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FINITE SIZE SCALING AND PHENOMENOLOGICAL RENORMALIZATION

by

B. Derrida, L. de Seze
SPT and SPSRM, CEN Saclay, 91190 Gif-sur-Yvette France

and
J. Vannimenus
GPS, Ecole Normale Supérieure, 24 rue Lhomond 75005 Paris, France

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Abstract : The basic equations of the phenomenological renormalization method are recalled. A simple derivation using finite-size scaling is presented. The convergence of the method is studied analytically for the Ising model. Using this method we give predictions for the 2d bond percolation. Finally we discuss how the method can be applied to random systems.

I Introduction

The phenomenological renormalization (PR) method, first introduced by Nightingale[1], is a very powerful tool for studying the critical properties of a large class of models in statistical mechanics. The method consists in calculating the thermodynamic properties of one dimensional systems (like infinite strips of finite width) and in extracting from this information the critical properties of infinite systems in higher dimension. The main interests of the method are the following :

- it gives satisfactory results with a very reasonable amount of calculation
- the results can be improved systematically by increasing the width of the strips
- the convergence of the results is much more rapid than that of the Monte Carlo renormalization (a power law convergence instead of a logarithmic one)
- in contrast to most real-space renormalizations where there is a proliferation of interactions, here, only one parameter is renormalized.

We first recall the basic equations of the PR method. Then we explain how these equations can be derived from the finite size scaling hypothesis. For the 2d Ising model, we calculate analytically how the estimations of the critical temperature and the critical exponent ν converge when the width of the strip increases. Numerical results on 2d bond percolation are presented. Finally the application of the method to random systems is discussed.

II The phenomelogical renormalization method

Suppose that one wants to calculate the critical temperature and the critical exponents of a two dimensional system (for example the Ising model). For simplicity, the only parameter in the model is the temperature T. Using the transfer matrix, one can calculate any thermodynamic quantity $Q_n(T)$ (like the correlation length $\xi_n(T)$, the magnetic susceptibility $\chi_n(T)$...) for an infinite strip of finite width n. Following Nightingale[1], let us write the fundamental equation of the PR method :

$$\frac{\xi_n(T)}{\xi_m(T')} = \frac{n}{m} \qquad (1)$$

This equation establishes a correspondence between two strips : the strip of width n at temperature T is related by a scaling transformation to a strip of width m at temperature T' : this transformation is a contraction of ratio n/m of the width of the strip. Thus the correlation length which is the characteristic length along the strip has to be contracted in the same ratio.

The main hypothesis of the PR method is to assume that the relation (1) between T and T' *depends only on the ratio* n/m. Therefore if $\xi_\infty(T)$ is the correlation length of the infinite system, one has

$$\frac{\xi_n(T)}{\xi_m(T')} = \frac{n}{m} = \frac{\xi_\infty(T)}{\xi_\infty(T')} \tag{2}$$

Then from (2), one can find the critical temperature T_c and the critical exponent ν ($\xi_\infty(T) \sim |T - T_c|^{-\nu}$) as usual with real space renormalizations

$$\frac{\xi_n(T_c)}{\xi_m(T_c)} = \frac{n}{m} \tag{3}$$

$$1 + \frac{1}{\nu} = \frac{\log\left[\frac{d\xi_n}{dT}(T_c) / \frac{d\xi_m}{dT}(T_c)\right]}{\log[n/m]} \tag{4}$$

Suppose that we want to calculate another critical exponent ω which describes the critical behaviour of a quantity $Q_\infty(T)$ of the 2d system. [$Q_\infty(T) \sim |T - T_c|^{-\omega}$]. Assuming again that $Q_n(T)/Q_m(T')$ depends only on the ratio n/m, one has :

$$\frac{Q_n(T)}{Q_m(T')} = \frac{Q_\infty(T)}{Q_\infty(T')} \tag{5}$$

Then the critical exponent ω can be obtained by :

$$\frac{\omega}{\nu} = \frac{\log[Q_n(T_c)/Q_m(T_c)]}{\log[n/m]} \tag{6}$$

where T_c and ν are given by (3) and (4).

III Derivation of the PR equations from the finite-size scaling hypothesis

We shall show now that the basic equations (2) and (5) of the PR method are consequences of finite-size scaling[2,3]. The content of the finite-size scaling hypothesis is to assume the existence of scaling functions F_Q such that

$$Q_n(T) \sim Q_\infty(T) F_Q[n/\xi_\infty(T)] \tag{7}$$

This relation is expected to be valid when the size n of the system is large and when T is in the neighbourhood of T_c. Notice that when T approaches T_c, $Q_\infty(T)$ and $\xi_\infty(T)$ are singular whereas $Q_n(T)$ remains regular. Therefore the function $F_Q(z)$ must behave like $z^{\omega/\nu}$ for $z \to 0$ in order to eliminate the singularities of Q_∞ and ξ_∞.

Consider now two temperatures T and T' such that

$$\frac{\xi_\infty(T)}{\xi_\infty(T')} = \lambda \tag{8}$$

For any choice of λ, equation (8) provides a relation between T and T'. Using the expression (7), we can write the ratio $Q_n(T)/Q_m(T')$ as :

$$\frac{Q_n(T)}{Q_m(T')} = \frac{Q_\infty(T)}{Q_\infty(T')} \frac{F_Q[n/\xi_\infty(T)]}{F_Q[m/\xi_\infty(T')]} \tag{9}$$

If we choose the ratio $\lambda = n/m$, the arguments of the two functions F_Q in (9) are identical. Then, the function F_Q is eliminated in (9) and one recovers equation (5). Equation (2) follows as a particular case of equation (5), when Q is equal to ξ.

IV Application to the Ising Model

It is never possible to make numerical calculations on very large strips. So the corrections to finite-size scaling are not negligible. This is why the estimations $T_c(n,m)$ and $\nu(n,m)$ which are calculated by the equations (3) and (4) differ from the exact values T_c and ν and converge to these exact values only when n or m becomes infinite. In order to estimate the accuracy of the method, it is interesting to know the convergence law of $T_c(n,m)$ and $\nu(n,m)$. This convergence law can be calculated analytically for the 2d Ising model. The Hamiltonian of the 2d Ising model is

$$H = -K \sum_{<ij>} \sigma_i \sigma_j \quad \text{with} \quad \sigma_i = \pm 1 \tag{10}$$

From the exact solution, one knows that :

$$K_c = \frac{1}{2} \log(1 + \sqrt{2}) \quad \text{and} \quad \nu = 1 \tag{11}$$

Using the exact expression (that one can find in reference 1) of the correlation lengths ξ_n for strips of width n with periodic boundary conditions, one can show that the estimations of K_c and ν obtained by solving (3) and (4) for large n and m are :

$$K_c(n,m) = \frac{1}{2} \log(1 + \sqrt{2}) - \frac{\pi^3}{192} \frac{\lambda(\lambda+1)}{n^3} + \dots \tag{12}$$

$$\nu(n,m) = 1 - \frac{\pi^2 \log 2}{24} \frac{\lambda^2 - 1}{n^2 \log \lambda} + \dots \tag{13}$$

where $m = n/\lambda$. One sees that if the ratio n/m remains finite when n and m increase, the convergence is rather rapid (a power law). This convergence is optimal for λ as close to 1 as possible. Therefore the best choice for m is $m = n-1$. One can notice that if only $n \to \infty$ and m remains finite, then $\lambda \sim n$ and the convergence for ν becomes logarithmic as in Monte Carlo renormalizations[16,17].

V Application to percolation

Following Nightingale[1], the Phenomenological Renormalization Method has been used to study a large class of models : generalized Ising models[4,5], Ising antiferromagnets in a magnetic field[6], lattice gas models[7,8], quantum spin systems[9], Lee and Yang singularities[10], percolation[11,12], directed percolation[illegible], self avoiding walks[14], lattice animals[12]...

We present in table I results recently obtained[12] for bond percolation on a square lattice. The correlation lengths were calculated for strips of width n with periodic boundary conditions by the transfer matrix method[11]. For each choice of n and m, the estimations of p_c and ν were obtained by solving the equation (3) and (4)

n	m	p_c	ν
2	1	.50260	1.2410
3	2	.48559	1.2015
4	3	.49133	1.2374
5	4	.49563	1.2710
6	5	.49774	1.2922
7	6	.49873	1.3047
8	7	.49921	1.3121
9	8	.49948	1.3169
Extrapolation		.5000 ±.0002	1.332 ± .003
Expected values		.5(exact)	4/3[15]

Table I

The extrapolations[12] were obtained by assuming a power law convergence of the results[7]. We see here the advantages of the PR method : first, even when n and m are small, the estimations of p_c or ν are rather satisfactory ; Second the extrapolation $(n \to \infty)$ is much easier than in Monte Carlo renormalizations[16,17] where the numbers to extrapolate are always obtained with errorbars. Moreover the convergence $[4/3 - \nu(n) \sim n^{-x}$ with $x = 2.3 \pm .4]$ for large n of the exponent $\nu(n)$ is more rapid than in the Monte Carlo renormalizations $[4/3 - \nu(n) \sim \log^{-1} n]$. Last but not least, the calculations do not cost much computer time (the PR for widths n = 9 and m = 8 took less than 5 minutes of IBM 3033).

VI Application to models with random interactions

For pure systems, the thermodynamic properties of infinite strips can be calculated from the eigenvalues of transfer matrices. For random systems (e.g. random magnets) the transfer matrices become random and the eigenvalues of the transfer matrices are replaced by the Liapounov exponents which describe the asymptotic behaviour of a product of a large number of transfer matrices. Although a lot is known about the existence of these Liapounov exponents, there does not exist rules to calculate them. Therefore one must use the brute-force method of multiplying a large number N of random matrices (the error on the Liapounov exponent is at least of order $N^{-1/2}$). These statistical errors make more difficult the calculations of the derivatives in equation (4). However the ideas of finite-size scaling remain in principle valid for random systems. Recently Pichard and Sarma[18] used this idea to study the problem of localization and could find rather accurate estimations of the critical behaviour of correlation lengths. In the same spirit, a polymer in a random medium has been studied recently[19].

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