PACS CLASSIFICATION NUMBER

03.20 + i,

46.10 + z

52.20.Dq

95.10.Ce

April 1981

Renormalized Lie Perturbation Theory

E. Rosengaus and R.L. Dewar

Plasma Physics Laboratory, Princeton University,

Princeton, New Jersey 08544.

A Lie operator method for constructing action-angle transformations continuously connected to the identity is developed for area preserving mappings. By a simple change of variable from action to angular frequency a perturbation expansion is obtained in which the small denominators have been renormalized. The method is shown to lead to the same series as the Lagrangian perturbation method of Greene and Percival, which converges on KAM surfaces. The method is not superconvergent, but yields simple recursion relations which allow automatic algebraic manipulation techniques to be used to develop the series to high order. It is argued that the operator method can be justified by analytically continuing from the complex angular frequency plane onto the real line. The resulting picture is one where preserved primary KAM surfaces are continuously connected to one another.

I. INTRODUCTION

The construction of action-angle transformations [1] for Hamiltonian systems is an important goal since it represents an essentially complete solution of the dynamics. An integrable system is one where these transformations are defined almost everywhere in phase space, with the possible exception of the separatrices between regions where the topologies of the orbits are different (a set of zero measure).

The Kolmogorov-Arnold-Moser theorem [2] guarantees that for non-integrable systems sufficiently close to integrability a set of invariant surfaces of finite measure exists (the KAM surfaces), where a form of action-angle variables is well defined. [3,4] As we increase the perturbation parameter which takes us away from integrability, the isolating integrals (actions) cease to exist locally. Accordingly, KAM surfaces are broken when action is no longer well defined. [5] When, in a given region of phase space, the last KAM surface with the same topology as the unperturbed surfaces is broken, connected stochasticity prevails throughout that region. This offers a more precise criterion for the onset of stochasticity than the "overlap-of-resonances" criterion,[6] provided we have a method for calculating the action-angle transformation and a rule for selecting the most robust surfaces.

In this paper we compare two perturbative approaches to the problem of constructing action-angle variables on the primary KAM surfaces. We limit ourselves to iterated 2-dimensional area-preserving mappings, in particular to the Standard mapping of Chirikov and Taylor and a variant, the Semistandard mapping of Greene and Percival.[5] These systems can be regarded as simplified models of Hamiltonian systems, sampled at integer multiples of a unit time. The perturbative approach can be readily extended to truly Hamiltonian systems and systems of dimensionality higher than two.

In Sec. II we set up the general problem of computing the action-angle transformation for primary KAM curves. The first method we examine is based on the assumption that there exists a global canonical transformation continuously connected to the identity such that the primary KAM surfaces (KAM curves in 2D) are reduced to straight lines. The methods are used to construct a formal power series in the perturbation parameter, ϵ,[7,8,9,10], which is justified by an argument based on analytic

The mapping $\underset{\sim}{T} = \underset{\sim}{T}_0 + \epsilon \underset{\sim}{T}_1$ is said to be *integrable* if phase space is covered with curves invariant under the application of $\underset{\sim}{T}$. In this case the invariant curves can be labelled by a new action J, constant on each curve, and parametrized along their length by a new angle Θ, so that

$$(\theta, I) = \underset{\sim}{C}(\Theta, J) \tag{8}$$

where $\underset{\sim}{C}$ is a canonical (unit Jacobian) transformation 2π-periodic in Θ. Since we are interested in constructing $\underset{\sim}{C}$ by a perturbation expansion in ϵ, we require $\underset{\sim}{C}:\mathbb{C}^2 \to \mathbb{C}^2$ to lie within the group of canonical transformations containing the identity. We also assume the inverse transformation to exist:

$$(\Theta, J) = \underset{\sim}{C}^{-1}(\theta, I), \tag{9}$$

and assume $\underset{\sim}{C}$ and $\underset{\sim}{C}^{-1}$ to be differentiable. That is, $\underset{\sim}{C}$ is a *diffeomorphism*. These restrictions on $\underset{\sim}{C}$ mean that Θ and J will correspond to action-angle variables, as ordinarily understood, only when the invariant curves $J = const.$ have the same topology as the unperturbed curves $I = const.$ For example, in the case of the physical pendulum, with gravity regarded as the perturbation, $\underset{\sim}{C}$ is an ordinary action-angle transformation only in the region of phase space corresponding to rotational motion.[1] The interpretation of $\underset{\sim}{C}$ for libratory motion will be discussed elsewhere.

In the new action-angle representation, the nth iterate of the point (Θ_0, J_0) is given by composing n times a mapping $\underset{\sim}{S}$, which is similar to $\underset{\sim}{T}_0$, but with a different angular frequency function $\Omega(J)$. That is:

$$(\Theta_n, J_n) = \underset{\sim}{S}(\Theta_{n-1}, J_{n-1}), \tag{10}$$

$$\underset{\sim}{S}(\Theta, J) = (\Theta + \Omega(J, \epsilon), J). \tag{11}$$

From Eqs. (1), (4) and (6) we see that

$$\underset{\sim}{T} = \underset{\sim}{C} \circ \underset{\sim}{S} \circ \underset{\sim}{C}^{-1} \tag{12}$$

The relation between $\underset{\sim}{T}$ and $\underset{\sim}{S}$ is represented pictorially by

$$\begin{array}{ccccccc}
\binom{\theta_0}{I_0} & \underset{\sim}{\overset{T}{\longrightarrow}} & \binom{\theta_1}{I_1} & \underset{\sim}{\overset{T}{\longrightarrow}} & \cdots & \underset{\sim}{\overset{T}{\longrightarrow}} & \binom{\theta_n}{I_n} \\
\uparrow \underset{\sim}{C} & & \uparrow \underset{\sim}{C} & & & & \uparrow \underset{\sim}{C} \\
\binom{\Theta_0}{J_0} & \underset{\sim}{\overset{S}{\longrightarrow}} & \binom{\Theta_1}{J_1} & \underset{\sim}{\overset{S}{\longrightarrow}} & \cdots & \underset{\sim}{\overset{S}{\longrightarrow}} & \binom{\Theta_n}{J_n}
\end{array}$$

The case when $\underset{\sim}{T}$ is integrable for finite ϵ is highly exceptional: in general, no $\underset{\sim}{C}$ exists such that $\underset{\sim}{S}$ is everywhere of the form given by Eq. (11). However, for sufficiently small ϵ there remains an infinity of invariant curves characterized by Eq. (11) with J =const. by the KAM theorem. Because $\underset{\sim}{C}$ is a diffeomorphism, these invariant KAM surfaces are topologically equivalent to the straight line invariant curves of $\underset{\sim}{T}_0$. We term these the *primary* KAM curves to distinguish them from other invariant curves surrounding elliptic fixed points of $\underset{\sim}{T}$.

In Sec. IV we construct a perturbation theory using ϵ as an expansion parameter by proceeding formally as if $\underset{\sim}{T}$ were integrable. If $\underset{\sim}{T}$ is restricted to $\mathbb{R}^2$ this expansion can at best converge only for values of J on the primary KAM curves. Unfortunately, there is no way of telling *a priori* which values of J to examine, since KAM curves are actually characterized by an irrational value of the winding number $2\pi/\Omega(J,\epsilon)$. We follow Greene [12] in assuming that the most robust KAM surfaces, those which disrupt last as ϵ increases, are those possessing inverse winding numbers which we can call *generalized golden means*. That is, the most robust surfaces will be those with values of Ω whose partial fraction expansion is terminated by an infinite number of ones, i.e.:

$$\Omega = 2\pi[a, b, \ldots, y, z, 1, 1, 1, \ldots] \tag{13}$$

with $a, b, \ldots, y, z$ integers. Since Ω selects the most stable KAM surfaces, it would seem preferable to treat Ω, rather than J, as the independent variable. Two such methods are discussed in Secs. V and VI.

From the above discussion it would appear that $\underset{\sim}{C}$ is defined only on a highly pathological, perhaps not even dense, set of values of J or Ω on which differentiation cannot be defined. This is true if Ω is restricted to be real, but by continuing $\underset{\sim}{T}$ and $\underset{\sim}{C}$ into C^2, we shall find that $\underset{\sim}{C}$ is defined in a large connected region of the complex Ω plane. By proceeding formally on the assumption that $\underset{\sim}{C}$ exists we are implicitly working in C^2. The transformation is obtained on KAM curves by analytic continuation back to the real Ω axis.

III. LIE THEORY

There is a one to one correspondence between canonical transformations, such as $\underline{T}$ and $\underline{C}$, which map phase space onto itself, and certain unitary operators which map the space of functions defined on phase space onto itself.[13] For instance, given the operator A corresponding to the canonical transformation $\underline{A}$ (denoted $A \mapsto \underline{A}$), we can construct the transformation as follows:

$$\underline{A}(\Theta, J) = (A\Theta, AJ) \tag{14}$$

Note that $\underline{A}$ is a nonlinear, vector-valued function of a vector, whereas A is a linear, scalar operator defined on functions of Θ and J. The unitary operator A can be shown [7] to have the property that

$$Af(\Theta, J) = f(A\Theta, AJ) = f(\underline{A}(\Theta, J)), \tag{15}$$

where f is any function on phase space. If we choose f to be another canonical transformation $\underline{B}$ with associated unitary operator B, then Eq. (15) gives $(AB\Theta, ABJ) = \underline{B} \circ \underline{A}(\Theta, J)$. Accordingly, we deduce the following *composition rule*: the unitary operator corresponding to the composition of two canonical transformations is the product of the individual unitary operators *in reverse order*. That is

$$AB \mapsto \underline{B} \circ \underline{A}. \tag{16}$$

Similarly, $ABC \mapsto \underline{C} \circ \underline{B} \circ \underline{A}$, and so on. In particular, the operator equation corresponding to Eq. (12) is:

$$T = C^{-1}SC \tag{17}$$

It is well known that a Hamiltonian flow generates a family of canonical transformations parametrized by the time variable. Analogously, by inventing a suitable generator W which plays the role of the Hamiltonian, we can obtain any canonical transformation that is continuously connected to the identity by "advancing" the coordinates in a new variable which plays the role of time (the Lie parameter, ϵ). The canonical transformation $\underline{C}$ is thus forced to be continuously connected to the identity at $\epsilon = 0$, while $\underline{T}$ and $\underline{S}$ are assumed to be continuously connected to the unperturbed mapping, i.e., $\underline{T}_0 = \underline{T}(\epsilon = 0) = \underline{S}(\epsilon = 0)$. Following Dewar [7] we seek "Lie generating functions" $U(\Theta, J, \epsilon)$, $V(\Theta, J, \epsilon)$ and

$W(\Theta, J, \epsilon)$ such that the unitary operators S,T and C obey the following operator equations and boundary conditions:

$$\partial_\epsilon T = L_W T, \quad T(\epsilon = 0) = T_0, \tag{18}$$

$$\partial_\epsilon S = L_U S, \quad S(\epsilon = 0) = T_0, \tag{19}$$

$$\partial_\epsilon C = L_W C, \quad C(\epsilon = 0) = 1, \tag{20}$$

where $\partial_\epsilon \equiv \partial/\partial\epsilon$. L_W denotes the Lie derivative operator, which takes the Poisson bracket of the function it is applied to and W:

$$L_W \equiv (\partial_J W)\partial_\Theta - (\partial_\Theta W)\partial_J, \tag{21}$$

and T_0 is the operator $\exp[\Omega_0(J)\partial_\Theta]$. Similarly, by Eq. (11)

$$S = \exp[\Omega(J, \epsilon)\partial_\Theta]. \tag{22}$$

Comparing Eqs. (19), (21) and (22) we see that $U \equiv U(J, \epsilon)$, such that

$$\Omega(J, \epsilon) = \Omega_0(J) + \int_0^\epsilon d\epsilon' \partial_J U(J, \epsilon'). \tag{23}$$

In similar fashion, we can determine V, since T is associated with the prescribed mapping $\underset{\sim}{T}$. For instance, in the special case of mappings defined by Eqs. (6) and (7), the generating function V is readily verified to be the potential $V(\Theta)$. What we need now is to find equations for W and U. Using Eqs. (17)-(20) and the identity: [7]

$$C^{-1} L_f C = L_{Cf}, \tag{24}$$

for any function f, we find

$$-L_{C^{-1}W} T + L_{C^{-1}U} T + L_{C^{-1}SH} = L_V T,$$

which can be satisfied by choosing the generating function W and an arbitrary constant in U so that

$$(S - 1)W = CV - U. \tag{25}$$

This is the analogue of the "Hamilton-Jacobi equation for the Lie generating function" derived previously for Hamiltonian systems,[7] with U playing the role of the "new Hamiltonian", K. In order to

determine $U(J,\epsilon)$, observe that

$$\langle [S-1]W\rangle = [S-1]\langle W\rangle = 0,$$

where the averaging operation $\langle\ \rangle$ is defined for any f by

$$\langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} d\Theta\, f.$$

The average of Eq. (25) yields

$$U = \langle CV\rangle. \tag{26}$$

Eq. (26) ensures that W will be a generator for an action-angle transformation, but it is still not unique, since we have not yet specified $\langle W\rangle$. The simplest choice is to take $\langle W\rangle = 0$; instead, to facilitate comparison with the Lagrangian perturbation theory, we prefer to require

$$\langle C\Theta - \Theta\rangle = 0. \tag{27}$$

This is not necessarily equivalent to specifying $\langle W\rangle = 0$, but the difference corresponds simply to a trivial action-dependent phase shift in the Θ coordinate. In the case of the Standard and Semistandard mappings, Eq. (27) is equivalent to $\langle W\rangle = 0$.

IV. CANONICAL PERTURBATION THEORY

Adding $(T_0 - S)W$ to both sides of Eq. (25) we write it in the form:

$$[T_0 - 1]W = CV + [T_0 - S]W - U \tag{28}$$

Since $(T_0 - 1)$ is independent of ϵ, Eq. (28) is in a form amenable to solution by simple power-series expansion of $W(\Theta, J, \epsilon)$, which we term a *primitive* perturbation expansion.

In order to avoid working with operators, it is convenient to define auxiliary variables:

$$v \equiv CV, \tag{29}$$

$$\theta \equiv C\Theta, \tag{30}$$

$$w \equiv [T_0 - S]W. \tag{31}$$

The variable $\beta(\Theta, J, \epsilon)$ is simply the old angle in terms of the new phase space coordinates Θ and J. It is needed to implement Eq. (27). The auxiliary variable $v(\Theta, J, \epsilon)$ is simply $V(\theta)$ in the case of the mappings defined by Eqs. (6) and (7).

We now expand W and U according to the convention:

$$W = \sum_{n=0}^{\infty} W_{n+1}\epsilon^n, \tag{32}$$

$$U = \sum_{n=1}^{\infty} U_{n+1}\epsilon^n, \tag{33}$$

(We assume $\langle V \rangle = 0$ so that Eq. (26) implies $U_1 = 0$). The convention for v is

$$v = \sum_{n=0}^{\infty} v_n\epsilon^n, \tag{34}$$

and similarly for θ and w. Differentiating Eqs. (29) and (30) with respect to ϵ, using Eq. (20), and equating coefficients of different powers of ϵ, we find the following simple nonlinear recursion relations for $n \geq 1$:

$$v_n = \frac{1}{n}\sum_{m=1}^{n}\{v_{n-m}, W_m\}, \tag{35}$$

$$\theta_n = \frac{1}{n}\sum_{m=1}^{n}\{\theta_{n-m}, W_m\}, \tag{36}$$

where $\{f, g\}$ denotes the Poisson bracket $L_g f$, and the recursion is initialized with

$$v_0 = V(\Theta), \tag{37}$$

$$\theta_0 = \Theta. \tag{38}$$

The recursion for w_n is much more complicated:

$$w_n = -T_0\sum_{k=1}^{\lfloor n/2 \rfloor}\sum_{l=0}^{\lfloor n/2-m \rfloor}\frac{C_{l,k}}{k!}\partial_\Theta^k W_{n-2k-2l+1}, \tag{39}$$

where

$$C_{l,k} = \frac{1}{lA_0} \sum_{\nu=1}^{l} (\nu k - l + \nu) A_\nu C_{l-\nu} \qquad \text{for } l > 0$$

$$C_{0,k} = A_0^k \tag{40}$$

$$A_\nu = \frac{1}{2(\nu+1)} \partial_J U_{2(\nu+1)}$$

and $[f]$ denotes the integer part of f. We have assumed that V involves only odd Fourier components (as for the Standard and Semistandard map cases), so that $U_{2n+1} = 0$ for all integers n. The coefficients $C_{l,k}$ are independent of Θ, so Eq. (39) does not couple Fourier components.

Denoting $(f - \langle f \rangle)$ by $\tilde{f}$, we split Eq. (28) into its average and fluctuating parts, and equating powers of ϵ, we obtain two more recurrence relations:

$$U_{n+1} = \langle v_n \rangle, \tag{41}$$

$$\tilde{W}_{n+1} = (T_0 - 1)^{-1} (\tilde{v}_n + \tilde{w}_n). \tag{42}$$

By Eqs. (27) and (38) we require

$$\langle \theta_n \rangle = 0 \quad \text{for } n \geq 1.$$

Equation (36) then gives a recursion relation for $\langle W_{n+1} \rangle$:

$$\partial_J \langle W_{n+1} \rangle = -\sum_{m=1}^{n} \langle \{ \theta_{n+1-m}, \tilde{W}_m \} \rangle. \tag{43}$$

Equations (35), (39)-(43) make up a complete set of recurrence relations for all unknowns. The inversion of $T_0 - 1$ in Eq. (42) is most easily accomplished in Fourier space:

$$W_{n+1}^m = \frac{(v_n^m + w_n^m)}{\exp[im\Omega_0(J)] - 1}, \tag{44}$$

for $m \neq 0$. Here, and in the following sections, the Fourier representation f^m of any function $f(\Theta)$ is defined by:

$$f(\Theta) = \sum_{m=-\infty}^{\infty} f^m \exp im\Theta. \tag{45}$$

Clearly, Eq. (44) will be undefined for some value of m whenever $\Omega_0(J)/2\pi$ is a rational fraction. Since $\Omega_0(J) \neq \Omega(J,\epsilon)$ in general we can expect this primitive perturbation series to be divergent even if J is chosen so that Ω is a "generalized golden mean". Also, the complicated nature of the recursion for w_n suggests that adding $(T_0 - S)W$ to Eq. (25) was not the best thing to do. In Sec. VI we present a method which allows $(S - 1)$ to be inverted directly.

V. LAGRANGIAN PERTURBATION THEORY

The work of Greene and Percival[5] based on the averaged Lagrangian variational principle of Percival [3] shows that there is indeed an alternative and simple perturbation method for primary KAM surfaces; this method is convergent (for ϵ small enough) within a strip containing the real axis in the complex angle variable (Θ) plane. Their starting point is the discretization of the Lagrangian differential equations of motion. The use of canonical variables and Poisson brackets is avoided, simplifying the recursion relations significantly. The method does not in general give rise to simple recursion relations. For this reason, we restrict ourselves in this section to an examination of the Standard and the Semistandard maps.

From Eqs. (1)-(5) and (8)-(11) we can show that:

$$\delta^2\theta \equiv \theta(\Theta + \Omega) - 2\theta(\Theta) + \theta(\Theta - \Omega) = -\epsilon V'(\theta). \tag{46}$$

Since J and ϵ can be regarded as constants in Eq. (46), we have suppressed the last two arguments of $\theta(\Theta, J, \epsilon)$.

For a concrete example let us consider the Standard mapping, given by Eq. (6). Defining auxiliary variables:

$$f(\theta, \Omega, \epsilon) = -V'(\theta) = -\sin\theta \tag{47}$$

and

$$v(\theta, \Omega, \epsilon) = V'(\theta) = -\cos\theta \tag{48}$$

we get

$$\partial_t f = -v\partial_\theta \theta, \qquad \partial_t v = f\partial_\theta \theta. \tag{49}$$

The derivatives with respect to t are taken at fixed Ω, rather than at fixed J. We now assume that ϵ is a small parameter, and expand f, v and θ as in Eq. (34). From Eqs. (46) and (49), by equating powers of ϵ, we obtain simple nonlinear recursion relations (c.f. Eqs. (35)-(42)):

$$\begin{aligned} n v_n &= \sum_{\nu=1}^{n} \nu f_{n-\nu}\theta_\nu, \\ n f_n &= -\sum_{\nu=1}^{n} \nu v_{n-\nu}\theta_\nu, \\ \xi^2 \theta_n &= f_{n-1}. \end{aligned} \tag{50}$$

Note that the existence of a finite set of quadratically nonlinear recursion relations is critically dependent on the specific form of V, in contrast to the canonical perturbation theory.

The second difference operator is inverted by Fourier expanding θ_j, v_j and f_j in Θ, getting:

$$\begin{aligned} v_n^m &= \frac{1}{n}\sum_{\nu=1}^{n}\sum_{\mu=-\infty}^{\infty} \nu f_{n-\nu}^{m-\mu}\theta_\nu^\mu, \\ f_n^m &= -\frac{1}{n}\sum_{\nu=1}^{n}\sum_{\mu=-\infty}^{\infty} \nu v_{n-\nu}^{m-\mu}\theta_\nu^\mu, \\ \theta_n^m &= -\frac{\exp im\Omega}{2(\exp[im\Omega]-1)^2} f_{n-1}^m. \end{aligned} \tag{51}$$

Equations (51) furnish us with a closed recursion procedure, provided we start with finite Fourier series in Θ for f_0 and v_0, because the μ summations truncate after a finite number of terms. Note that the recursion formulae are completely algebraic in nature, involving no differential operators. In particular, for the Standard mapping we start with:

$$\begin{aligned} f_0^1 &= \frac{1}{2i}, \qquad f_0^{-1} = -\frac{1}{2i}, \\ v_0^1 &= \frac{1}{2} \qquad v_0^{-1} = \frac{1}{2}, \end{aligned} \tag{52}$$

and $f_0^m = g_0^m = 0$ for $m \neq 1, -1$. The indicated power coefficients form a triangular array, such that the m^{th} power contains Fourier modes ranging from $-m$ through m, and only for those terms where m has the same parity as m are nonzero. I.e.:

$$\begin{array}{ccccccc} \theta_1^1 & 0 & \theta_1^{-1} & & & & \\ \theta_2^2 & 0 & \theta_2^0 & 0 & \theta_2^{-2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_m^m & 0 & \theta_m^{m-2} & \cdots & \theta_m^{-m+2} & 0 & \theta_m^{-m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{53}$$

and similarly for f and ν.

A similar procedure can be followed for the Semistandard mapping. In this case all harmonic coefficients except those that make the only terms that survive in the Fourier power coefficient series are those on the left edge of the triangular array (53). Remarkably, for this case *the Fourier series is the perturbation series*. This allows for a very efficient recursion procedure.

VI. RENORMALIZED CANONICAL PERTURBATION THEORY

A simple renormalization of the resonant denominators in Eq. (44) can be effected by changing from the canonical set $\{\Theta, J, \epsilon\}$ to the noncanonical set $\{\Theta, \Omega, \epsilon\}$ of independent variables [14]. Therefore ∂_J and ∂_{Θ} will *henceforth imply that Ω, rather than J, is to be held fixed*. Where ∂_J and ∂_Θ occurred previously, they must be replaced according to the rule:

$$\partial_J \rightarrow (\partial_\Omega J)^{-1} \partial_\Omega \tag{54}$$

$$\partial_\Theta \rightarrow (\partial_\Omega J)^{-1} [(\partial_\Theta J)\partial_\Omega - (\partial_\Omega J)\partial_\Theta] \tag{55}$$

We again need the auxiliary variables ν and θ, but it does not matter since $S = 1$ is now independent of ϵ, and Eq. (25) is a convenient form for recursively generating $\vec{W}(\Theta, \Omega, \epsilon)$ as an assumed power series in ϵ. Substitution of the forms of Eqs (32) and (54).

An equation for J is obtained using Eqs. (20), (29), (54) and (55):

$$[(\partial_\alpha J)\dot{\Phi} - (\partial_\beta J)\dot{\Theta}_\beta]_t = [\![v, W]\!] \tag{56}$$

where $[\![\ ,\ ]\!]$ denotes the *modified Poisson bracket*

$$[\![f, g]\!] = (\partial_\alpha f)(\partial_\beta g) - (\partial_\beta f)(\partial_\Theta g) \tag{57}$$

An equation similar to Eq. (56) holds for θ. Averaging, and applying Eq. (27), we find an equation for W:

$$\partial_\alpha \langle W \rangle = -\langle [\![\tilde{\theta}, \tilde{W}]\!] \rangle. \tag{58}$$

An equation for J is obtained by differentiating Eq. (25) with respect to ϵ at constant J and using Eqs. (54) and (55):

$$J(\Omega, \epsilon) = J_0(\Omega) - \int_0^\epsilon d\epsilon' \partial_\beta U(\Omega, \epsilon'), \tag{59}$$

where $J_0(\Omega)$ is the solution of the equation

$$\Omega_0(J_0) = \Omega. \tag{60}$$

Inserting Eq. (59) in Eq. (56), using Eqs. (25), (32)–(34) and the expansions:

$$J = J_0(\Omega) - \sum_{n=1}^{\infty} J_n \epsilon^n, \tag{61}$$

$$\theta = \Theta + \sum_{n=1}^{\infty} \theta_n \epsilon^n, \tag{62}$$

and finally expanding all functions in Fourier series over Θ (for the invert $S = 1$), we obtain a closed set of recursive formulae:

$$W^m_{n+1} = \frac{1}{\exp(i m \Omega_0) - 1} \tilde{t}^m_n \quad \text{for } m \neq 0, \tag{63}$$

$$v^{\nu}_{n} = \frac{1}{n}\sum_{m=0}^{n-1}\Big(\sum_{\nu}\nu(j-\nu)[U_{n-m}\partial_{\theta}W^{\nu}_{m+1} - \partial_{\theta}U^{\nu}_{n-m}W^{j-\nu}_{m+1}] - \partial_{\Omega}\zeta_{n-m}\partial_{\theta}U_{m} + \frac{n-m+1}{m}\,v^{j}_{n-m}\partial^{j}_{\Omega}U_{m}\Big) \tag{64}$$

where W and ζ have been expanded according to Eqs. (32), (34) and (45). θ obeys an equation identical to Eq. (64) with ζ replaced by θ. W^{0}_{n} is obtained from Eq. (58):

$$W^{0}_{j+1} = \int d\Omega \sum_{\nu=0}^{j-1}\sum_{m\neq 0}[\![\theta^{m}_{j-\nu}\,W^{\nu-m}_{\nu+1}]\!] \tag{65}$$

The recursion formulae are fairly straightforward, and the resonant denominators now involve only Ω directly. This achieves the desired renormalization. In fact the theory described in this section reproduces the results of the Lagrangian theory of Sec. V for the Standard and Semistandard mappings.

VII. PERTURBATION SERIES AND A RESONANCE OVERLAP CRITERION

We have constructed different perturbation theories which share certain characteristics: they are recursive, do not involve any approximations beyond those inherent in the perturbative formulation itself, and have relatively simple recursion formulae. Their practical value depends strongly on our ability to obtain high-order results. The recursion relations we have described so far are simple enough to permit this, in contrast to other possible avenues of attack, such as superconvergent expansions.[2]

The primitive perturbation theory yields, at best, an asymptotic series. While reasonable results can often be obtained from such series,[15] a convergent series is clearly preferable (for the special case of the Semistandard mapping, the primitive perturbation theory is equivalent to the renormalized theory, since in this case $J = \Omega$ identically). The renormalized theory is simple and convergent on KAM surfaces for small enough values of ϵ, in both its canonical and Lagrangian formulations. The formulae of the renormalized canonical theory are of wider applicability than those of the Lagrangian formulation, since the recursion relations are closed through the use of a generating function, instead of the properties of the potentials used. We pay a price for this, however: the canonical theory is more difficult to

use, because it involves differential operators in its recursion formulae, as opposed to purely algebraic recursion relations for the Lagrangian theory.

The recursive formulations obtained allow us to implement the perturbation theories using a computer. We can obtain analytic results to relatively high order in ϵ through the use of an automatic algebraic manipulator.[16] We have used MACSYMA to obtain results to order ϵ^{16} before the complexity of the expressions involved made it impractical to proceed further. The analytic results provide us with significant insight into the resonance-resonance interaction mechanism. The first few orders in the primitive perturbation theory for W' and U are given by:

$$W' = \epsilon \frac{\Xi}{2(\Upsilon-1)} + \epsilon^2 \frac{\Xi^2}{4(\Upsilon-1)^3(\Upsilon+1)} + \epsilon^3\left(\frac{\Upsilon(\Upsilon^2+1)(\Upsilon^2+3\Upsilon+1)\Xi}{16(\Upsilon-1)^5(\Upsilon+1)^2} + \frac{\Upsilon^2(\Upsilon^2+4\Upsilon+1)\Xi^3}{16(\Upsilon-1)^5(\Upsilon+1)^2(\Upsilon^2+\Upsilon+1)}\right) + O(\epsilon^4) \tag{66}$$

$$U = -\frac{\epsilon\Upsilon}{2(\Upsilon-1)^2} - \frac{\epsilon^3\Upsilon^2(\Upsilon^4+2\Upsilon^3+4\Upsilon^2+2\Upsilon+1)}{8(\Upsilon-1)^6(\Upsilon+1)^2} + O(\epsilon^5) \tag{67}$$

where $\Xi \equiv \exp(i\Theta)$ and $\Upsilon \equiv \exp(iJ)$. By using Eq.(23) we can attempt to determine the angular frequency for a given value of J, but the denominators in U will produce undefined values at the resonances. A selective resummation of the primitive perturbation series could provide us with an alternative renormalization,[17] but that requires a much better picture of the structure of the series than that which we have been able to obtain analytically.

In contrast to the primitive theory, the renormalized perturbation theory in either its Lagrangian or canonical formulations uses Ω as the expansion variable. The resonant denominators in the renormalized theory will never be zero provided we choose an irrational value for $\Omega/2\pi$. The perturbation theory will converge for small enough ϵ on a surface characterized by one such value. As ϵ is increased a point will be reached where the perturbation series will cease to converge. This point marks the destruction of the particular KAM surface corresponding to the chosen value of Ω. It is sufficient to consider the perturbation series for W' since if W' diverges, so will all other quantities.[5]

The perturbation theories can provide information about the local properties of the mapping close

to a resonance. By expanding every term in the ϵ series for W in partial fractions we obtain a series whose general form is:

$$W = \sum_{\mu} W_{\mu}$$

with

$$W_{\mu} = \sum_{n} \sum_{m} \epsilon^{n} \exp(im\Theta) \sum_{\nu=0}^{\nu_{max}} \frac{a_{nm\mu\nu}}{(Y - Y_{\mu})^{\nu}}, \tag{68}$$

where $Y \equiv \exp(i\Omega)$, ν_{max} is an integer which depends on n,m and μ, $a_{nm\mu\nu}$ is some constant, and $Y_{\mu} = \exp(2i\pi\mu)$ for $\mu = q/p$, where q and p are any two mutually prime integers such that $0 \leq \mu < 1$. Y_{μ} is termed a *primitive p'th root of unity*.[18] We approximate W_{μ} by retaining only the most divergent term, $\nu = \nu_{max}$, and get an approximation which is good near the $Y = Y_{\mu}$ resonance

$$W_{\mu} \approx \sum_{m} \sum_{n} \frac{a_{nm\mu} \epsilon^{n} \exp(im\Theta)}{(Y - Y_{\mu})^{\nu_{max}}}. \tag{69}$$

To examine a simple case, we restrict ourselves to the Semistandard map, since in this case $a_{nm\mu\nu} = a'_{n\mu\nu}\delta^{m}_{n}$, we thus get:

$$W_{\mu} \approx \sum_{n} \frac{a'_{n\mu} \epsilon^{n} \exp(in\Theta)}{(Y - Y_{\mu})^{\nu_{max}}} \tag{70}$$

A table of $a'_{n\mu}$ and ν_{max} for the Semistandard map is shown in table I, while table II shows the values for the coefficients $a_{nm\mu\nu}$ for one particular resonance, $\mu = 0$. A more refined approximation than Eq. (69) can be obtained by considering more terms in Eq. (68) than those with $\nu = \nu_{max}$ (c.f. table II).

The "*radius of divergence*" of the W_{μ} series in the complex Y-plane can be determined by taking ratios of terms in Eq. (69) for a fixed value of ϵ. The detailed scaling of the size of the divergent regions can be extracted from table I. We take ratios of those terms in Eq. (70) where the differences in ν_{max} are 2. By following this procedure, we obtain fixed ratios for each resonance, from which we can conclude that the radius of divergence at $Y = Y_{\mu}$ scales as $\epsilon^{\frac{\lambda}{2}}$, where λ is the order in ϵ at which the resonance at Y_{μ} first appears. This means that the size of the resonance at $Y = 1$ scales as $\sqrt{\epsilon}$, at $Y = -1$ it scales as ϵ, etc.: this is the scaling we would expect from direct mode-coupling between the primary resonances.

The partial fraction decomposition provides a more rigorous version of the overlapping resonances picture of the breakup of KAM surfaces than does Chirikov's criterion.[5] Consider an irrational value of $\Omega/2\pi$. Define $\rho_n(\Omega)$ as the "radius of divergence" associated with the nth convergent of $\Omega/2\pi$, $\frac{q_n}{p_n} \equiv q_n/p_n$ [19] (n-th order truncations of the continued fraction expansion). The invariant curve associated with Ω will be preserved only if $\rho_n(\Omega) \to 0$ faster than $q_n/p_n \to \Omega/2\pi$ as $n \to \infty$. As ϵ is increased, $\rho_n(\Omega)$ drops in size more and more slowly as $n \to \infty$, and therefore the invariant curve will be destroyed when ϵ is large enough. Even if an invariant curve is destroyed on the real Ω axis, "surfaces" will still be preserved for complex Ωs far enough away from the axis, to avoid being overlapped by any of the circles of divergence (see Fig. 1). Because $\rho_n(\Omega)$ decreases rapidly with n, the region of convergence will have "tendrils", touching the real line at the preserved KAM curves, if any of them are left. An alternative method for extrapolating the behavior of $\rho_n(\Omega)$ at infinitely high order has been suggested by Escande and Doveil,[20] through the use of renormalization group techniques.

VIII. THE STRUCTURE OF THE FOURIER SERIES

Assuming that the ϵ series for the Fourier coefficients converges, and summing over different orders in ϵ, we can examine the existence of the transformation by studying the convergence of the Fourier series, without using the partial fraction decomposition. This is the approach used by Greene and Percival.[5] Because of the superposition of all the different resonances, the Fourier coefficients have a very irregular dependence on m, the mode index, necessitating the calculation of the series to very high order. The Lagrangian formulation seems ideally suited to this task, because the algebraic nature of its recursion relations permits us to obtain extremely high order results ($O(\epsilon^{2000})$ for the Semistandard mapping) in a purely numerical fashion.

The coefficients of the Fourier series must decay exponentially if the series is to converge. We can estimate the decay rate of the Fourier coefficients by least-squares fits of exponentials or exponential-polynomial products to the values of the coefficients of the series for θ obtained from the perturbation theory. This yields the value for the critical ϵ needed to break the golden-mean surface ($\Omega/2\pi = (\sqrt{5} -$

[1,2] to within 1% of the value Greene [12] obtained for the Standard map ($\epsilon_{crit} = 0.97$), considering only the first 80 Fourier coefficients. Fig. 2 shows a least squares fit of $\alpha m^{\beta} \exp m\gamma$ to the Fourier coefficients for the Standard mapping for the golden mean, where we adjust α, β and γ. We show two such fits, for different values of ϵ; γ changes sign for $\epsilon = 0.96\ldots$, which marks the point where the Fourier series starts to diverge. Higher accuracy can be obtained simply by considering a larger number of Fourier coefficients.

The Fourier coefficients for both the Standard and the Semistandard maps show some very interesting structure for real Ω. The structure is a direct consequence of the resonant denominators. If we construct the convergents [19] q_n/p_n of any irrational value of Ω, we can see that the Fourier coefficients with mode index $m = p_n$ are the slowest decaying ones. Further subsequences can be picked by a simple algorithm. It is easy to see that the subsequence $m = p_n + p_{n-1}$ again decays slowly, though not as slow as the subsequence $m = p_n$. In general, coefficients with mode index corresponding to linear combinations of p_n's with integer coefficients form slowly decaying subsequences, as can again be expected from parametric mode coupling arguments. Subsequences formed by modes where m cannot be expressed as one such linear combination will decay the fastest, and in general three-mode combinations decay faster than two-mode combinations. It is sufficient to fit exponential-polynomial decay curves to those coefficients decaying the slowest to determine convergence or divergence of the whole series. Figure 3 shows the absolute value of the Fourier coefficients of θ for the Semistandard mapping. The exponential-power fit to each subsequence is now much better than in Fig. 2.

When we analytically continue the mappings into the complex Ω plane, the spiky structure of the Fourier coefficients is preserved close to the real axis. As we get farther from the real axis, the magnitude of the Fourier coefficients begins to drop dramatically, again showing the convergence of the Fourier expansion in the same domain where the ϵ expansion converges. Fig. 4 shows a comparison of the Fourier coefficients on and off the real axis.

By scanning over complex values of Ω, we determine numerically the regions of divergence of the Fourier series for fixed values of ϵ by looking at the exponential decay of the Fourier coefficients. Fig.

5 shows such a scan for the Semistandard mapping, for Ω in the vicinity of the golden mean. The "tendrils" structure is again obtained. In the case of the Semistandard map this is not surprising, since the Fourier series is equivalent to the perturbation series in ϵ; this is not true in the Standard mapping case, where the convergence studies can be carried out independently of each other.

IX. SUMMARY

The renormalized perturbation theory, both in its Lagrangian as well as its canonical formulation appears as a much more rigorous tool to study non-integrable mappings than many others hitherto used. The theory is convergent on KAM surfaces and is remarkably useful both in understanding the analytical structure of resonance-resonance interactions as well as obtaining numerical estimates (such as critical values for the breakup of the KAM surfaces).

The canonical formulation has a simpler recursive structure in general than the Lagrangian theory, but it involves differential operators that limit its use. Even though these differential operators are normal to the KAM surfaces, their presence is not of great concern because they can be understood as derivatives taken in the complex Ω plane; since a "tendril" of analyticity reaches down to a preserved KAM surface, the differential operators can be thought of as the analytic continuation to the real line of well defined operators in the complex plane.

The process of analytic continuation allows us to understand the process of the breakup of KAM surfaces. Since we can compute the size of the divergent regions in the complex angular frequency plane, we can see clearly that KAM surfaces are destroyed by an overlapping process reminiscent of that of Chirikov.[5] In contrast to the Chirikov picture, however, the resonances that ultimately destroy a given KAM surfaces *are not the primary resonances*; they are the high-order resonances lying nearby. The scaling of the widths of these resonances with ϵ seems to be consistent with a picture where the primary resonances beat directly against each other, i.e., where we neglect the interaction between resonances that have themselves been generated by beating. This fact is substantiated by a comparison between the

divergent regions predicted by the Fourier series convergence studies and the analytical estimations of the radii of divergence of the perturbation series.

The values of the perturbation parameters necessary to break any given KAM surface can be obtained with arbitrary accuracy given enough computer time, using the theories described in this paper to compute the regions of divergence associated with high-order resonances. This is a significant achievment for a perturbation theory, since, as discussed above, the existence of KAM surfaces is determined by the interaction of resonances of arbitrarily high order.

A study of the libratory motion of the physical pendulum, to be reported elsewhere, suggests that secondary (islandlike) KAM surfaces correspond to certain complex values of Ω. If this is true as well for nonintegrable systems, then the perturbative theories, coupled with the analytic continuation methods, may allow us to explore the nature of stochastic motion in a region where the quasilinear diffusion approximation fails.[21]

ACKNOWLEDGMENTS

We gratefully acknowledge useful discussions with Drs. J.M. Greene, Y.C. Lee and D. Escande. This work was supported by the United States Department of Energy under contract number DE-AC02–76–CH03073. The high order calculations were done with the aid of MACSYMA, a large symbolic manipulation program developed at the MIT Laboratory for Computer Science and supported by the National Aeronautics and Space Administration under grant NSG 1323, by the Office of Naval Research under grant N00014–77–C–0641, by the U. S. Department of Energy under grant ET–78–C–02–4687 and by the U. S. Air Force under grant F49620–79–C–020.

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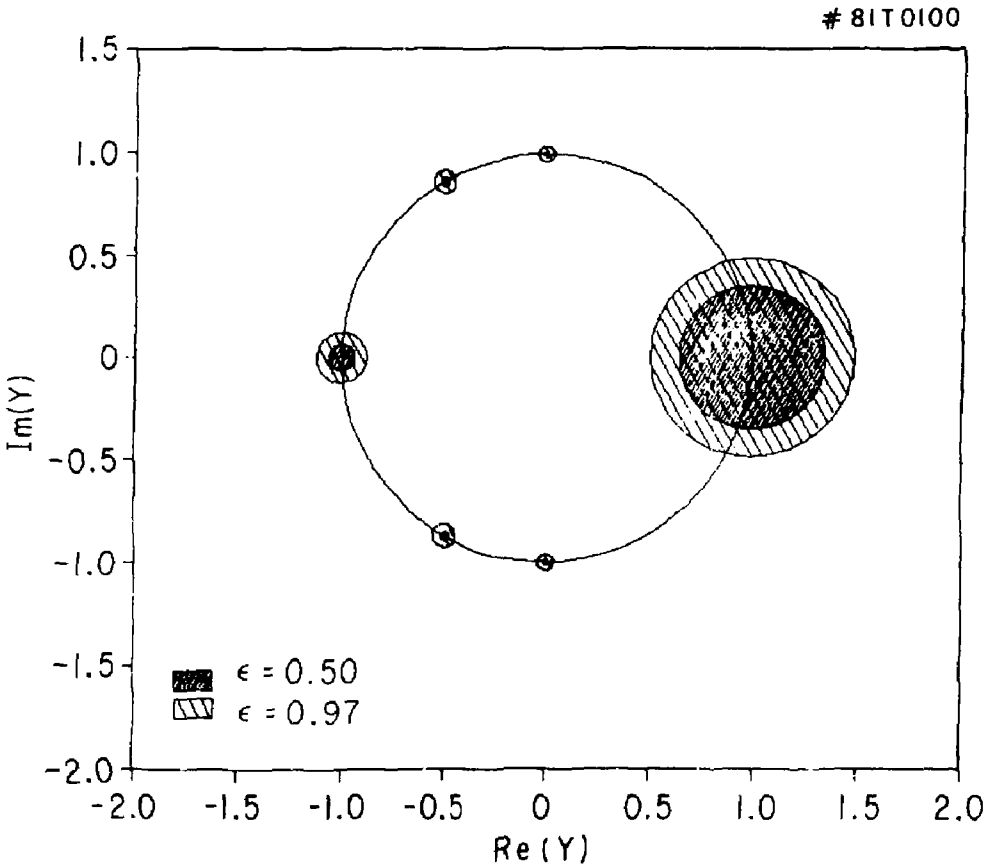
Tables

TABLE I. Local approximations to the generating function W for the Semistandard mapping at selected resonances. Notation corresponds to Eq. (70).

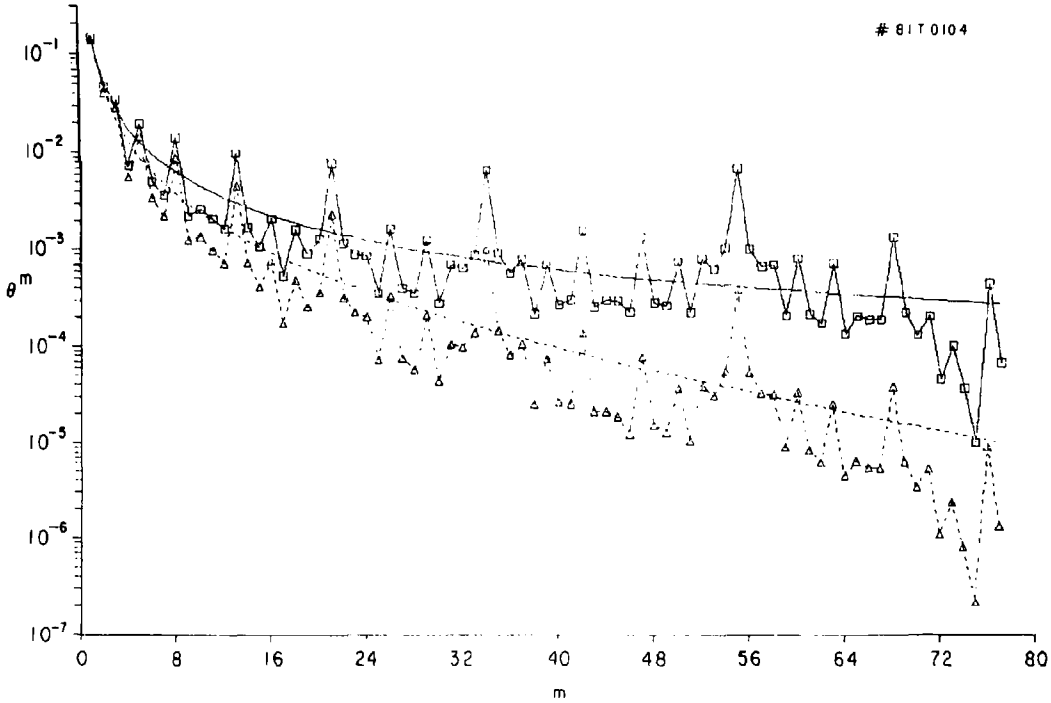
	$Y = 1$		$Y = -1$		$Y = e^{(2i\pi/3)}$		$Y = i$	
n	ν_{max}	$a'_{n\mu}$	ν_{max}	$a'_{n\mu}$	ν_{max}	$a'_{n\mu}$	ν_{max}	$a'_{n\mu}$
1	1	$\frac{-1}{2}$						
2	3	$\frac{1}{8}$	1	$\frac{1}{32}$				
3	5	$\frac{-1}{32}$	2	$\frac{-1}{256}$	1	$\frac{-e^{i\pi/3}-1}{144}$		
4	7	$\frac{1}{128}$	3	$\frac{-1}{2048}$	2	$\frac{2e^{i\pi/3}-1}{2592}$	1	$\frac{-5i}{1536}$
5	9	$\frac{-1}{512}$	4	$\frac{1}{16384}$	2	$\frac{e^{i\pi/3}-1}{7776}$	2	$\frac{-5i-5}{23328}$
6	11	$\frac{1}{2048}$	5	$\frac{1}{131072}$	3	$\frac{1}{41472}$	2	$\frac{5}{373248}$
7	13	$\frac{-1}{8192}$	6	$\frac{-1}{1048576}$	4	$\frac{-5(e^{i\pi/3}-1)-1}{1492992}$	2	$\frac{-5i-5}{3359232}$
8	15	$\frac{1}{32768}$	7	$\frac{-1}{8388608}$	4	$\frac{7e^{i\pi/3}}{4478976(e^{i\pi/3}+2)}$	3	$\frac{-25i}{40310784}$
9	17	$\frac{-1}{131072}$	8	$\frac{1}{67108864}$	5	$\frac{-(e^{i\pi/3}-1)-1}{11943936}$	4	$\frac{-25}{362318438(i-1)}$

TABLE II. Coefficients for the partial fraction expansion of W near the primary resonance ($Y_\mu = 1$) for the Semistandard map. Notation corresponds to Eq. (68).

	$a_{n\nu}$										
n	$\nu = 1$	2	3	4	5	6	7	8	9	10	11
1	$-\frac{1}{2}$										
2	$-\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$								
3	$\frac{-23}{4608}$	0	[illegible]	$-\frac{1}{32}$	$-\frac{1}{32}$						
4	$\frac{-125}{165888}$	$\frac{-313}{110592}$	[illegible]	$-\frac{1}{256}$	$-\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{128}$				
5	[illegible]	$\frac{-451}{165888}$	$\frac{451}{165888}$	0	[illegible]	$\frac{3}{1024}$	$\frac{1}{1024}$	$\frac{-1}{256}$	$\frac{-1}{512}$		
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$\frac{51}{32768}$	$\frac{1}{768}$	[illegible]	[illegible]	[illegible]	$\frac{1}{2048}$



[illegible] Bilinear Mapping. [illegible] plane.


#81T0104
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
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θ^m
0
8
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72
80
m

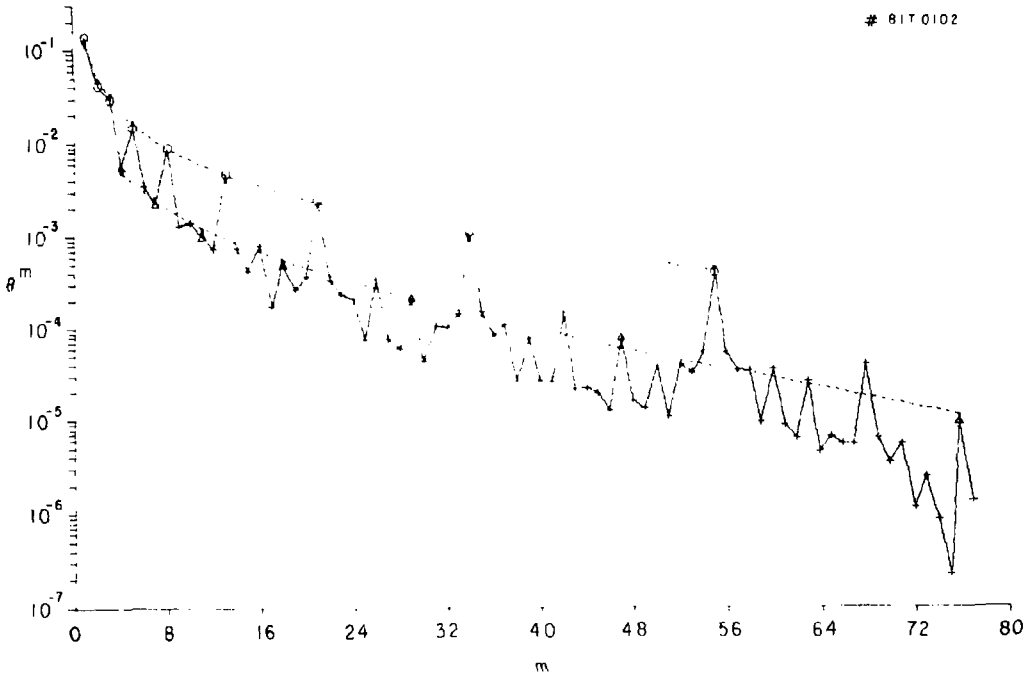

FIG. 3 — Slowly decaying subsequences for the Standard Mapping. We show a case for $\Omega/2\pi = 2/3 - \sqrt{5} = 0.2111...$. Compare the positions of the peaks with those in fig. 2, where $\Omega/2\pi = 0.111...$. The dashed curves pass through the "slowly decaying" subsequences of Fourier coefficients.

Fig. 30

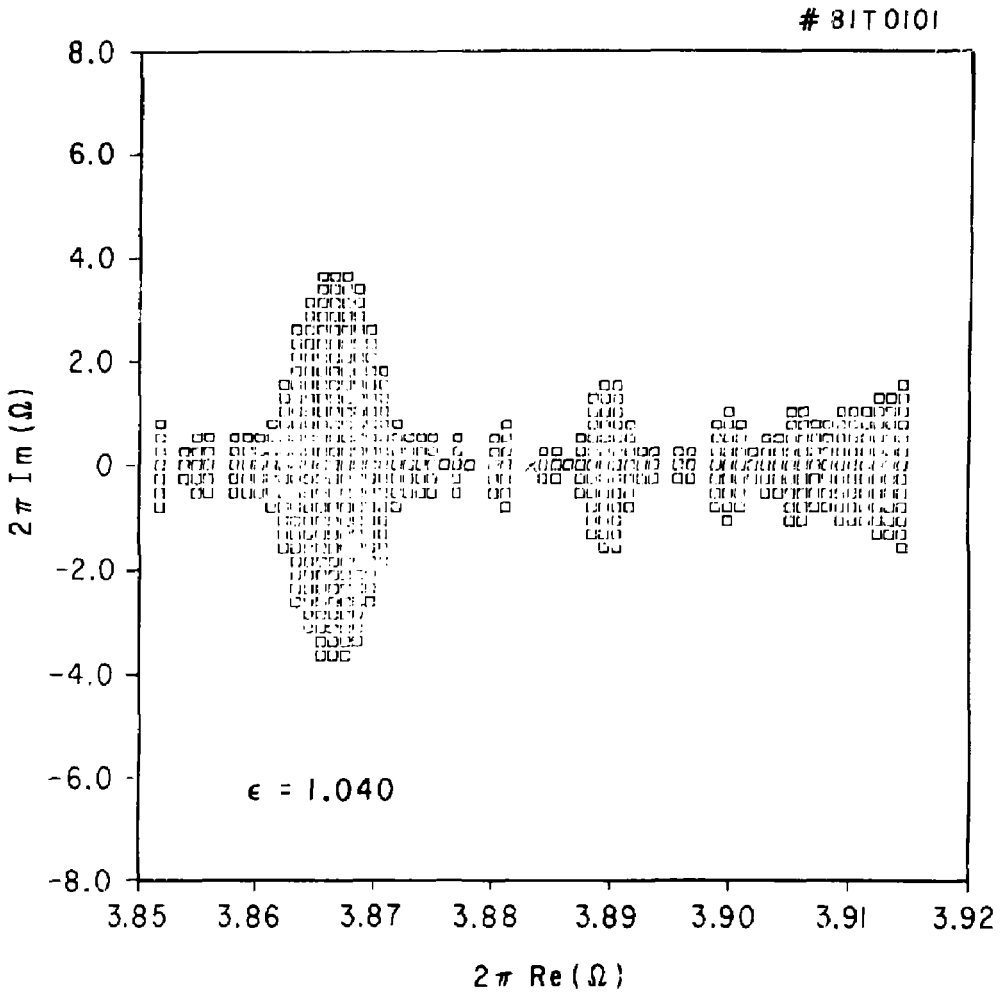

FIG. 5. Numerical scan about $\Omega/2\pi = (\sqrt{5}-1)/2$. The dots correspond to cases where the numerical fit of exponential decay curves to the Fourier coefficients of the Semistandard Mapping indicate divergence of the perturbation series, using $O(\epsilon^{100})$ perturbation theory. Tendrils of convergence can be seen reaching down to the real axis at the preserved KAM surfaces.