \mathcal{D}_2^{(1,2)} \otimes \mathcal{D}_2^{(3,4)} \otimes \ldots \otimes \mathcal{D}_2^{(n-1,n)} \, . \tag{A3.7}$$

An element of (3.7) will have the form

$$(\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_n^{p_n}) \tag{A3.8}$$

and the multiplication law will be

$$(\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_n^{p_n})(\lambda_1^{q_1}, \lambda_2^{q_2}, \ldots, \lambda_n^{q_n}) = (\lambda_1^{\overline{p_1+q_1}}, \lambda_2^{\overline{p_2+q_2}}, \ldots, \lambda_n^{\overline{p_n+q_n}}). \tag{A3.9}$$

Therefore the one-to-one mapping

$$(\gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_n^{p_n}) \in G(n)/Z(n) \leftrightarrow (\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_n^{p_n}) \in D \qquad \text{(A3.10)}$$

stablishes the isomorphism

$$G(n)/Z(n) \sim D_2^{(1,2)} \otimes D_2^{(3,4)} \otimes \cdots \otimes D_2^{(n-1,n)} \quad \text{, for } n \text{ even} \qquad \text{(A3.11)}$$

For n odd one has $Z = \{\pm I, \pm\gamma_1\gamma_2\cdots\gamma_n\}$ and hence $\gamma_1\gamma_2\cdots\gamma_n Z = Z$. This implies

$$\gamma_2\cdots\gamma_n Z = \gamma_1 Z \Rightarrow \cdots \Rightarrow \gamma_n Z = \gamma_1\gamma_2\cdots\gamma_{n-1} Z \, . \qquad \text{(A3.12)}$$

Due to that one can write the elements of $G(n)/Z(n)$ for n odd as

$$\gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_{n-1}^{p_{n-1}} Z \qquad \text{(A3.13)}$$

and the composition law will be just (A3.9) with γ_n omitted.

Therefore, for n odd, the one-to-one mapping

$$\gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_{n-1}^{p_{n-1}} Z \in G(n)/Z(n) \leftrightarrow (\lambda_1^{p_1},\lambda_2^{p_2},\ldots,\lambda_{n-1}^{p_{n-1}}) \in D_2^{(1,2)} \otimes D_2^{(3,4)} \otimes \cdots \otimes D_2^{(n-2,n-1)} \tag{A3.14}$$

establishes the isomorphism

$$G(n)/Z(n) \sim D_2^{(1,2)} \otimes D_2^{(3,4)} \otimes \cdots \otimes D_2^{(n-2,n-1)} \text{, for } n \text{ odd} \tag{A3.15}$$

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